q-Opers, QQ-System and Bethe Ansatz

(BPS/CFT conference) conference CIRM 09/13/19

w/ D Sage, A Zeitlin $((SL(n), q)$-opers$)$

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Motivation 1) BPS/CFT correspondence, dualities between integrable systems and gauge/string theory

In particular, quantum/classical duality [PK, Gaiotto 2013]

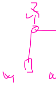
quantum/quantum

XXZ / tRS or 3d $\mathcal{N}=4$ quiver gauge theory / 4d $\mathcal{N}=2^*$ theory w BPS boundary conditions

$T[U(Q)]$, $U(N)$ on $\mathbb{R}^2 \times S^1$

Moduli space of flat connections on $T^2 \setminus \{pt\}$

$ABA^{-1}B^{-1} = $

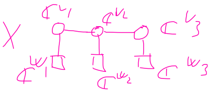


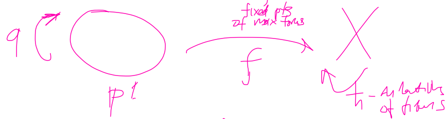
2) Recent progress in enumerative AG (Okounkov et al [AFO], [AO]) [PSZ], [KPSZ]

Quantum equivariant K-theory of Nakajima quiver varieties

In physics terms $V_{\vec{p}} = \left(\text{equiv K-th push forward of moduli space of quasimaps from } \mathbb{P}^1 \text{ to } X\right) = Z^{3d}_{\text{vort}}$ of quiver gauge theory corresponding to X ($\mathcal{N}=2^*$)

k-th vertex function of X



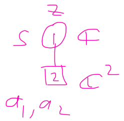


$V \underset{q\to 1}{\sim} e^{\frac{\widetilde{W}_{YY}(\vec{a},\vec{z})}{\log q}}$ Yang-Yang function for

$K_T(X)$ — tautological bundles of tautological bundles over X

T: $\mathbb{C}^*_\hbar \times \mathbb{C}^*_{a_1} \cdots \mathbb{C}^*_{a_n}$

Quantum version of the duality $H^{tRS}_i(2)\, V_{\vec{p}} = e_i(a_1, \dots a_n)\, V_{\vec{p}}$



$X = T^*\mathbb{F}l_{n+1}$ in type A, $n=2$

$H_1 = \frac{\hbar z_1 - z_2}{z_1 - z_2} p_1 + \frac{\hbar z_2 - z_1}{z_2 - z_1} p_2$ — quantum multiplication by $\hat{V}_1^*$, V_1

$p_1 f(z_1, z_2) = f(qz_1, z_2)$

$p_2 f(z_1, z_2) = f(z_1, qz_2)$

$X = T^*\mathbb{P}^1$, $z = \frac{z_1}{z_2}$

$p_1 = p_2^{-1} = p$

Classically $(q \to 1)$ $V_{\vec{p}} \to e^{\frac{\widetilde{W}_{YY}(\vec{a},\vec{z},\vec{s})}{\log q}} \Rightarrow \left\{ \frac{\partial \widetilde{W}(\vec{s})}{\partial s} = 0 \right\} \Leftrightarrow \{ H_i = e_i \}$

- Note this only works for type-A (albeit one can use 'orbifolding' tricks to get some insight about types B and C)
- Also spin chains exist for all possible weights, no constraints on ranks of bundles V_i, W_i

$\mathfrak{sl}_N$ Need to extend the duality $X = T^*Gr_{k,n}$, $k \le n$

It turns out that the proper language to formulate such duality is geometric q-Langlands correspondence

3) Langlands quantum Correspondence. Categorical equivalence $D_\varkappa(Bun\, G) \simeq D_{-\frac{1}{m\varkappa}}(Bun\, {}^LG)$ on Riemann surface X

we will be working at critical level $\varkappa \to 0$

(Automorphic) Bethe equations — Gaudin model

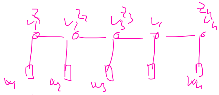
A-side $D\text{-mod}(Bun\, G) \simeq Loc_{^LG}$ Opers (Galois) B-side

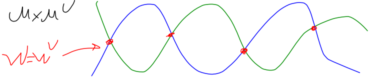
Consider difference version of the above relationship

geometric q-Langlands correspondence

A	B
QQ-system ${}^L\mathfrak{g}$	(G,q)-opers

Categorical conjecture made by [Elliot-Pestun]





$T[U(Q)]_{\rho}^{\rho^\vee}$, $N = \sum_i w_i$

$U(N)$ on $\mathbb{R}^2 \times S^1 \times I^1$

Moduli space of flat $GL(N, \mathbb{C})$ connections on $T^2 \setminus \{pt\}$

$ABA^{-1}B^{-1} = C$

ρ - Nahm pole Br. L

$\rho^\vee$ generalized Nahm BI R

Vacua of 3d quiver theory $\mathcal{L} = \mathcal{L}_{L,a}^{\rho} \cap \mathcal{L}_{R,z}^{\rho^\vee}$ → Solution of XXZ Bethe equations in weight zero sector

$\mathcal{L}_{L,a}^{\rho}$ fixes e.v.s of A and ρ + Slodowy slices on B

$\mathcal{L}_{R,z}^{\rho^\vee}$ fixes eigenvalues of B and $\rho^\vee$ + Slodowy slices on A

looks like geometric Langlands + S duality

function
torsion poly
of trunc bundles
over X

(X)

Riemann
X

(Galois)

$^L g$ | $\frac{B}{(G, q)\text{-pcs}}$

Q-opers & Quantum/Classical duality — Talk @ UC Davis QMAP seminar 11/9/18 — almost complete w/ A Zeitlin, D Sage (LSU)

⓪ Motivation and goals

• (quantum) (q-) geometric Langlands correspondence

$D_\kappa(Bun_G) \leftrightarrow D_{-\frac{1}{m\kappa}}(Bun_{{}^LG})$ — Bun_G on X - compact Riemann surface

Classical (critical level) $\kappa \to 0$

$D_{mod}(Bun_G) \leftrightarrow Loc_{{}^LG}$ — moduli stack of flat ${}^LG$-bundles

In this talk we'll be talking about difference version of this

q-Geometric Langlands

• Recent progress in enumerative algebraic geometry and integrable systems (I gave my AG talk last quarter) — Okounkov et al, my recent work w/ Zeitlin

Quantum K-theory of symplectic resolutions, dualities between integrable systems which arise in gauge/string theory — learned it with enumerative geometry

Brief history

• Ed Frenkel pointed out importance of opers to Langlands (07')
• Drinfeld & Sokolov (85) studied them locally as gauge equivalence classes of certain diff operators valued in $\mathfrak{g}$ — L-operators of generalized KdV
• Beilinson, Drinfeld (05') made a coordinate-independent definition
• Frenkel (07') Miura structure

① SL(2) opers

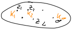
1.1 **Def** A $Gl(2)$-oper on $\mathbb{P}^1$ is a triple $(E, \nabla, \mathcal{L})$, where E - rank-2 vector bundle on $\mathbb{P}^1$, connection ∇ $E \to E \otimes K$, $\mathcal{L}$ - line subbundle such that induced map $\bar{\nabla}: \mathcal{L} \to E/\mathcal{L} \otimes K$ is an iso.

If the structure group can be reduced to $SL(2)$ we can get an $SL(2)$ oper.

Pick a trivialization on E. In the vicinity of any point z we have

$$s(z) \wedge \nabla_z s(z) \neq 0, \qquad \nabla_z = \partial_z \cdot \nabla$$

Talk about $SL(2)$-opers w/ regular singularities and trivial monodromies around singular points



An $SL(2)$ q-oper w/ sing at z_1, z_2, z_l, ∞ of weights k_1, k_l, k_∞ is the same as above, when $\bar{\nabla}$ is an iso everywhere except z_i, ∞ where it has order of vanishing k_i

$$s(z) \wedge \nabla_z s(z) \sim (z - z_i)^{k_i} \qquad (1)$$

Trivial monodromy condition ⇒ after an appropriate gauge transformation A is trivial on $\mathbb{P}^1 \setminus \infty$ subbundle $\mathcal{L}$ has section

$$s(z) = \begin{pmatrix} q_+(z) \\ q_-(z) \end{pmatrix}, \quad q_+, q_- \text{ have no common roots}$$

then (1) reads

$$q_+ \partial_z q_- - \partial_z q_+ q_- = \rho(z), \quad \rho(z) = \prod_{i=1}^{L} (z - z_i)^{k_i} \qquad (2)$$

By a gauge transformation we can make $\deg(q_-) < \deg(q_+)$ and q_- is monic (always possible)

Nondegenerate ⇒ none of z_i's are roots of q_- ⇒ each root of q_- has multiplicity 1

Let $k = \sum_{i=1}^{L} k_i = \deg \rho(z)$ ⇒ $\deg(q_-) + \deg(q_+) = k + 1$ ⇒ $\deg(q_-) = \ell_- < \frac{k}{2}$

Rewrite (2) as

$$\partial_z \left(\frac{q_+(z)}{q_-(z)} \right) = - \frac{\rho(z)}{q_-(z)^2}$$

Compute residues at each w_i we get

$$\sum_{m=1}^{L} \frac{k_m}{z_m - w_i} = \sum_{j \neq i} \frac{2}{w_j - w_i}, \quad i = 1, \ldots, \ell$$

Bethe equations for $sl(2)$ Gaudin model at level $k - 2\ell > 0$

to study ∞ do analytical transform $z \to \frac{1}{z}$

Theorem 1 There is a 1-1 correspondence between spectrum of the sl_2 Gaudin model and space of non-degenerate sl_2 opers w/ trivial monodromy and regular singularities

1.2 Miura structures

Recall the condition of nonnegative level $K - 2\ell_-$.

Miura oper $(E, \nabla, \mathcal{L})$ together with additional subbundle $\hat{\mathcal{L}}$ preserved by ∇

$\mathcal{L}, \hat{\mathcal{L}}$ span E everywhere except maybe finitely many points.

Thus for any oper we have a family of Miura opers parameterized by G/B flag [Frenkel]

Pick $\hat{\mathcal{L}}$ being generated by $\hat{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Theorem 2 There is a 1-1 relation between solutions of Bethe Ansatz equations of the Gaudin model and $SL(2)$ Miura opers w/ reg sing & trivial monodromies with weights at finite points z_1, z_2

1.3 Connections and differential operators

Eigenvalues of Gaudin Hamiltonian directly from oper

Apply $SL(2)$-gauge transformation $g(z) = \begin{pmatrix} q_- & -q_+ \\ 0 & q_-^{-1} \end{pmatrix}$ so that $g(z)s(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to a trivial connection

$$A = -(\partial_z g)g^{-1} = \begin{pmatrix} -\frac{\partial_z q_-}{q_-} & -\rho(z) \\ 0 & \frac{\partial_z q_-}{q_-} \end{pmatrix} \xrightarrow[\text{gauge transform } \begin{pmatrix} \rho^{-1/2} & 0 \\ 0 & \rho^{1/2} \end{pmatrix}]{\text{diagonal}} A(z) = \begin{pmatrix} -u(z) & -1 \\ 0 & u(z) \end{pmatrix},$$

where $u(z) = -\frac{\partial_z \rho}{2\rho} + \frac{\partial_z q_-}{q_-} = -\sum_m \frac{k_m/2}{z - z_m} + \sum_i \frac{1}{z - w_i}$

Apply Miura transform $g(z) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ so the new connection read $B(z) = \begin{pmatrix} 0 & -1 \\ -t(z) & 0 \end{pmatrix}$,

where $t(z) = \partial_z u(z) + u^2(z) = \sum_m \frac{k_m(k_m+2)/4}{(z - z_m)^2} + \sum \frac{c_m}{z - z_m}$ ← eigenvalues of Gaudin Hamiltonians

$$c_m = k_m \left(\sum_{n \neq m} \frac{k_n}{z_m - z_n} - \sum_{i=1}^{\ell} \frac{1}{z_m - w_i} \right)$$

Horizontal section $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ to $B(z)$ is determined by $\left(\partial_z - B(z)\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0$

$\left(\partial_z^2 - t(z)\right) f_1(z) = 0$ ← projective connection

1.4 Irregular singularities [FFTL, FFR]

$A \sim d + \begin{pmatrix} a & \\ & -a \end{pmatrix} dz$ on $\mathbb{P}^1 \setminus \infty$ section $s(z) = e^{-\begin{pmatrix} a & \\ & -a \end{pmatrix} z} \begin{pmatrix} q_+ \\ q_- \end{pmatrix}$

then Bethe equations will take the following form $-2a + \sum_m \frac{k_m}{z_m - w_i} = \sum_{j \neq i} \frac{2}{w_j - w_i}$

Theorem 3 There is a 1-to-1 correspondence between solutions of inhomogeneous Bethe equations and nondegenerate $SL(2)$ opers w/ regular singularities at $z_1, \ldots, z_L$ and w/ a double pole w/ 2-residue $-\begin{pmatrix} a & \\ & -a \end{pmatrix}$

2) Q-opers

2.1

Let $q \in \mathbb{C}^\times$, consider E^q – pull back of E – vector bundle over $\mathbb{P}^1$ under map $z \mapsto qz$

Consider map of vector bundles $A \quad E \to E^q$ gives linear map of given bases $E_z \longrightarrow E_{qz}$

q-gauge transformations $\quad A(z) \mapsto g(qz) A(z) g^{-1}(z)$

change of trivialization by $g(z)$

Let $D_q \quad E \to E^q$ be $\quad S(z) \mapsto S(qz)$ then $D_q(S) = AS$

Def A meromorphic $(G(N), q)$ – connection over $\mathbb{P}^1$ is (E, A), where E is a trivializable vect bundle of rank N over $\mathbb{P}^1$ and A is a meromorphic section of the sheaf $Hom_{\mathcal{O}_{\mathbb{P}^1}}(E, E^q)$ for which $A(z)$ is invertible. If there exists a trivialization where $\det A(z) = 1$ then we have an $(SL(N), q)$ oper

Now look at $SL(2)$

Def 1) $A\,(GL(2), q)$ – oper is a triple $(E, A, \mathcal{L})$, where (E, A) – $GL(2)$ q-connection, $\mathcal{L}$ is a line subbundle such that the induced map $\bar{A} \quad \mathcal{L} \to (E/\mathcal{L})^q$ is an iso

2) $(SL(2), q)$ – oper w/ regular singularities $z_1, z_2, \; z_L \neq 0, \infty$ w/ weights $k_1, \; k_L$ is a meromorphic $(SL(2), q)$ – oper for which $\bar{A}$ is an iso everywhere on $\mathbb{P}^1 \setminus \{0, \infty\}$ except the points $z_m, q^{-1} z_m, q^{-2} z_m, \quad q^{-k_m+1} z_m, \quad m = 1, .. L$ $\quad S(qz) \wedge A(z) S(z) \neq 0$

3) An $(SL(2), q)$ – oper $(E, A, \mathcal{L})$ w/ reg singularities is called a Z-twisted q-oper if A is gauge equivalent to Z^{-1}, where $Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$

4) Miura q-oper $(E, A, \mathcal{L}, \hat{\mathcal{L}})$, where $\hat{\mathcal{L}}$ is preserved by A

2.2 Q-Wronskian and Bethe Ansatz

Theorem 4 There is a 1-to-1 correspondence between the set of non-degenerate solutions of SL_2 XXZ Bethe equations and the set of non-degenerate Z-twisted $(SL(2), q)$ – opers w/ regular sing at $z_1, \; , z_L \neq 0, \infty$ w/ weights $k_1, \; , k_L$ and can be represented by meromorphic q-connection

$$\left(D_q^2 - T(qz) D_q - \frac{\rho(qz)}{\rho(z)}\right) f_1 = 0$$

Proof $\quad S(qz) \wedge A(z) S(z) = \rho(z)$

$$\zeta^{-1} Q_+(z) Q_-(qz) - \zeta Q_+(qz) Q_-(z) = \prod_{m=1}^{L} \prod_{j=0}^{k_m - 1} (z - q^{-j} z_m)$$

Can always make $Q_-(z) = \prod_{i=1} (z - w_i)$

assume that q-lattices of z_a, w_i do not overlap

Then

$$\prod_{m=1}^{L} \frac{w_i - q^{1-k_m} z_m}{w_i - q z_m} = \zeta^{-2} q^{\ell - k} \prod_{j=1}^{\ell} \frac{q w_i - w_j}{w_i - q w_j}, \quad i = 1, \; , \ell$$

Analogously get XXZ Hamiltonians (transfer matrix)

$$A(z) = \begin{pmatrix} a(z) & \rho(z) \\ 0 & a^{-1}(z) \end{pmatrix}, \quad a(z) = \zeta^{-1} Q_-(qz) Q_-^{-1}(z) \implies \hat{A}(z) = \begin{pmatrix} 0 & \rho(z) \\ -\rho^{-1}(z) & T(qz)\rho^{-1}(qz) \end{pmatrix}$$

$$T(z) = \zeta^{-1} \rho(q^{-1} z) \frac{Q_-(qz)}{Q(z)} + \zeta \rho(z) \frac{Q_-(q^{-1} z)}{Q_-(z)}$$ – eigenvalue of the XXZ transfer matrix

23 Trigonometric Ruijsenaars-Schneider Model w/ 2 degrees of freedom

Take $\deg Q_\pm(z) = 1$, $\rho(z) = (z-z_+)(z-z_-)$

$$Q_- = z - p_-, \quad Q_+ = c(z-p_+), \quad c = q^{-1}(\zeta-\zeta^{-1})$$

From quantum Wronskian relation

$$z^2 - \frac{z}{q}\left[\frac{\zeta-q\zeta^{-1}}{\zeta-\zeta^{-1}}p_+ + \frac{q\zeta-\zeta^{-1}}{\zeta-\zeta^{-1}}p_-\right] + \frac{p_+p_-}{q} = (z-z_+)(z-z_-)$$

from where tRS Hamiltonians
$$H_1 = q(z_+ + z_-)$$
$$H_2 = z_+z_-$$

$$qWr \Leftrightarrow \det\left(z\mathbb{1} - L_{tRS}\right) = \rho(z)$$

24 SL(N) Generalization

25 Opers hierarchy

26 Enumerative geometry