4d/2d Correspondence with Eight and Four Supercharges

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HEP seminar McGill University June 19th 2012

How rich are N=2 gauge theories in 4d?

Dynamics of low energy effective theories is quite well understood [Seiberg Witten ...]

However dynamics of non-BPS sector seems to be complicated

Still a full partition function of N=2 d=4 theory can be computed by localization [Nekrasov]

Recently a solid connection to non-SUSY CFTs was outlined [Alday, Gaiotto, Tachikawa]

and connection to relatively simple 2d sigma models

[Dorey, Hollowod, Lee] [Shifman, Yung] [Gaiotto, Moore, Neitzke]...

This talk: last two points

Outline

- 4d/2d w/ 8 supercharges: what and why?
  - ⋆ *Vortices in field theory vs. type IIA string theory*
  - ⋆ *(2,2) GLSM, NLSM*
  - ⋆ *The Dictionary of 4d/2d*
- AGT duality vs 4d/2d correspondence
  - ⋆ Omega Background
  - ⋆ Liouville at large central charge
  - ⋆ 4d/2d duality in NS limit and duality
- Less Supersymmetry (4 supercharges)
  - ⋆ *Heterotic deformation and Large-N solution*

4d/2d

4d / 2d duality

[Dorey Hollowood, Tong]

$\mathcal{N}=2$ $SU(N)$ SQCD $N_f = N + \tilde{N}$ fund hypers w/ masses $m_1, \ldots, m_N$ $\mu_1, \ldots, \mu_{\tilde{N}}$ $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ on baryonic Higgs branch	(2,2) $U(1)$ GLSM e N chiral +1 $\tilde{N}$ chiral -1 w/ *twisted* masses $m_1, \ldots, m_N$ $\mu_1, \ldots, \mu_{\tilde{N}}$ $\tau = ir + \frac{\theta}{2\pi}$ vortex moduli space
BPS dyons (Seiberg-Witten)	kinks interpolating between different vacua

BPS spectra (as functions of masses, Lambda) are the same

Goal: understand it from field theory constructions

$U(N_c)$ $\mathcal{N}=2$ $d=4$ SQCD w/ N_f quarks

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\delta^{IJ} P_{\alpha\dot\beta} + 2\delta^{IJ} Z_{\alpha\dot\beta}$$

$$\{Q^I_\alpha, Q^J_\beta\} = 2Z^{IJ}_{\alpha\beta}$$

strings

monopoles domain walls

$$\mathcal{L} = \mathrm{Im}\left[\tau \int d^4\theta\, \mathrm{Tr}\left(Q^{i\,\dagger} e^V Q_i + \tilde{Q}^{i\,\dagger} e^V \tilde{Q}_i + \Phi^\dagger e^V \Phi\right)\right]$$

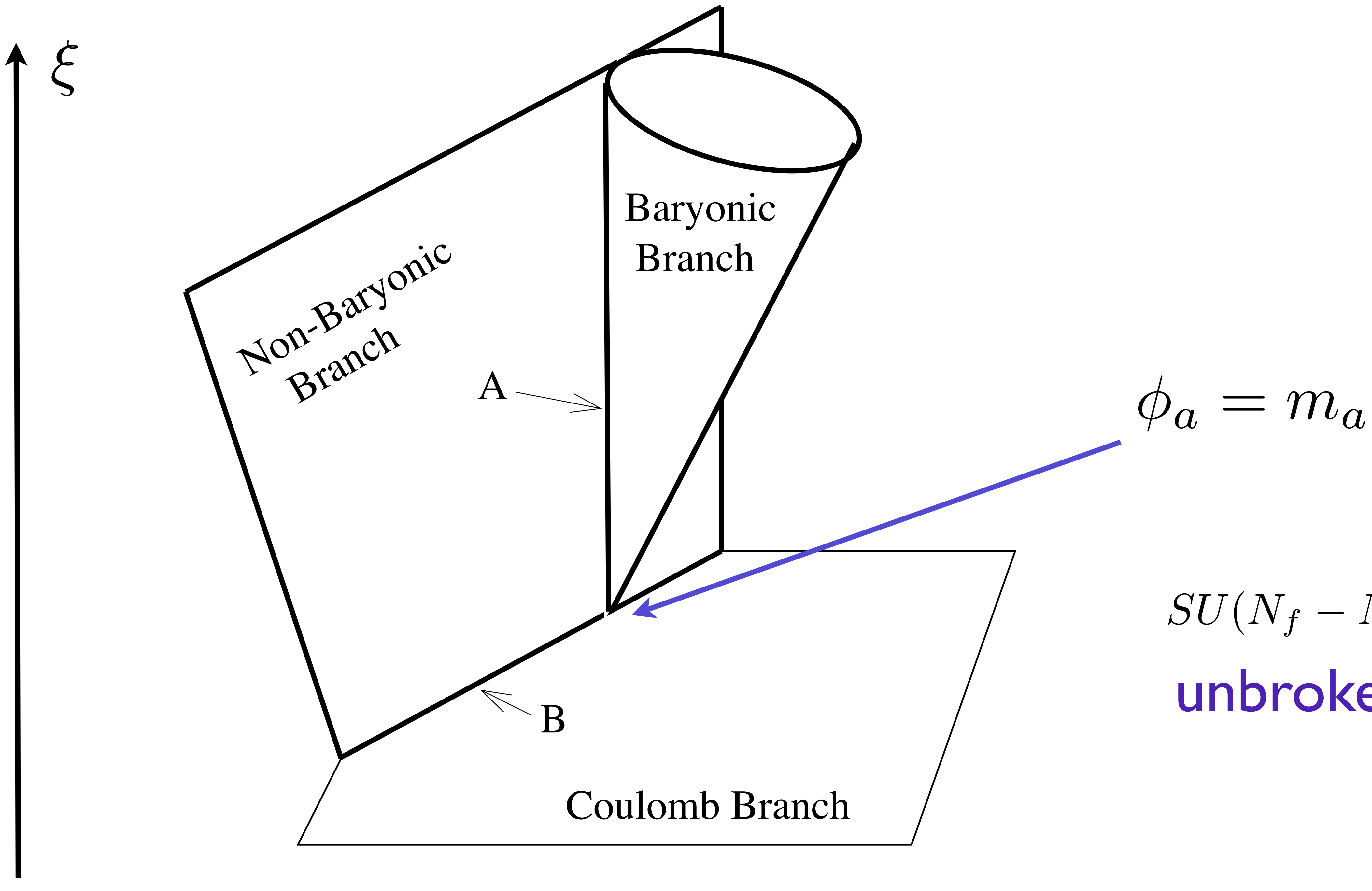
$$+\mathrm{Im}\left[\tau \int d^2\theta \left(\mathrm{Tr} W^{\alpha\,2} + m^i_j \tilde{Q}_i Q^j + Q_i \Phi \tilde{Q}^i\right)\right]$$

bosonic part

FI term

$$S = \int d^4x\, \mathrm{Tr}\left\{\frac{1}{2g^2} F^2_{\mu\nu} + \frac{1}{g^2}|D_\mu \Phi|^2 + |\nabla_\mu Q|^2 + \frac{g^2}{4}(Q\bar{Q} - \xi)^2 + |\Phi Q + QM|^2\right\}$$

Coulomb vs Higgs branches

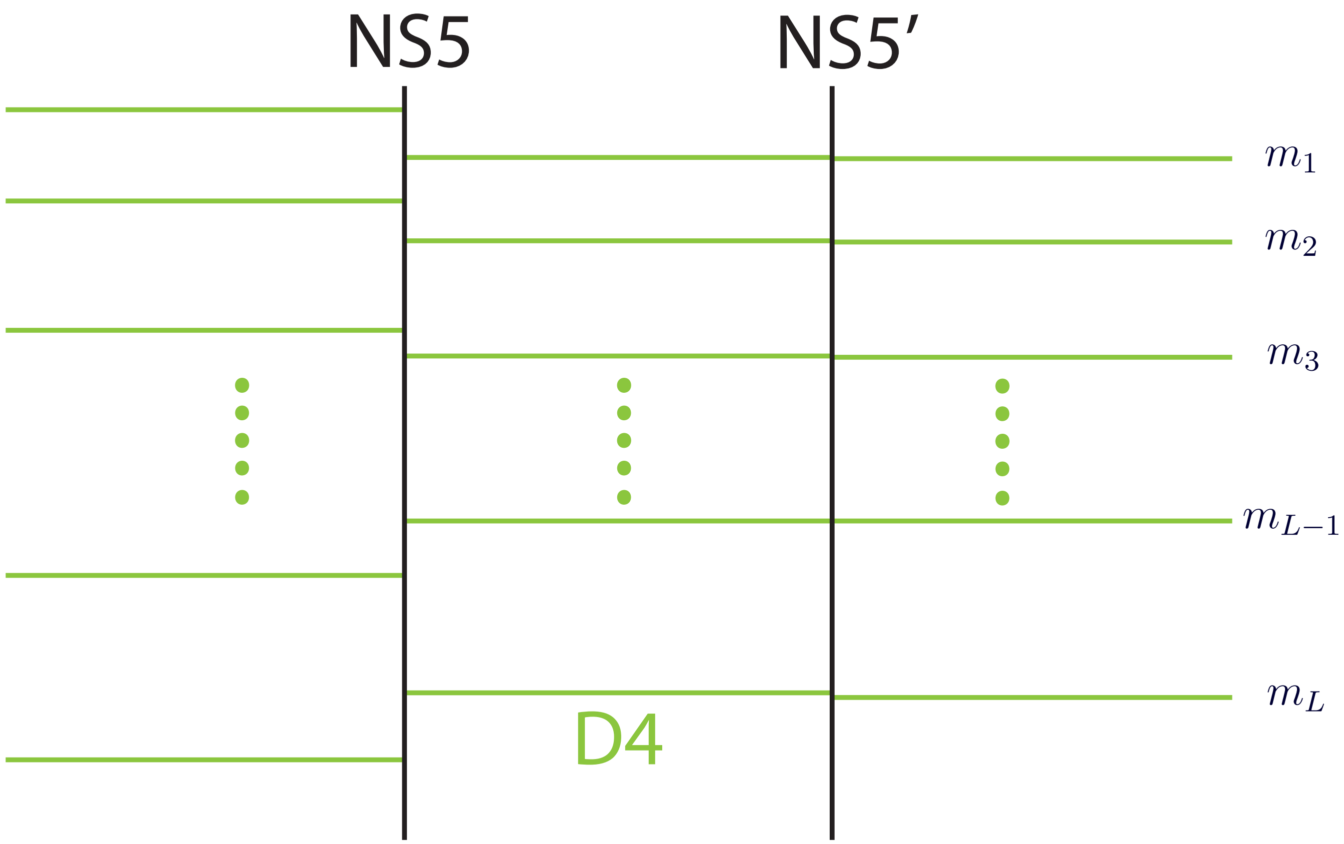


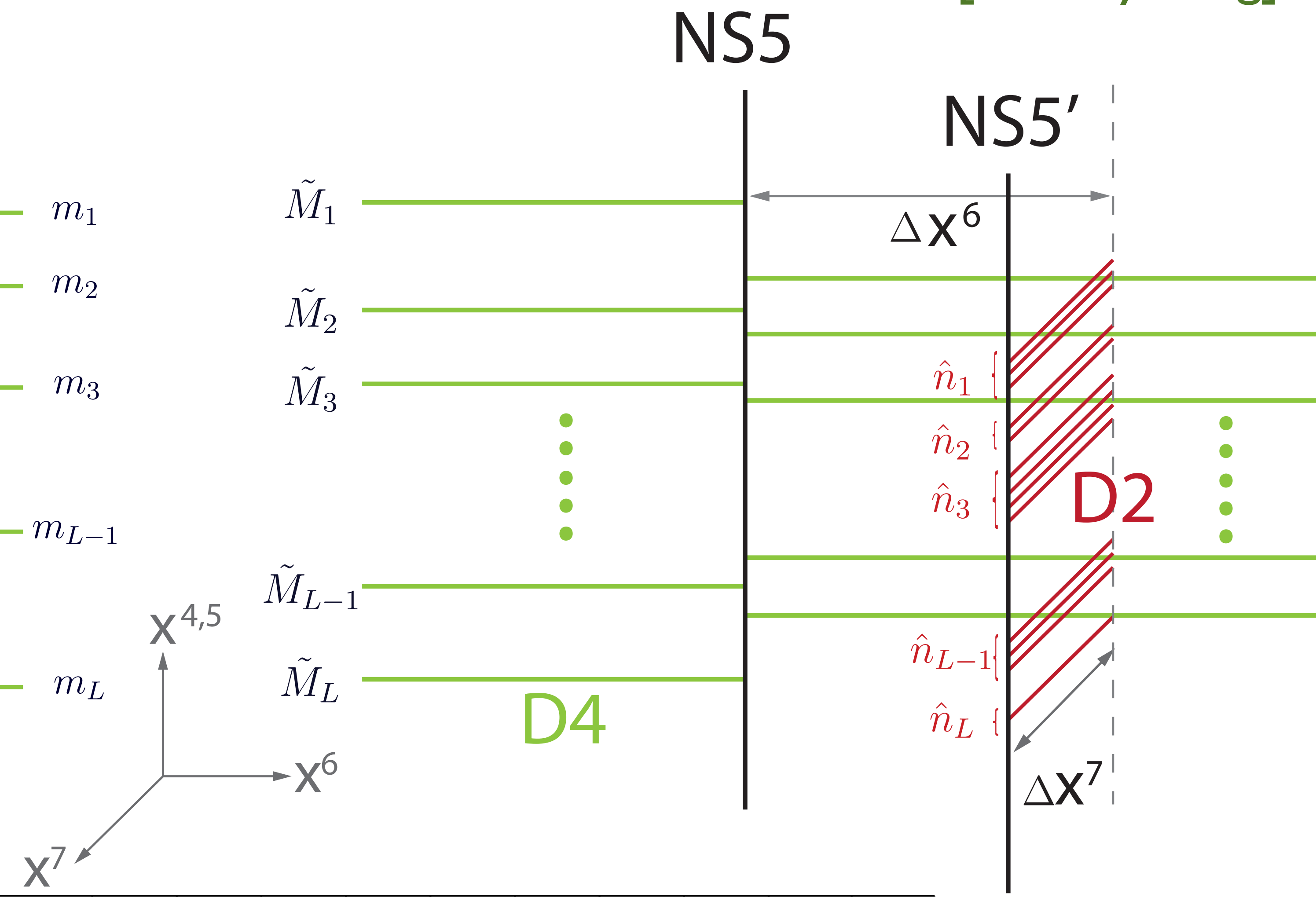
$SU(N_f - N_c)$

unbroken

Hanany-Witten construction

[Witten]
[Hanany Tong]





SQCD $N_f = 2N_c$

Higgs branch root

Color-flavor locked phase of SQCD

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D4	x	x	x	x			x			
D2	x			x				x		

2d FI parameter $r = \frac{\Delta x^6}{2\pi g_s l_s} = \frac{4\pi}{e^2}$

$$V_{2d}(\sigma, Z)$$

$$\sigma = X^4 + iX^5 \quad , \quad Z = X^1 + iX^2$$

@Dorey

Understanding 2d theory: 'ANO' String

$U(N)$ gauge theory with fundamental matter $\quad q \to UqV \qquad U \in U(N)_G, \quad V \in SU(N)_F$

$N_f = N_c$

$$S = \int d^4x \; \mathrm{Tr}\left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu\phi)^2\right) + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2$$

$$- \sum_{i=1}^{N_f} q_i^\dagger \phi^2 q_i - \frac{e^2}{4}\,\mathrm{Tr}\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\, 1_N)^2$$

Vacuum

$$\phi = 0 \quad , \quad q^a{}_i = v\delta^a{}_i$$

breaks symmetry

(color-flavor locking)

$$U(N)_G \times SU(N)_F \to SU(N)_{\mathrm{diag}}$$

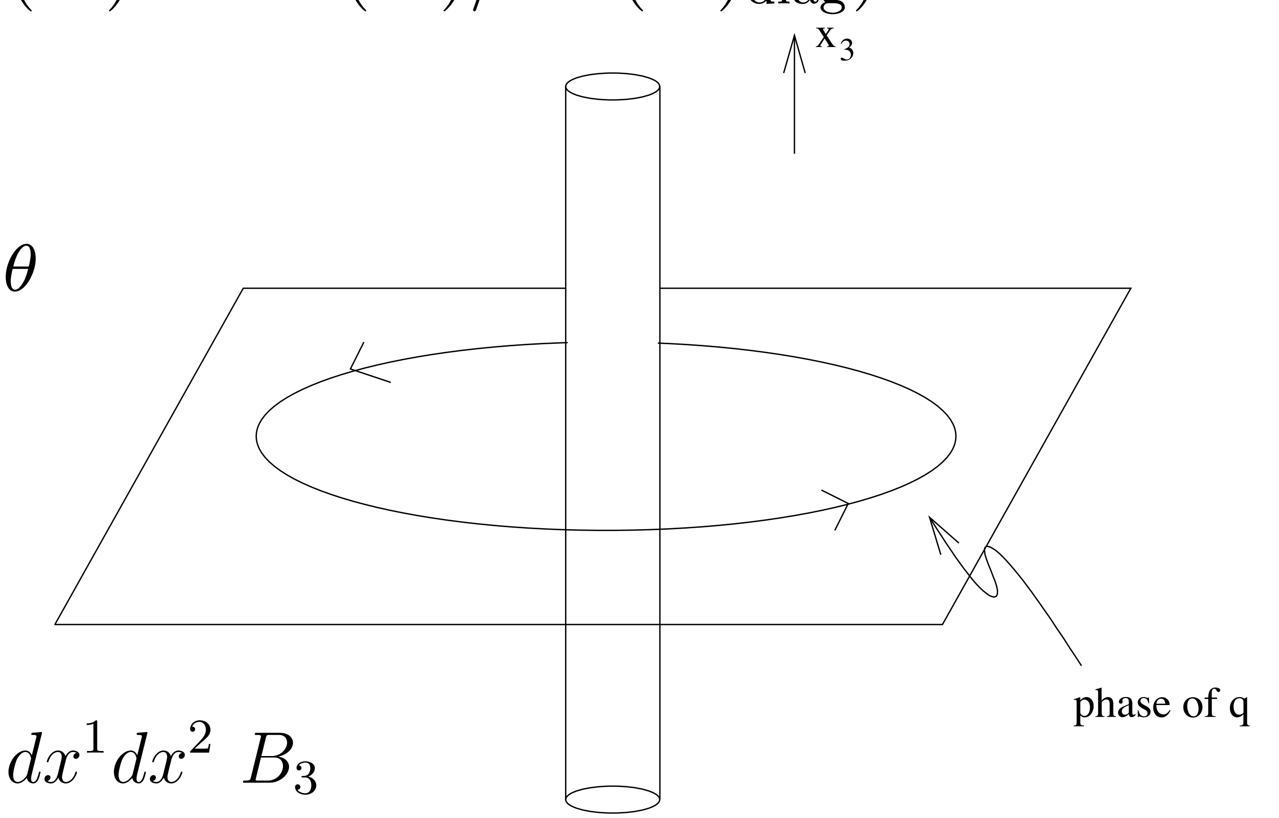
Induces nontrivial topology on moduli space

$$\Pi_1\left(U(N) \times SU(N)/SU(N)_{\mathrm{diag}}\right) \cong \mathbf{Z}$$

To find a string need winding at infinity

$$q_N \sim q\, \mathrm{e}^{ik\theta}$$

$$A_\theta \sim \frac{k}{\rho}$$



$$2\pi k = \mathrm{Tr}\oint_{\mathbf{S}^1_\infty} i\partial_\theta q\; q^{-1} = \mathrm{Tr}\oint_{\mathbf{S}^1_\infty} A_\theta = \mathrm{Tr}\int dx^1 dx^2\; B_3$$

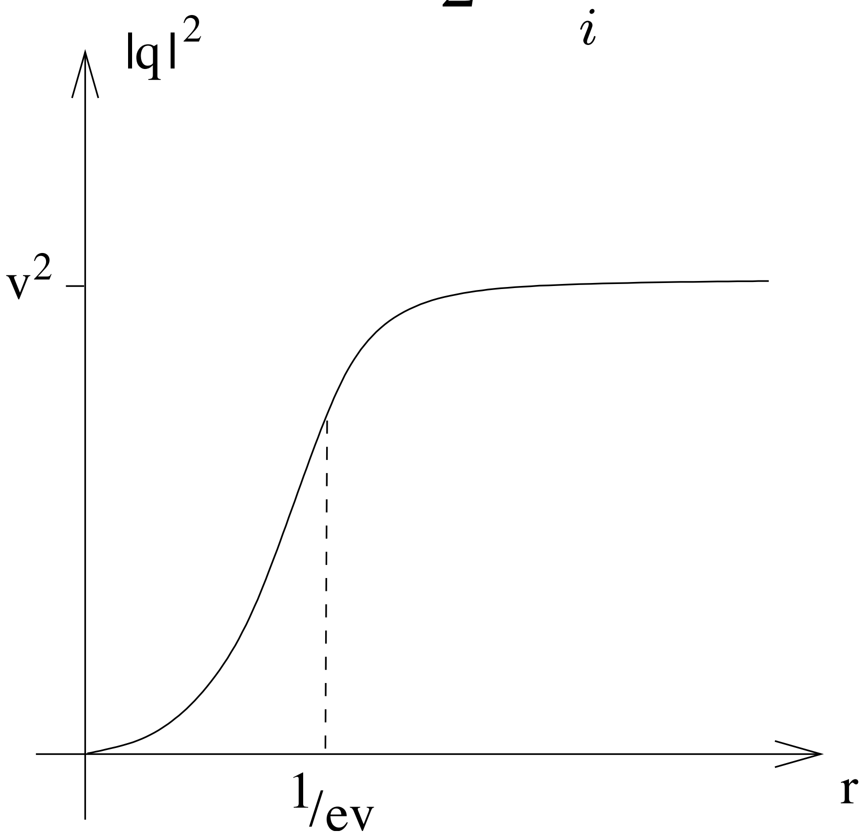
BPS equations for vortex

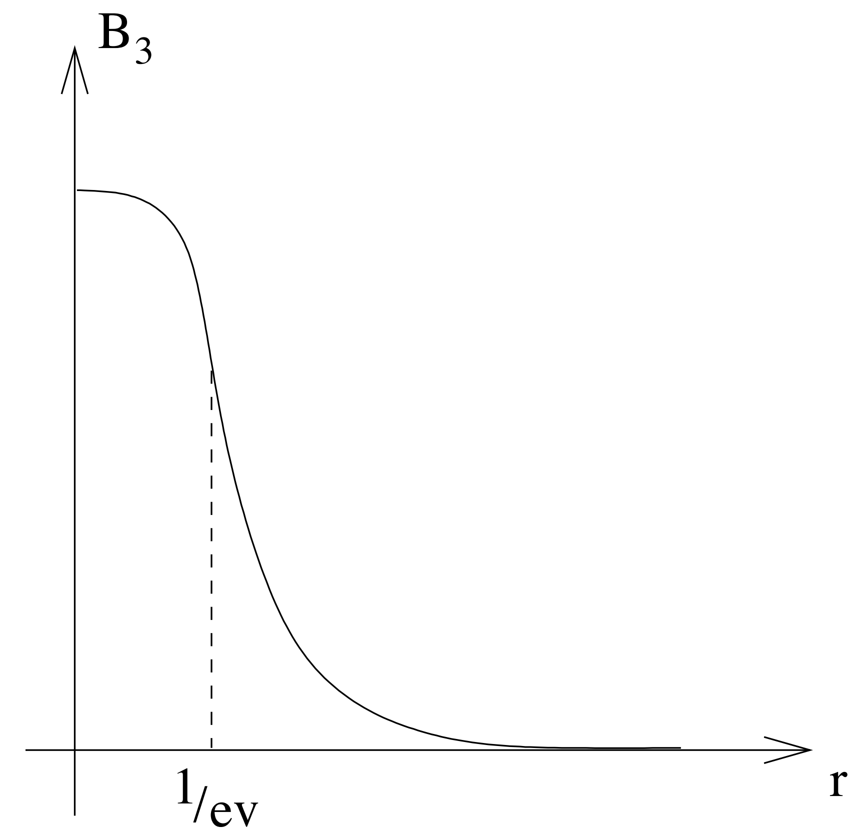
$$T_{\rm vortex} = \int dx^1 dx^2 \ {\rm Tr} \left(\frac{1}{e^2} B_3^2 + \frac{e^2}{4} (\sum_{i=1}^{N} q_i q_i^\dagger - v^2 \, 1_N)^2 \right) + \sum_{i=1}^{N} |\mathcal{D}_1 q_i|^2 + |\mathcal{D}_2 q_i|^2$$

$$= \int dx^1 dx^2 \ \frac{1}{e^2} {\rm Tr} \left(B_3 \mp \frac{e^2}{2} (\sum_{i=1}^{N} q_i q_i^\dagger - v^2 \, 1_N) \right)^2 + \sum_{i=1}^{N} |\mathcal{D}_1 q_i \mp i \mathcal{D}_2 q_i|^2$$

$$\mp v^2 \int dx^1 dx^2 \ {\rm Tr} \, B_3 \ \ \geq \mp v^2 \int d^2x {\rm Tr} B_3 \ \ = 2\pi v^2 |k| \qquad ($$

gives $$B_3 = \frac{e^2}{2} (\sum_i q_i q_i^\dagger - v^2 \, 1_N) \qquad (\mathcal{D}_x - i \mathcal{D}_y) q_i = 0$$





Vortices

Simple vortex w/ N=1, k=1 (ANO) has two collective coordinates-translations in x,y directions

U(N) vortex has more moduli

$$A_z = \begin{pmatrix} A_z^\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad , \quad q = \begin{pmatrix} q^\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}$$

Moduli space (k=1)

$$SU(N)_{\rm diag}/S[U(N-1)\times U(1)] \cong \mathbb{CP}^{N-1}$$

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1}$$

For higher k

$$\dim(\mathcal{V}_{k,N}) = 2kN$$

Again:

$T \geq 2\pi v^2 |k|$ bound saturates for BPS states

Non-Abelian String

[Auzzi, Bolognesi, Evslin, Konishi, Yung]

[Shifman Yung]

$$\varphi = U \begin{pmatrix} \phi_2(r) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \phi_2(r) & 0 \\ 0 & 0 & \dots & \phi_1(r) \end{pmatrix} U^{-1} ,$$

Take Abelian string solution
Make global rotation

$$A_i^{\mathrm{SU}(N)} = \frac{1}{N} U \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} U^{-1} \, (\partial_i \alpha) \, f_{NA}(r)$$

Matrix U parameterizes orientational modes

$$A_i^{\mathrm{U}(1)} = -\frac{1}{N} \, (\partial_i \alpha) \, f(r) \, , \qquad A_0^{\mathrm{U}(1)} = A_0^{\mathrm{SU}(N)} = 0 \, ,$$

Gauge group is broken to $\mathbb{Z}_N$

All bulk degrees of freedom massive $M^2 \sim \xi$

Theory is fully Higgsed

Vortex moduli space

Nf=Nc color-flavor locked phase
single SUSY vacuum

$$U(N_c) \times SU(N_f) \rightarrow SU(N)$$

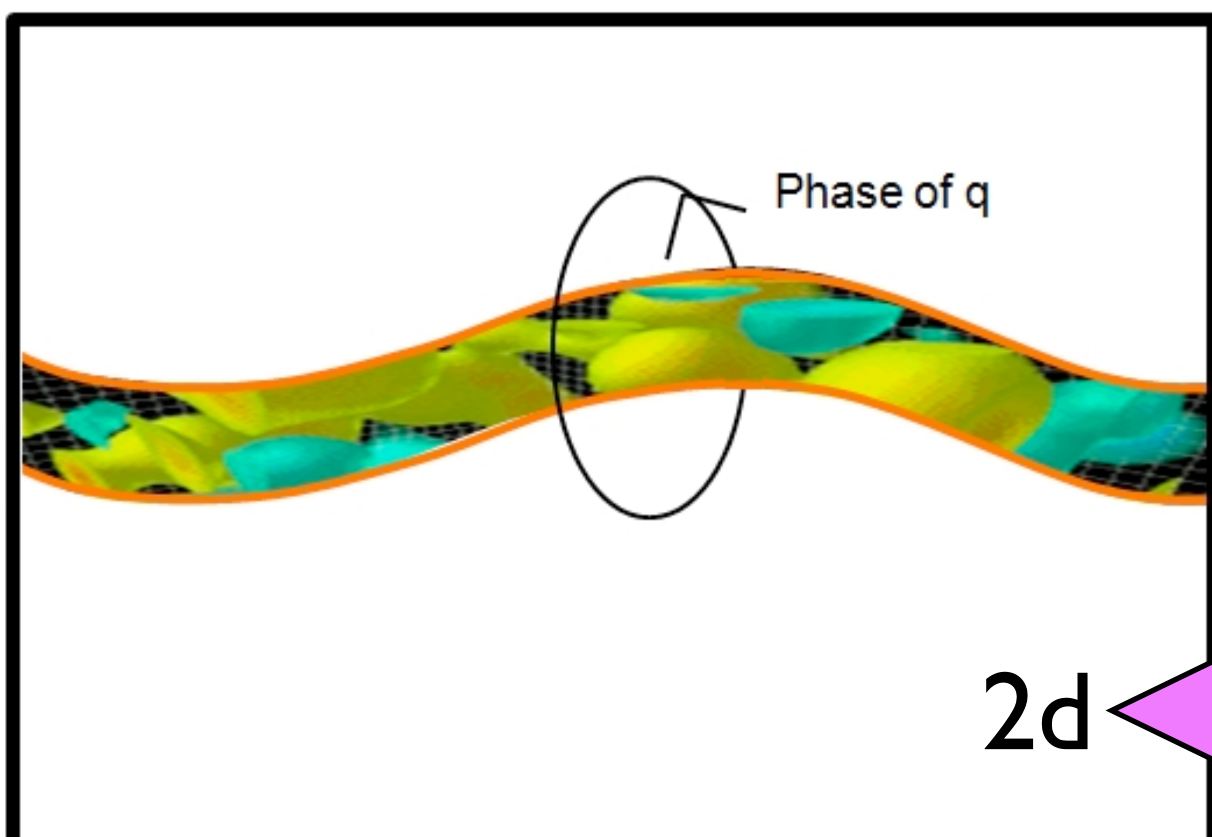
Nf=Nc local vortex

$$\frac{SU(N)}{SU(N-1) \times U(1)} = \mathbb{CP}^{N-1}$$

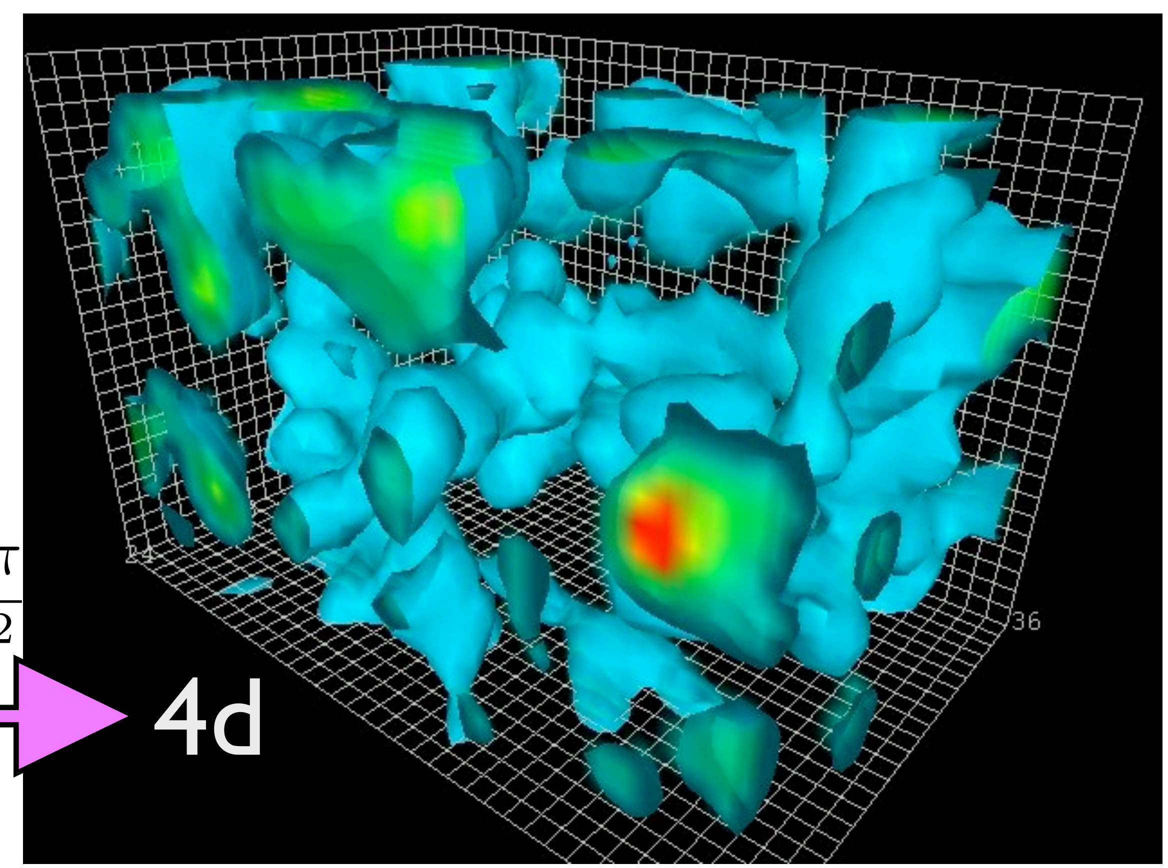
Nf>Nc semilocal
(+size moduli)

$$\pi_2(\mathcal{M}_{vac}) = \pi_2\left(\frac{SU(N+\tilde{N})}{SU(N) \times SU(\tilde{N}) \times U(1)}\right) = \mathbb{Z}$$

Duality between two strongly coupled theories



$$r = \frac{4\pi}{g^2}$$

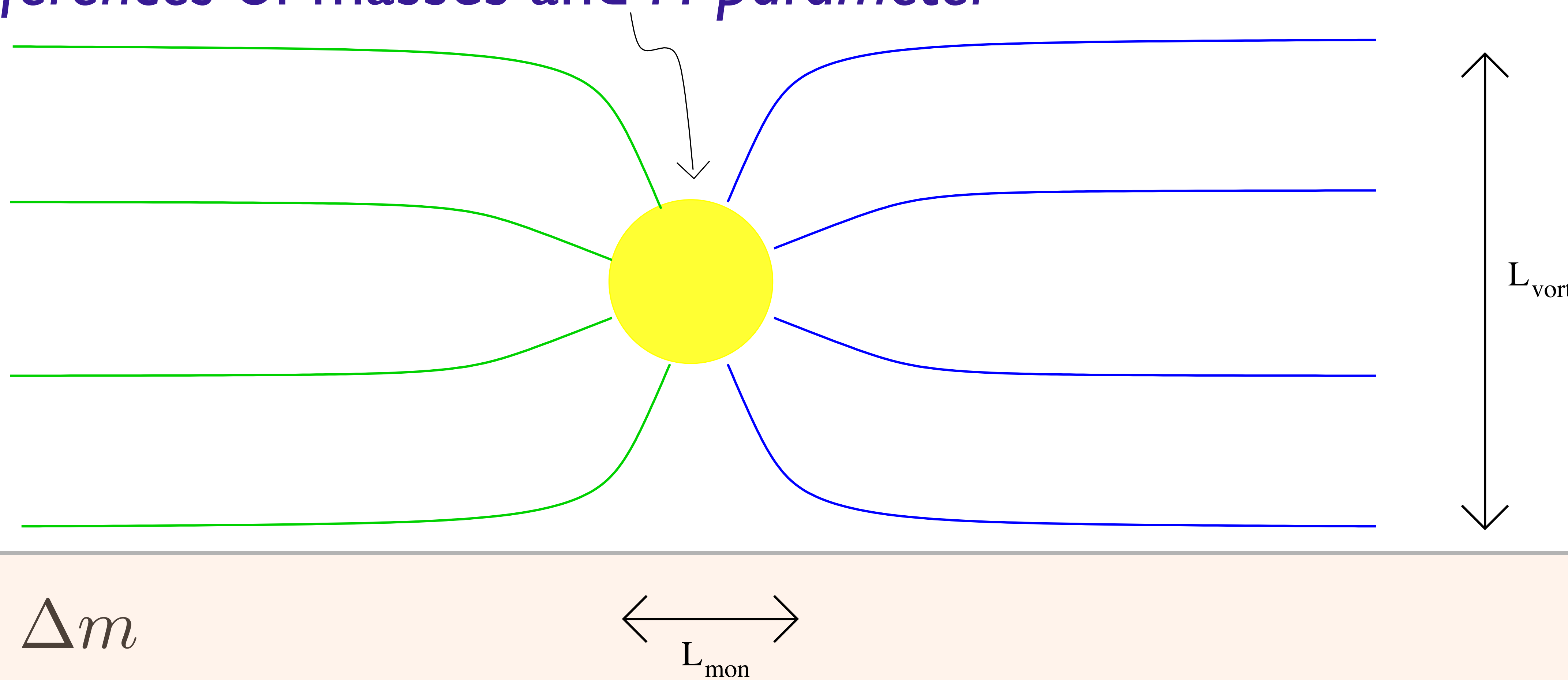


Monopoles in Higgs Phase

[Shifman, Yung]
[Tong]

Add masses. New vacuum $\phi = \mathrm{diag}(m_i) \quad , \quad q^a{}_i = v\delta^a{}_i \quad , \quad \tilde{q}^a{}_i = 0$

Pattern of symmetry breaking depends on the relationship between the *differences* of masses and *FI parameter*



$ev \gg \Delta m$

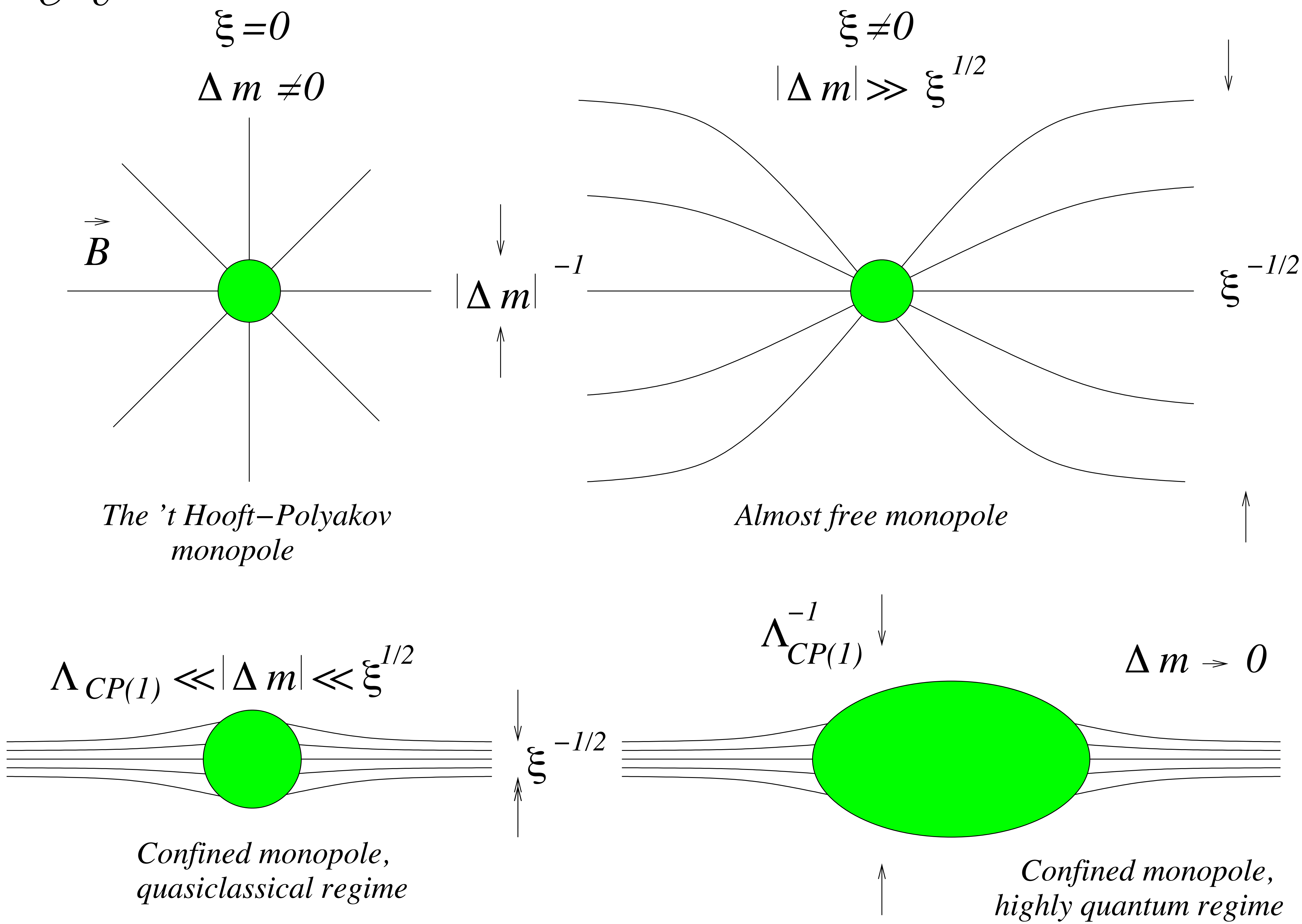
$$U(N)_G \times SU(N)_F \xrightarrow{v} SU(N)_{\mathrm{diag}} \xrightarrow{m} U(1)^{N-1}_{\mathrm{diag}}$$

$ev \ll \Delta m$

$$U(N)_G \times SU(N)_F \xrightarrow{m} U(1)^N_G \times U(1)^{N-1}_F \xrightarrow{v} U(1)^{N-1}_{\mathrm{diag}}$$

Confined monopoles

$$\xi = e^2 v^2$$



$\frac{(\Delta m)^2}{\xi}$ becomes 2d FI term r

BPS dyons in 4d N=2

$$Z = \sum_{a=1}^{N_c} \phi_a (j_a + \tau h_a) + \sum_{i=1}^{N_f} m_i s_i$$

Central charge

$$Z = \sum_{i=1}^{N_c} m_i (S_i + \tau h_i)$$

At baryonic root of Higgs branch

$$F(t,u) = \left(t - \prod_{i=1}^{N_c} (u - m_i) \right) (u - \Lambda^{N_c})$$

SW curve degenerates
has Nc branching pts

$$Z = \sum_{i=1}^{N_c} (m_i S_i + m_{Di} h_i)$$

All quantum corrections in mD

Integrating from one branching point to another

$$m_{Dl} - m_{Dk} = \frac{1}{2\pi} N_c (e_l - e_k) + \frac{1}{2\pi} \sum_{i=1}^{N_c} m_i \log \left(\frac{e_l - m_i}{e_k - m_i} \right)$$

(2,2) 2d GLSM

[Witten]

Consider U(1) gauge theory

$$\mathcal{L}_{\text{vortex}} = \frac{1}{2g^2}\left(F_{01}^2 + |\partial\sigma|^2\right) + \sum_{i=1}^{N_c}\left(|\mathcal{D}\psi_i|^2 + |\sigma - m_i|^2|\psi_i|^2\right) + \frac{g^2}{2}\left(\sum_{i=1}^{N_c}|\psi_i|^2 - r\right)^2$$

$$\text{Vacuum } i: \quad \sigma = m_i \quad , \quad |\psi_j|^2 = r\delta_{ij}$$

for vortex embedded into i's U(1) subgroup

FI term runs

$$r(\mu) = r_0 - \frac{N_c}{2\pi}\log\left(\frac{M_{UV}}{\mu}\right) \Rightarrow \Lambda = \mu\exp\left(-\frac{2\pi r(\mu)}{N_c}\right)$$

Effective twisted superpotential

$$\mathcal{W}(\Sigma) = \frac{i}{2}\tau\Sigma - \frac{1}{4\pi}\sum_{i=1}^{N_c}(\Sigma - m_i)\log\left(\frac{2}{\mu}(\Sigma - m_i)\right)$$

→ Vacua

$$\exp\frac{\partial\widetilde{\mathcal{W}}}{\partial\sigma} = 1$$

Central charge

$$Z = -i\sum_{i=1}^{N_c}(m_i S_i + m_{D\,i}T_i)$$

$$m_{D\,i} = -2i\mathcal{W}(e_i) = \frac{1}{2\pi i}N_c e_i + \frac{1}{2\pi i}\sum_{j=1}^{N_c}m_j\log\left(\frac{e_i - m_j}{\Lambda}\right)$$

Hanany-Tong model as U(1) GLSM

$$\mathcal{L} = \int d^4\theta \left[\sum_{i=1}^{N_c} \Phi_i^\dagger \, \mathrm{e}^{\mathcal{V}} \Phi_i + \sum_{i=1}^{\tilde{N}} \widetilde{\Phi}_i^\dagger \, \mathrm{e}^{-\mathcal{V}} \widetilde{\Phi}_i - r\mathcal{V} + \frac{1}{2e^2} \Sigma^\dagger \Sigma \right]$$

$$V = \theta^+ \bar{\theta}^+ (A_0 + A_3) + \theta^- \bar{\theta}^- (A_0 - A_3) - \theta^- \bar{\theta}^+ \sigma - \theta^- \bar{\theta}^+ \bar{\sigma} + \bar{\theta}^2 \theta \lambda + \theta^2 \bar{\theta} \bar{\lambda} + \bar{\theta} \theta \bar{\theta} \theta D$$

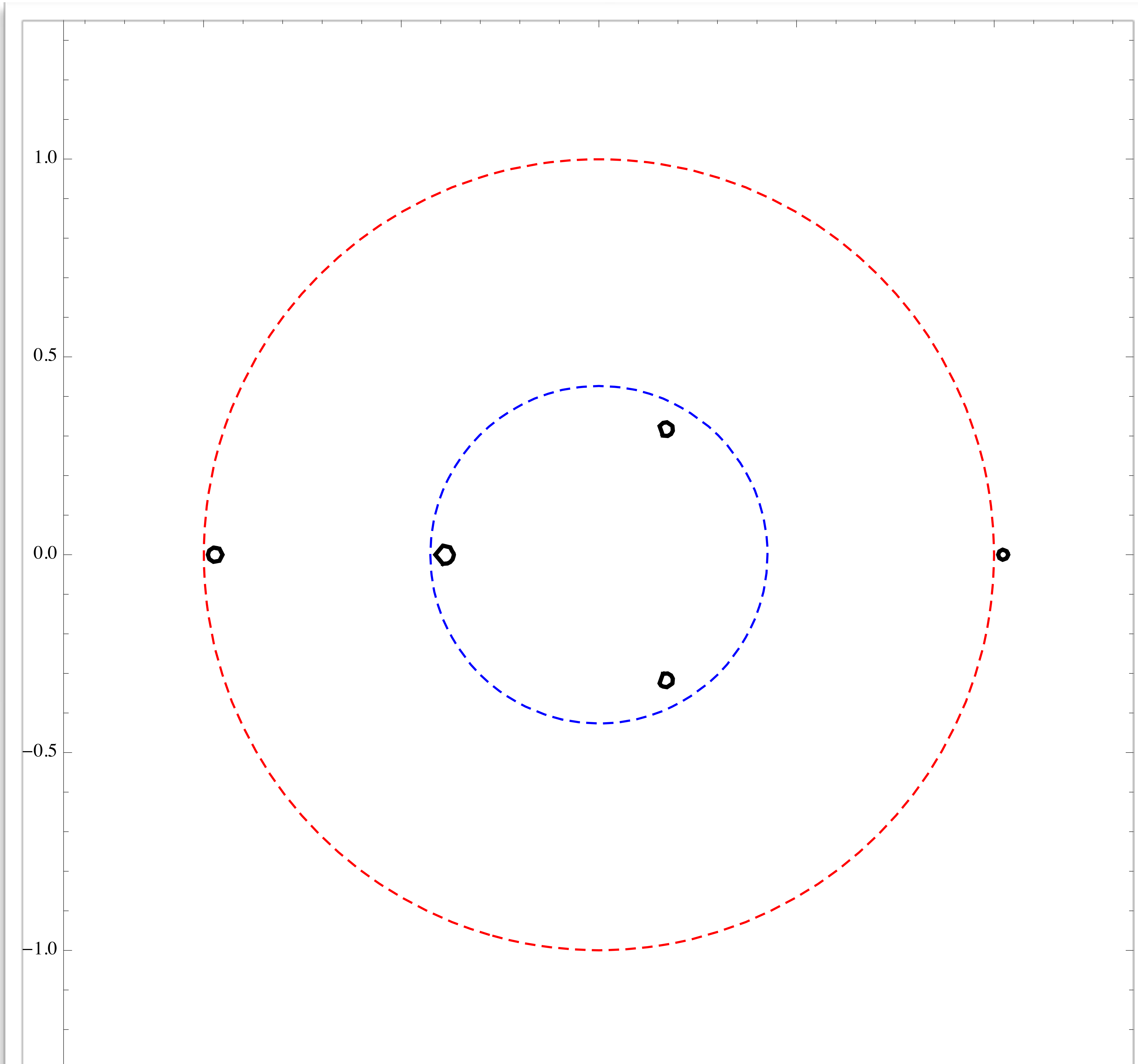
One loop twisted effective superpotential is exact in (2,2)

$$\begin{aligned} \widetilde{W}_{\text{eff}} &= -\frac{1}{2\pi} \sum_{i=1}^{N} (\sqrt{2}\sigma + m_i) \left(\log \frac{\sqrt{2}\sigma + m_i}{\Lambda} - 1 \right) + \\ &+ \frac{1}{2\pi} \sum_{j=1}^{\widetilde{N}} (\sqrt{2}\sigma + \widetilde{m}_j) \left(\log \frac{\sqrt{2}\sigma + \widetilde{m}_j}{\Lambda} - 1 \right) . \end{aligned}$$

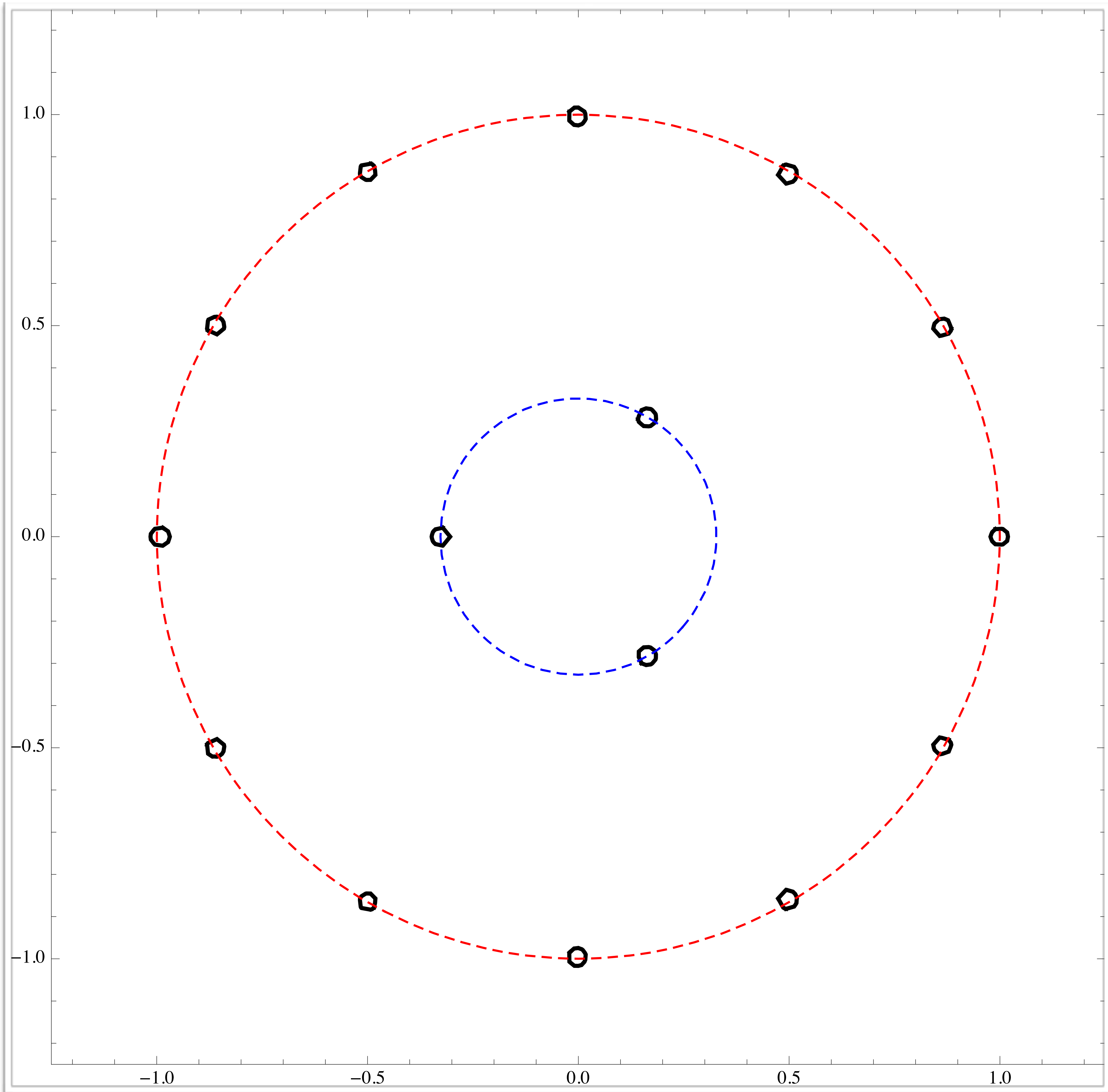
gives vacua of the theory and its BPS spectrum !!

[PK Monin Vinci]

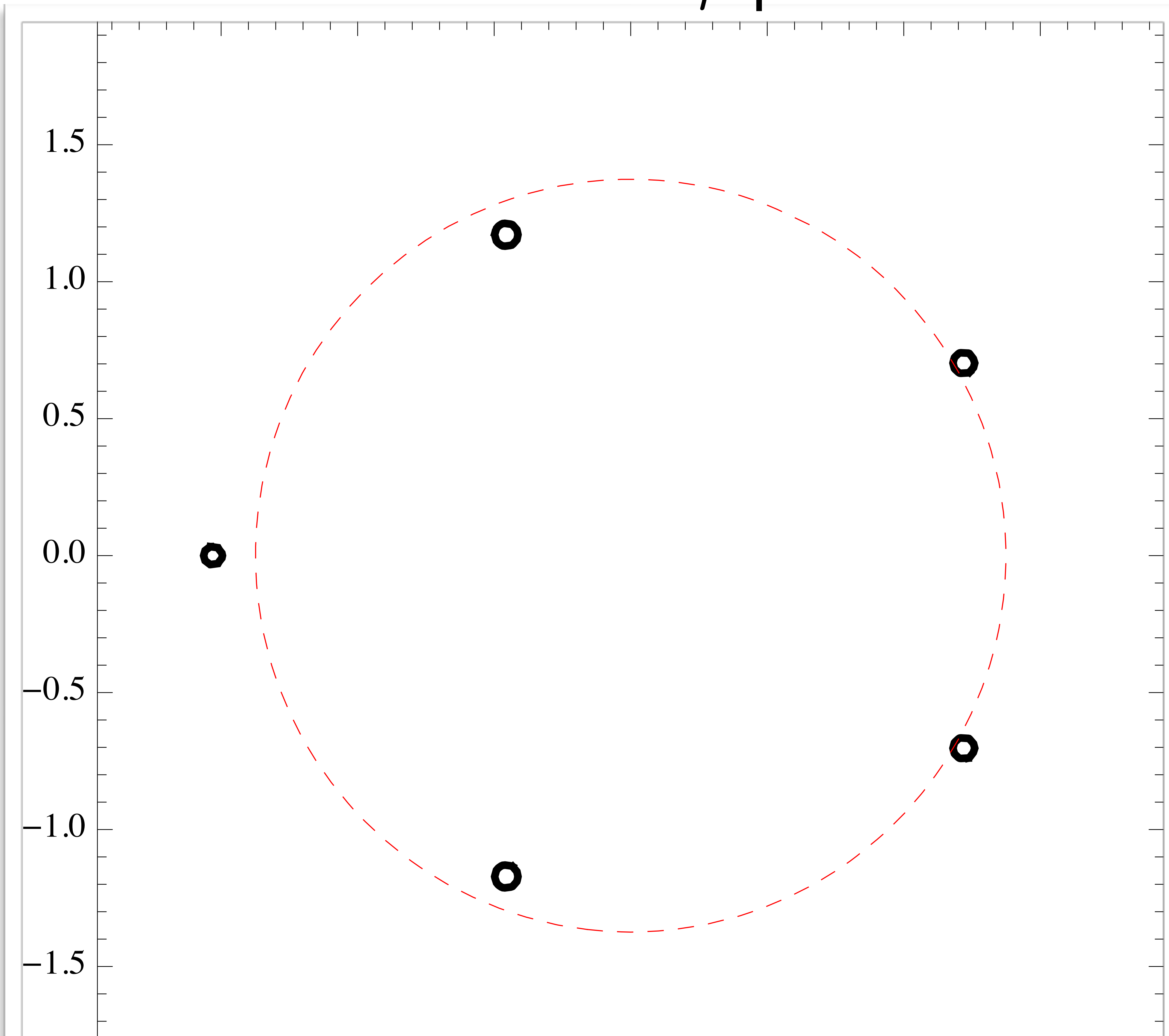
N=5 Nf=8

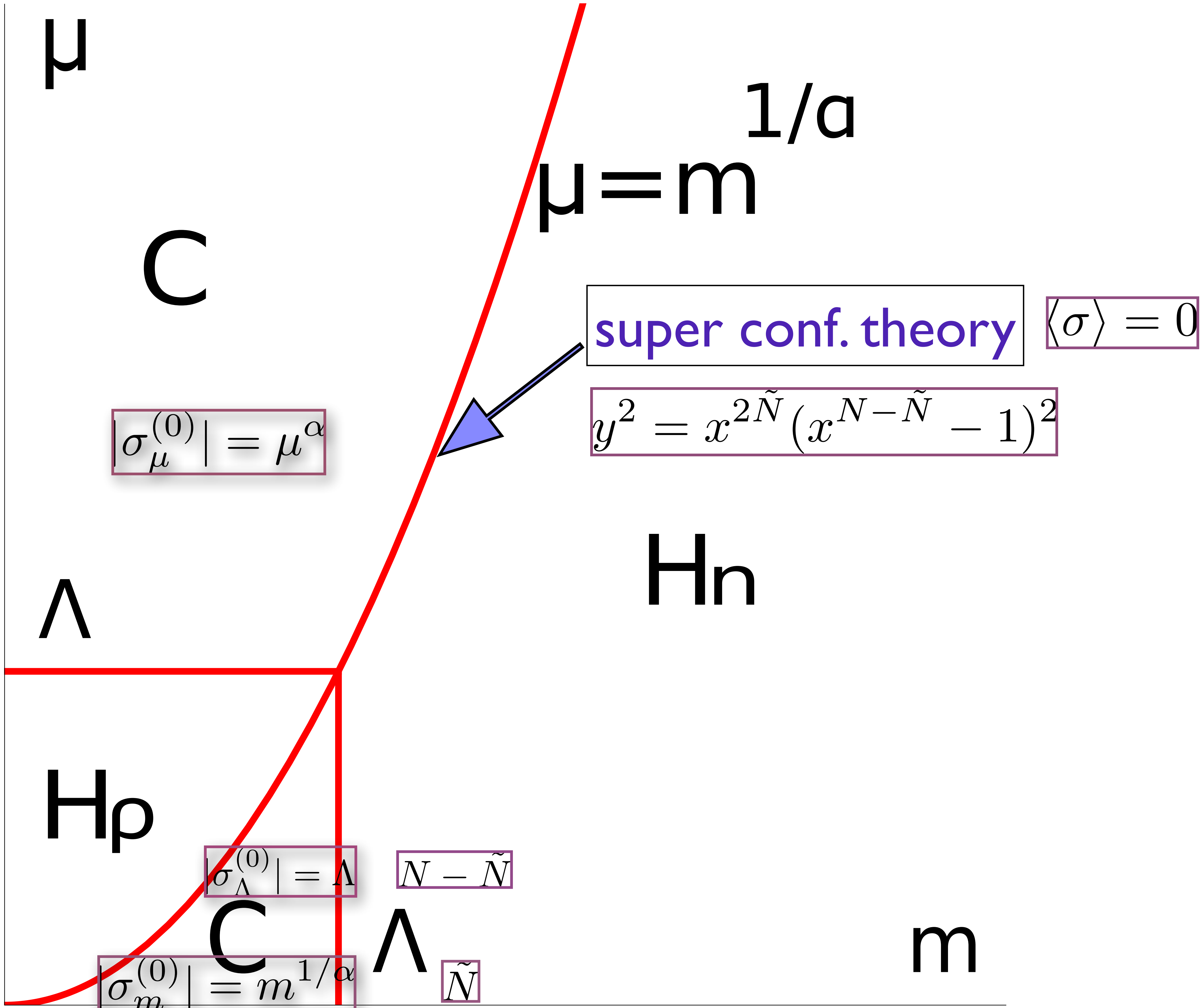


N=15 Nf=18



Nf=5 Cμ phase




μ
1/a
μ=m
C
super conf. theory
$\langle\sigma\rangle = 0$
$y^2 = x^{2\tilde{N}}(x^{N-\tilde{N}} - 1)^2$
$|\sigma_\mu^{(0)}| = \mu^\alpha$
Hn
Λ
Hp
$|\sigma_\Lambda^{(0)}| = \Lambda$
$N - \tilde{N}$
C
Λ
$\tilde{N}$
m
$|\sigma_m^{(0)}| = m^{1/\alpha}$

Subtlety #1

[Shifman Vinci Yung]

Brane construction is not sensitive to IR physics

Blind to deformations within the same universality class

Need to know explicit metric on the vacuum manifold in order to go beyond BPS sector

Let's see if GLSMs from brane picture are the same as sigma models which live on a vortex

From GLSM

$$\mathcal{L} = \int d^4\theta \left(\left(|X_1|^2 + |X_2|^2\right) e^V - rV + \frac{1}{e^2}|\Sigma|^2 \right)$$

Take limit $e \to \infty$ solve for V

Kahler potential $K = r\log(1+|X|^2)$ $X = X_2/X_1$

For HT model

$$\mathcal{L}_{\rm HT} = \int d^4\theta \, \left(|\mathcal{N}_i|^2 {\rm e}^V + |\mathcal{Z}_j|^2 {\rm e}^{-V} - rV\right)$$

Limit $e \to \infty$ defines vacuum manifold

$$\mathcal{O}(-1)^{\tilde N} \downarrow \mathbb{CP}^{N-1}$$

Kahler potential

$$K_{\rm HT} = \sqrt{r^2 + 4r|\zeta|^2} - r\log\left(r + \sqrt{r^2+4r|\zeta|^2}\right) + r\log(1+|\Phi_i|^2)$$

$$|\zeta|^2 \equiv |\mathfrak{z}_j|^2(1+|\Phi_i|^2) \qquad \mathfrak{z}_j \;=\; r^{-1/2}\mathcal{N}_N\mathcal{Z}_j\,, \quad j = 1,\dots,\widetilde{N}$$

Let's see what is the metric on the vortex sigma model

ZN model vs HT model

$$K_{\rm HT} = \sqrt{r^2 + 4r|\zeta|^2} - r\log\left(r + \sqrt{r^2+4r|\zeta|^2}\right) + r\log(1+|\Phi_i|^2)$$

$\zeta \to 0$

$$K_{zn} = r|\zeta|^2 + r\log(1+|\Phi_i|^2)$$

$$K_{\rm HT} = K_{zn} + \mathcal{O}(|\zeta|^2)$$

IR physics of ZN and HT models is the same

BPS spectra are the same, but otherwise **different**

Subtlety #2: Perturbation theory

Gel-Mann-Low function

$$\beta_{i\bar{\jmath}} = a^{(1)} R^{(1)}_{i\bar{\jmath}} + \frac{1}{2r} a^{(2)} R^{(2)}_{i\bar{\jmath}} + \ldots$$

$$R^{(1)}_{i\bar{\jmath}} = R_{i\bar{\jmath}}\,,$$

$$R^{(2)}_{i\bar{\jmath}} = R_{i\bar{k}l\bar{m}} R^{\bar{k}\;\;l\bar{m}}_{\;\bar{\jmath}}$$

Kaehler metric $g_{i\bar{\jmath}} = \partial_i \bar{\partial}_{\bar{\jmath}} K(z_i, \bar{z}_i)$

Ricci tensor $R_{i\bar{\jmath}} = -\partial_i \bar{\partial}_{\bar{\jmath}} \log\det(g_{i\bar{\jmath}})$

for Hanany-Tong model N=2, Nf=3

$$-\log\det(g^{(\mathrm{HT})}_{i\bar{\jmath}}) = \log(1 + |\Phi_i|^2) - \log\left(1 + \frac{r}{\sqrt{r^2 + 4r|\zeta|^2}}\right)$$

FI term renormalization (GLSM)

$$r_{\rm ren}(\mu) = r_0 - \frac{N-\widetilde{N}}{2\pi}\log\frac{M}{\mu} \qquad r_{\rm ren} = 0 \Longrightarrow r_0 = \frac{N-\widetilde{N}}{2\pi}\log\frac{M}{\Lambda}$$

$$c_1(M_{\rm HT})\Big|_{\mathbb{CP}^{N-1}} = (N-\widetilde{N})\,[\omega_{\mathbb{CP}^{N-1}}]$$

Kaehler class is renormalized only at one loop, hence the result above should be the full answer for the coupling renormalization

If so what does the extra term in the last formula on the previous slide mean?

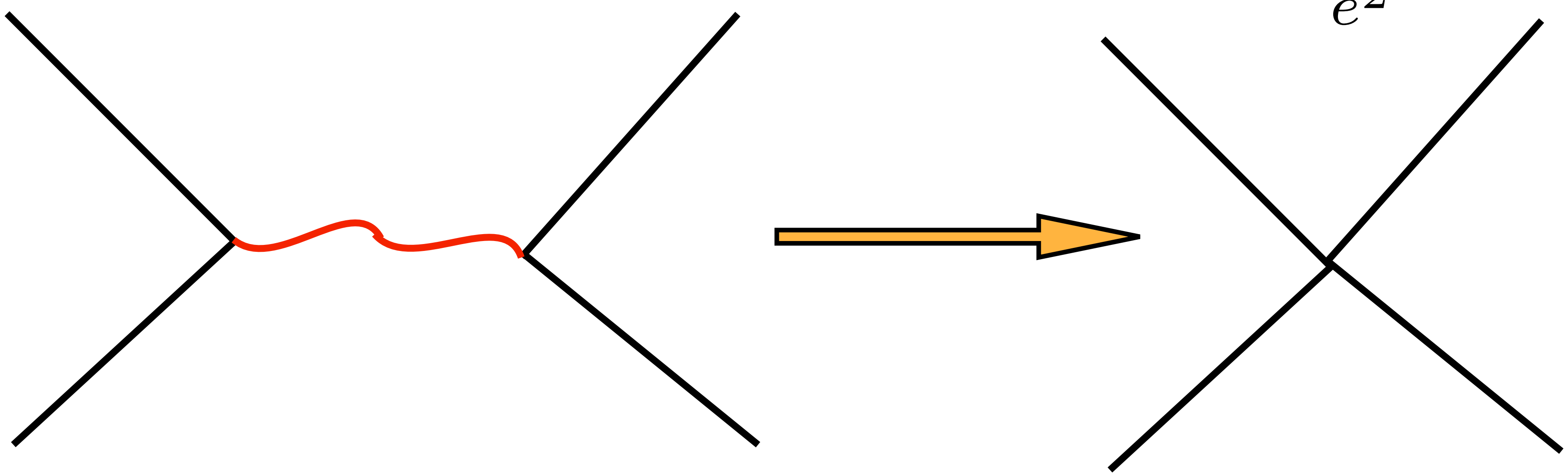
To understand why we need to compare renormalization schemes used in both calculations

GLSM vs NLSM

$$\int d^2x \int d^4\theta \left(|\Phi|^2 e^V - rV + \frac{1}{e^2}|\Sigma|^2 \right)$$

V-massive vector field w/ propagator

$$\frac{1}{\frac{p^2}{e^2} - M^2} \xrightarrow{p \ll e} \frac{1}{-M^2}$$



Integrating out V

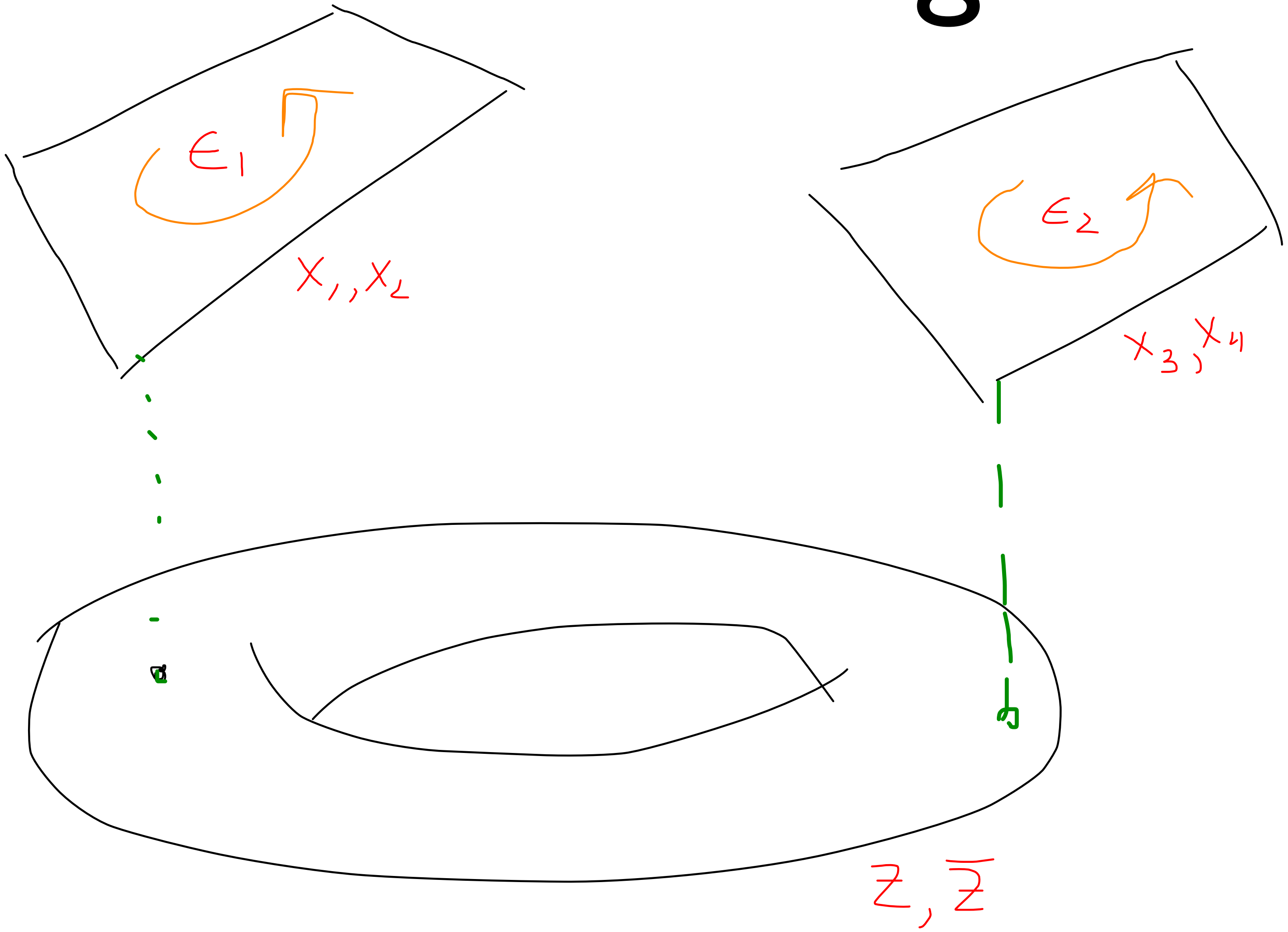
$$-\log\det(g_{i\bar{\jmath}}) = (N - \widetilde{N})\log(1 + |\Phi_i|^2) - (N-1)|\zeta|^2 + \mathcal{O}(|\zeta|^4)\,.$$

Dimensional regularization (GLSM perturbation theory) mixes up UV and IR divergencies. Need to single out the UV piece out, IR contribution is not seen in the GLSM limit

AGT in NS limit

Omega background

[Nekrasov et al]



Rotational symmetry broken to maximal torus

$$SO(4) \to SO(2) \times SO(2)$$

6d Metric

$$G_{AB}dx^A dx^B = Adzd\bar{z} + (dx^m + \Omega^m dz + \bar{\Omega}^m d\bar{z})^2$$

We will be interested in Nekrasov-Shatashvili limit

$$\Omega^m = (-i\epsilon x^2, i\epsilon x^1, 0, 0)$$

$$\epsilon_2 \to 0$$

The AGT duality

$3g-3+n$ Coulomb branch

Liouville theory on 2-sphere with 4 punctures at $\infty, 1, q, 0$	4d U(2) SQCD w/ 4 flavors with masses m_1, m_2, m_3, m_4

central charge

$$c = 1 + 6Q^2\,, \qquad Q = b + \frac{1}{b}$$

conformal dimensions of chiral operators

$$\Delta_1 = \alpha_0(Q-\alpha_0)\,, \quad \Delta_2 = \mu_0(Q-\mu_0)\,, \quad \Delta_3 = \mu_1(Q-\mu_1)\,, \quad \Delta_4 = \alpha_1(Q-\alpha_1)$$

$$\alpha_0 = \tfrac{1}{2}Q + \widetilde{\mu}_0\,, \quad \alpha = \tfrac{1}{2}Q + a\,, \quad \alpha_1 = \tfrac{1}{2}Q + \widetilde{\mu}_1$$

Conformal block matches with instanton partition function

$$\mathcal{Z}_{\text{inst}}(a, \mu_0, \widetilde{\mu}_0, \mu_1, \widetilde{\mu}_1) = (1-q)^{2\mu_0(Q-\mu_1)}\mathcal{F}_{\alpha_0\ \alpha\ \alpha_1}^{\ \mu_0\ \mu_1}(q)$$

$b = \epsilon_1 = 1/\epsilon_2$ In NS limit $b \to \infty$

But the proof already exists! [Mironov, Morozov]

at large c conformal block becomes a hypergeometric function

[Zamolodchikov]

$$B_{\Delta;\Delta_1\Delta_2\Delta_3\Delta_4}(x) \stackrel{c\to\infty}{\longrightarrow} {}_2F_1\Big(\Delta+\Delta_1-\Delta_2, \Delta+\Delta_3-\Delta_4; 2\Delta; x\Big) =$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \prod_{k=0}^{n-1} \frac{(\Delta+\Delta_1-\Delta_2+k)(\Delta+\Delta_3-\Delta_4+k)}{2\Delta+k}$$

Only chiral Nekrasov functions contribute

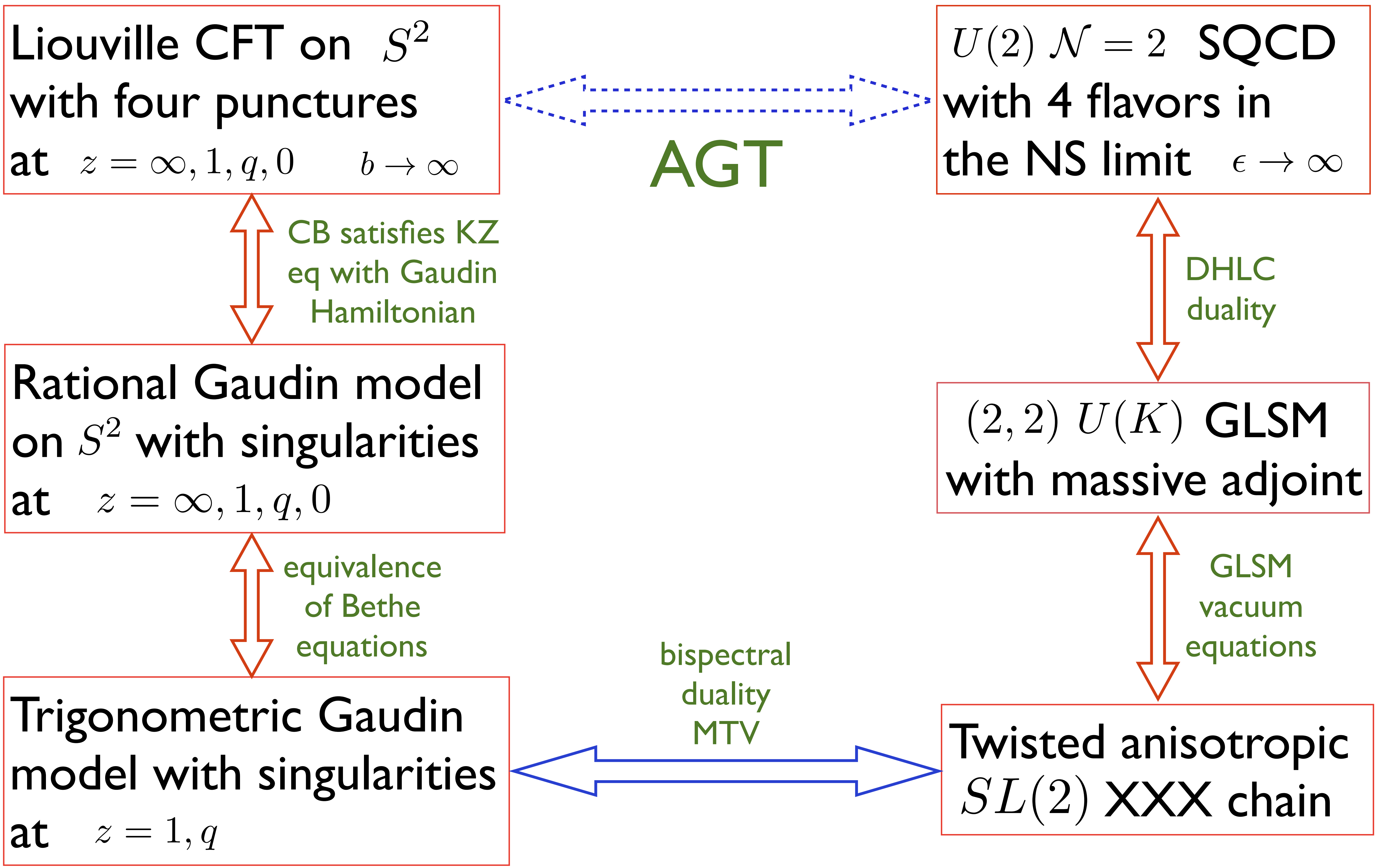
$$(Y, Y') = ([1^n], \emptyset) \text{ or } (\emptyset, [1^n])$$

One can identify each term of the expansion in the instanton number with the Taylor series in x for 2F1

Similar to Fateev-Litvinov conformal blocks

Both proofs are rather formal and deal with each term in the series. Need more physical understanding...

Roadmap to proof



4d/2d in Omega background

[Dorey Hollowood Lee]

N=2 SQCD in Omega background
in NS limit with Nf=2Nc

$$\vec{a} = \vec{m}_F - \vec{n}\epsilon \qquad \vec{n} = (n_1, \ldots, n_L) \in \mathbb{Z}^L$$

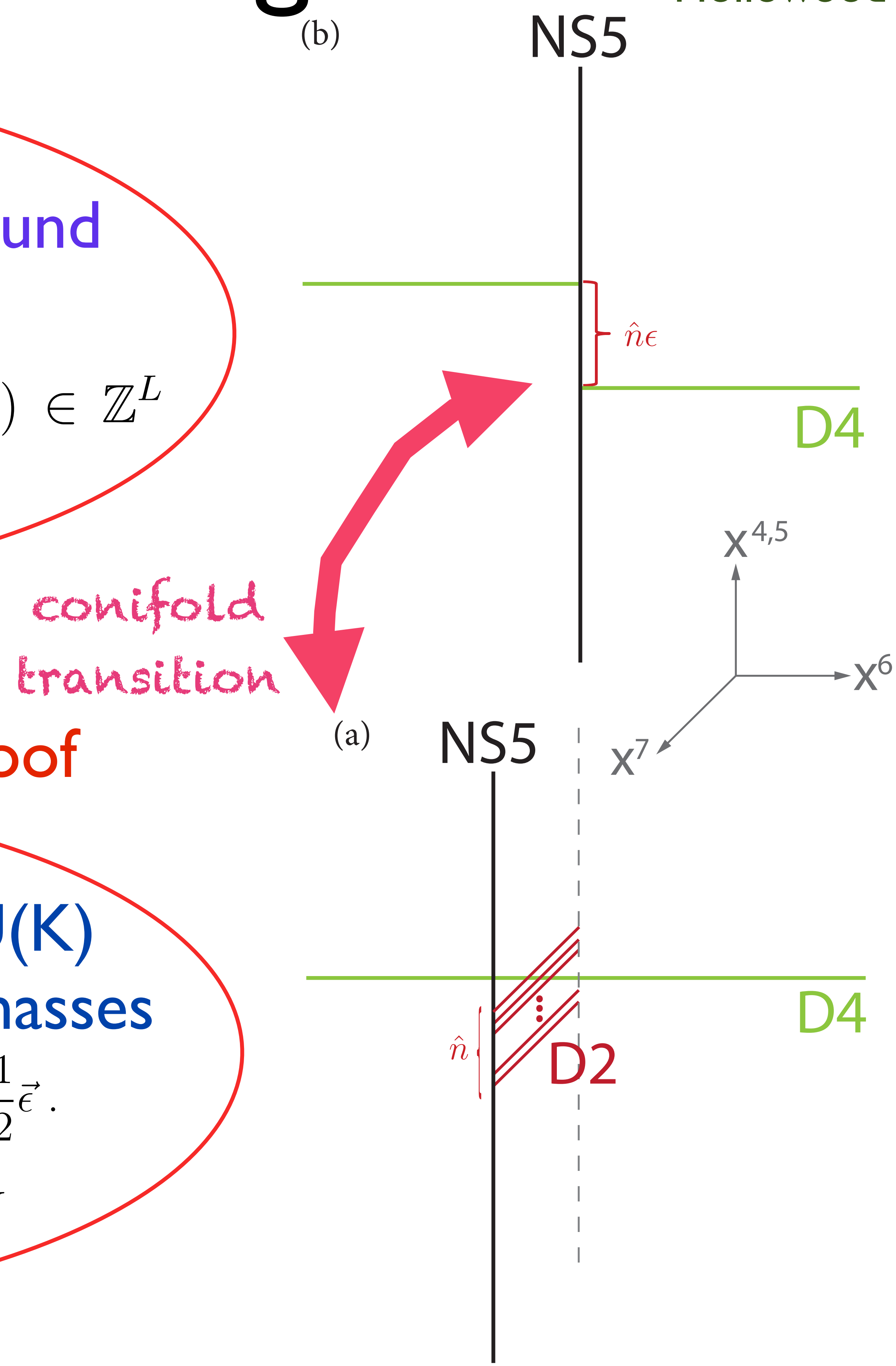
$$\mathcal{W}^{(I)} \overset{\text{on-shell}}{\equiv} \mathcal{W}^{(II)}$$

exact proof

(2,2) GLSM w/ gauge group U(K)
massive adjoint and twisted masses

$$\vec{M}_F = \vec{m}_F - \frac{3}{2}\vec{\epsilon}\,, \qquad \vec{M}_{AF} = \vec{m}_{AF} + \frac{1}{2}\vec{\epsilon}\,.$$

$$M_{adj} = \epsilon \qquad K = \sum_{i=1}^{N} n_i - N$$



XXX vs Gaudin

[Nekrasov Shatashvili]

Effective twisted superpotential

$$\widetilde{W}^{2d}_{\text{eff}}(\lambda) = \epsilon \sum_{a=1}^{K} \sum_{i=1}^{N} f\left(\frac{\lambda_a - M_i}{\epsilon}\right) - \epsilon \sum_{a=1}^{K} \sum_{i=1}^{N} f\left(\frac{\lambda_a - \widetilde{M}_i}{\epsilon}\right) + \epsilon \sum_{a,b=1}^{K} f\left(\frac{\lambda_a - \lambda_b - \epsilon}{\epsilon}\right) + 2\pi i \hat{\tau} \sum_{a=1}^{K} \lambda_a \,,$$

Ground state equations

Heisenberg SL(2) magnet

twisted and anisotropic

$$\prod_{a=1}^{N} \frac{\lambda_i - \nu_a + \frac{\epsilon}{2} S_a}{\lambda_i - \nu_a - \frac{\epsilon}{2} S_a} = q \prod_{\substack{j=1 \\ j \neq i}}^{K} \frac{\lambda_i - \lambda_j - \epsilon}{\lambda_i - \lambda_j + \epsilon}$$

$$\lambda_i \mapsto x\lambda_i, \qquad \nu_a \mapsto x\nu_a, \qquad \hat{\tau} \mapsto \frac{\hat{\tau}}{x}$$

Large anisotropy limit

rational Gaudin model

$$\frac{\log q}{\epsilon} - \sum_{a=1}^{N} \frac{S_a}{\lambda_i - \nu_a} = \sum_{\substack{j=1 \\ j \neq i}}^{K} \frac{2}{\lambda_i - \lambda_j}$$

Bethe equations obtained by diagonalizing (4 sites)

$$S(u) = \sum_{a=1}^{4} \frac{\mathcal{H}_a}{u - z_a} + \sum_{a=1}^{4} \frac{\Delta(\nu_a)}{(u - z_a)^2}$$

Gaudin Hamiltonians

$$\mathcal{H}_a = \sum_{b \neq a} \sum_{\alpha,\beta=1}^{\dim(\mathfrak{g})} \frac{\mathfrak{J}^{(b)}_\alpha \mathfrak{J}^{\alpha\,(b)}}{z_a - z_b}$$

Bispectral duality

[Mukhin Tarasov Varchenko]

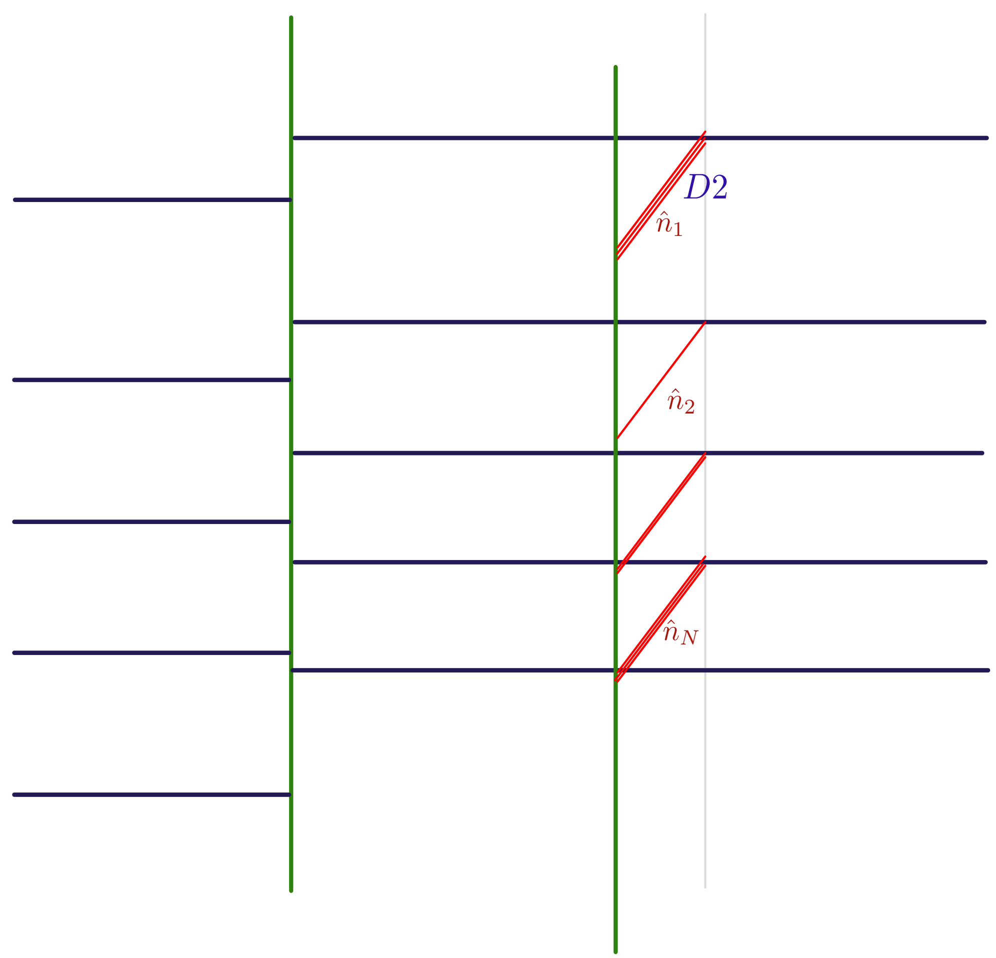
Trigonometric Gaudin vs XXX magnet

$$\frac{\mathcal{M}_1 - \mathcal{M}_2 - \epsilon}{t_i} + \sum_{b=1}^{2} \frac{\nu_b \epsilon}{t_i - z_b} - \sum_{\substack{j=1 \\ j \neq i}}^{\kappa_2} \frac{2\epsilon}{t_i - t_j} = 0, \quad i = 1, \ldots, \kappa_2 \,,$$

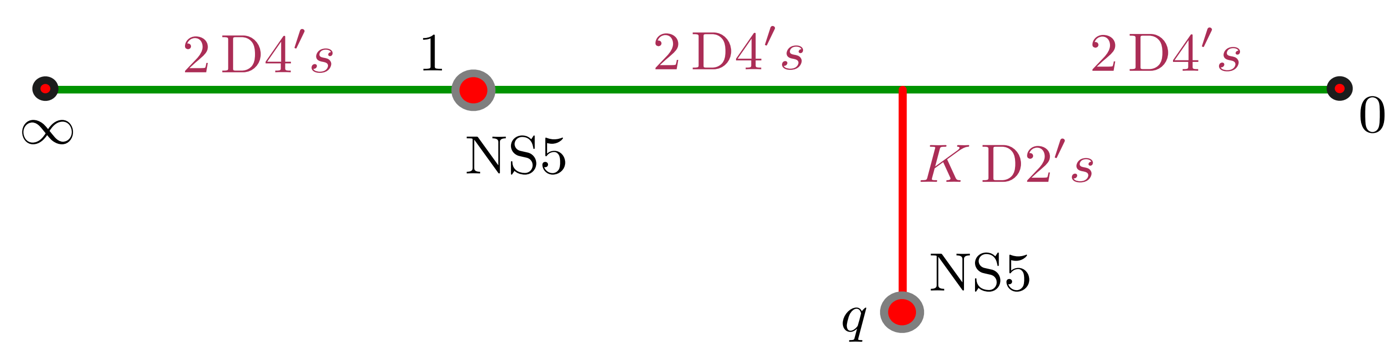
$$\prod_{a=1}^{2} \frac{\lambda_i + \mathcal{M}_a}{\lambda_i + \mathcal{M}_a + \kappa_a \epsilon} = \frac{z_2}{z_1} \prod_{\substack{j=1 \\ j \neq i}}^{\nu_2} \frac{\lambda_i - \lambda_j - \epsilon}{\lambda_i - \lambda_j + \epsilon}\,, \quad i = 1, \ldots, \nu_2$$

Equations have isomorphic spaces of solutions

$$\kappa_1 + \kappa_2 = \nu_1 + \nu_2$$



Nice brane interpretation
Rotation by 90 degrees



De Liouville à Gaudin

Gaudin Hamiltonian
in KZ equation

$$b^2 \frac{d\Psi(z_i)}{dz_i} = \mathcal{H}_{Gaud} \Psi(z_i)\,, \quad i = 1, \ldots, L$$

[Babujian Flume]

Dual WZNW model

$$b^2 = -(k+2)^{-1}$$

[Teschner]

NS limit - critical level

$$k \to -2$$

rescale conf dims

$$\delta_i = -\frac{\Delta_i}{b^2}$$

$$\delta_1 = -\left(\frac{\widetilde{\mu}_0}{b} - \frac{1}{2}\right)\left(\frac{\widetilde{\mu}_0}{b} + \frac{1}{2}\right)$$

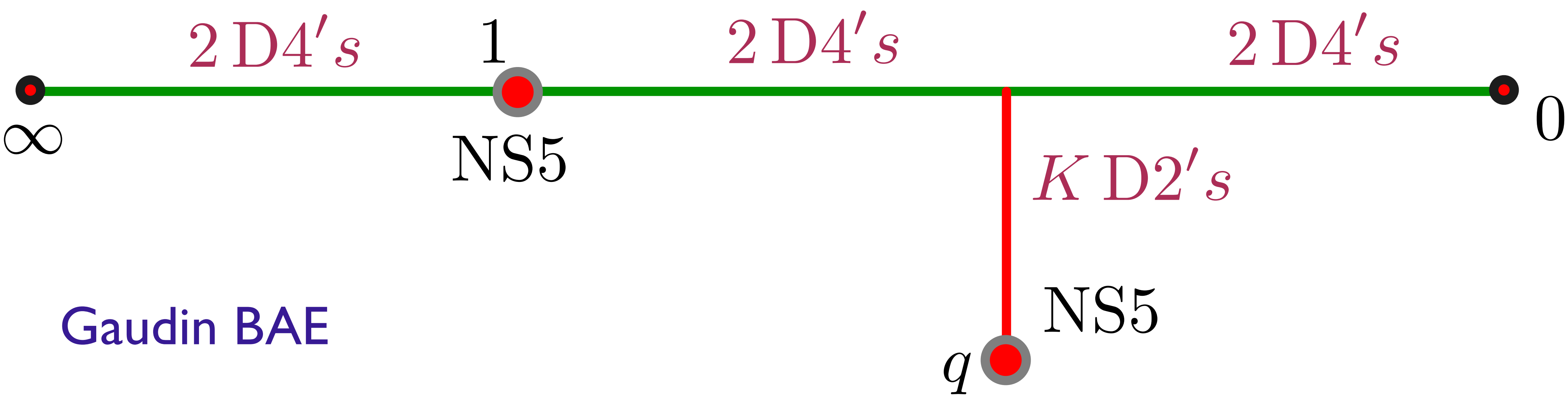
$$\delta_2 = -\left(\frac{\mu_0}{b} - 1\right)\frac{\mu_0}{b}\,,$$

$$\delta_3 = -\left(\frac{\mu_1}{b} - 1\right)\frac{\mu_1}{b}\,,$$

$$\delta_4 = -\left(\frac{\widetilde{\mu}_1}{b} - \frac{1}{2}\right)\left(\frac{\widetilde{\mu}_1}{b} + \frac{1}{2}\right)$$

take home message: CB in Liouville
- wave function in Gaudin

The Duality



Gaudin BAE

$$\sum_{b=1}^{4} \frac{\nu_b \epsilon}{t_i - z_b} - \sum_{\substack{j=1 \\ j\neq i}}^{\kappa_2} \frac{2\epsilon}{t_i - t_j} = 0$$

$$\epsilon\nu_1 = 0, \quad \epsilon\nu_2 = K, \quad \epsilon\nu_3 = m_3 - m_4 - \epsilon = 2\widetilde{\mu}_1 - \epsilon$$

Higgs branch root

$$a_a = m_{2+a} - n_a \epsilon\,, \quad a = 1, 2$$

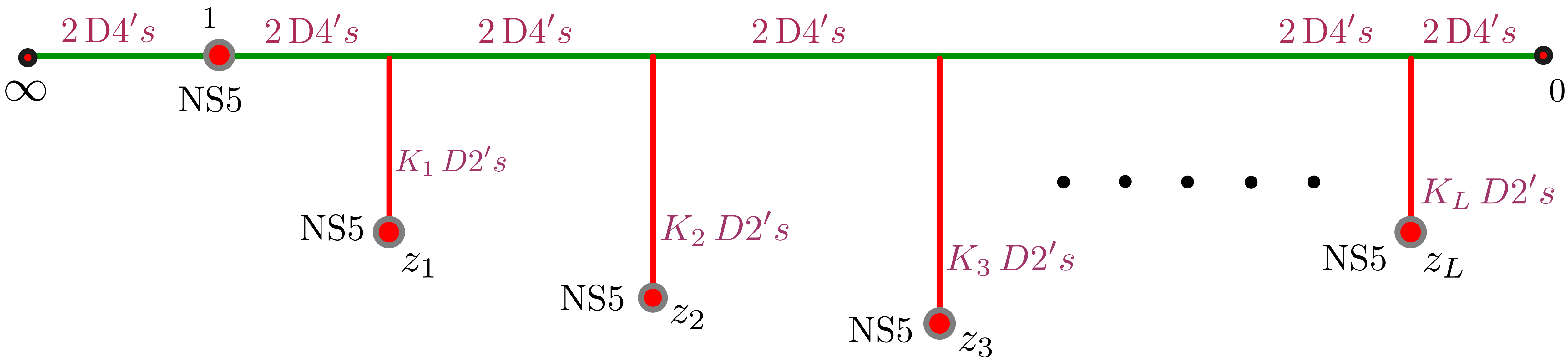
U(1) condition

$$\frac{\mu_1}{\epsilon} = \frac{n_1 + n_2}{2}$$

AGT in NS limit

Liouville conformal block at $b \to \infty$ on S^2 with four punctures	$U(2)\,, N_f = 4$ SQCD instanton partition function in the NS limit
Rational Gaudin model from KZ equation on conformal blocks	$SL(2)$ spin chain from the ground state equation for the 2d GLSM dual to 4d theory
Puncture's positions z_2/z_1	Instanton number q
$\mathfrak{sl}_2$ spin at $z = q$	$U(1)$ condition
Conformal dimensions of chiral operators at points $z = 1$, $z = q$ at points $z = \infty$, $z = 0$	Quadratic $\mathfrak{sl}(2)$ Casimir eigenvalues on spin 0, $\frac{1}{2}\hat{n}_1 + \frac{1}{2}\hat{n}_2$ representations spin 0, $\frac{1}{2}\hat{n}_1 - \frac{1}{2}\hat{n}_2 - \frac{1}{2}$, representations
Gaudin Hilbert space sectors with different number κ_a of Bethe roots	Higgs branch lattice $\{n_a\}$

Quiver Generalizations



Vortices in Omega background [PK Gorsky Chen] in progress

SUSY transform pure SYM

$$\delta\Lambda^I_\alpha = \zeta^I_\beta((\sigma^{mn})^\beta_\alpha F_{mn} + i[\phi,\bar\phi]\delta^\beta_\alpha + \nabla_m(\bar\Omega^m\phi - \Omega^m\bar\phi)\delta^\beta_\alpha) + \bar\zeta^I_{\dot\beta}(\sigma^m)^{\dot\beta}_\alpha(\nabla_m\phi - F_{mn}\Omega^n)$$

String central charge current

$$\zeta_3 = \frac{1}{2}\partial_m\left((\phi^a\bar\Omega^m - \bar\phi^a\Omega^m)B^a_3\right)\sigma^3_{\alpha\dot\alpha}\delta^{IJ} = \frac{i}{2}B^a_3\partial_\varphi(\phi^a\bar\epsilon - \bar\phi^a\epsilon)\sigma^3_{\alpha\dot\alpha}\delta^{IJ}$$

yields for a string of tension ~ epsilon

$$\mathcal{L} = \frac{1}{2}|B^a_z + \phi\tau^a\bar\phi - i\nabla_m(\Omega^m\bar\phi^a - \bar\Omega^m\phi^a)|^2 + \frac{1}{2}|\mathcal{D}_1\phi^a + i\mathcal{D}_2\phi^a - (\Omega_2 - i\Omega_1)B^a_z|^2 + \partial_m(B^a_z(\Omega^m\bar\phi^a - \bar\Omega^m\phi^a)) \geq \partial_m(B^a_z(\Omega^m\bar\phi^a - \bar\Omega^m\phi^a))\,.$$

Symmetry breaking pattern

$$SU(2)_c \times SU(2)_R \times SU(2)_{\mathcal{R}} \to U(1)_c \times SU(2)_{R+\mathcal{R}}$$

Searching for the field theoretical explanation of the new duality

Less SUSY:
Heterotic deformation

(0,2) Theory

[Gorsky Shifman Yung]
[Distler Kachru]
[Edalati Tong][Shifman Yung]

In 4d introduce masses $\int d^2\theta\, \mu^2 (\Phi^a)^2$

breaks $\mathcal{N}=2$ to $\mathcal{N}=1$

On the flux tube
$(2,2) \longmapsto (0,2)$

obtain heterotic sigma model

$$\mathcal{L} = \int d^4\theta \left(\Phi_i^\dagger e^V \Phi^i - rV - \mathcal{B}V \right)$$

Note: Cannot be (1,1) since then it's automatically (2,2)

B-right handed superfield

can be treated as model w/ field dependent FI term

$$K = (r+\mathcal{B}) \log(1+|\phi^i|^2)$$

(0,2) deformation of HT [PK Monin Vinci]

$$\int d^4\theta \left[\sum_{i=1}^{N_c} \Phi_i^\dagger \, e^V \, \Phi_i + \sum_{i=1}^{N_c - N_f} \tilde{\Phi}_i^\dagger \, e^{-V} \, \tilde{\Phi}_i - (r + \mathcal{B})V + \frac{1}{2e^2} \Sigma^\dagger \Sigma \right]$$

$$\Phi^i = n^i + \bar{\theta}\xi^i + \theta\bar{\xi}^i + \bar{\theta}\theta F^i \,, \quad i = 1, \ldots, N_c$$
$$\widetilde{\Phi}^j = \rho^j + \bar{\theta}\eta^j + \theta\bar{\eta}^j + \bar{\theta}\theta \tilde{F}^j \,, \quad j = 1, \ldots, \tilde{N}$$

$$\Sigma = \sigma + i\theta^+ \bar{\lambda}_+ - i\bar{\theta}^- \lambda_- + \theta^+ \bar{\theta}^- (D - iF_{01})$$

$$\mathcal{B} = \omega(\bar{\theta}\zeta_R + \bar{\theta}\theta\bar{\mathcal{F}}\mathcal{F})$$

deformation adds

$$\mathcal{L}^{het} = \mathcal{L} + \bar{\zeta}_R \partial_L \zeta_R - |\omega|^2 |\sigma|^2 - [i\omega\lambda_L\zeta_R + \text{H.c.}]$$

Not enough SUSY

non-pert. corrections out of control

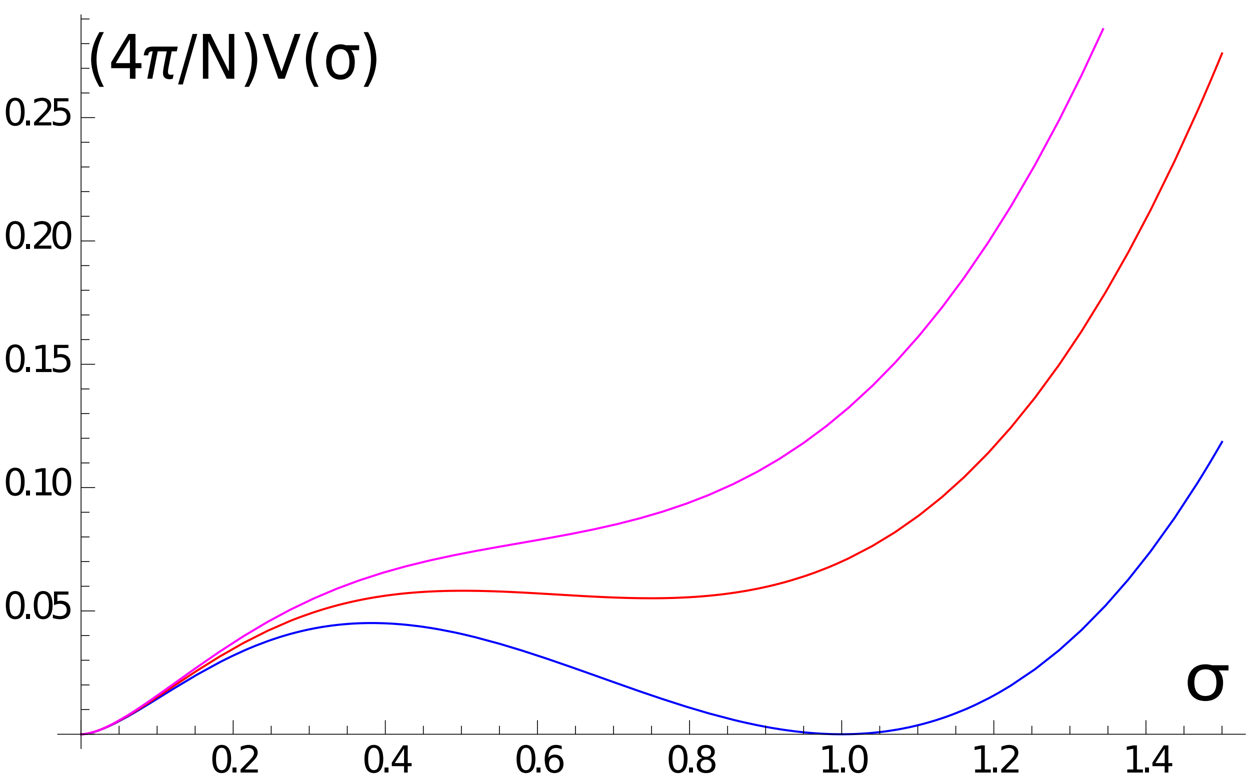
Have to dwell on large-N approach

Large-N solution of (0,2)

$$V_{1-loop} = \frac{1}{4\pi}\sum_{i=1}^{N-1}\left(-\left(D+|\sigma-m_i|^2\right)\log\frac{|\sigma-m_i|^2+D}{\Lambda^2}+|\sigma-m_i|^2\log\frac{|\sigma-m_i|^2}{\Lambda^2}\right)$$
$$-\frac{1}{4\pi}\sum_{j=1}^{\tilde{N}-1}\left(-\left(D-|\sigma-\mu_j|^2\right)\log\frac{|\sigma-\mu_j|^2-D}{\Lambda^2}-|\sigma-\mu_j|^2\log\frac{|\sigma-\mu_j|^2}{\Lambda^2}\right)$$
$$+\frac{N-\tilde{N}}{4\pi}D\,.$$

$$V_{eff} = V_{1-loop} + \left(|\sigma-m_0|^2+D\right)|n_0|^2 + \left(|\sigma-\mu_0|^2-D\right)|\rho_0|^2 + \frac{uN}{4\pi}|\sigma|^2$$

for zero masses



Symmetric masses

$$m_k = m\,e^{2\pi i\frac{k}{N}}\,,\quad k=0,\ldots,N-1\,,$$
$$\mu_l = \mu\,e^{2\pi i\frac{l}{\tilde{N}}}\,,\quad l=0,\ldots,\tilde{N}-1\,.$$

Vacuum equations

$$\left(|\sigma - m_0|^2 + D\right) n_0 = 0\,, \quad \left(|\sigma - \mu_0|^2 - D\right) \rho_0 = 0\,,$$

$$\frac{1}{4\pi} \sum_{i=1}^{N-1} \log \frac{|\sigma - m_i|^2 + D}{\Lambda^2} - \frac{1}{4\pi} \sum_{j=1}^{\tilde{N}-1} \log \frac{|\sigma - \mu_j|^2 - D}{\Lambda^2} = |n_0|^2 - |\rho_0|^2\,,$$

$$\frac{1}{4\pi} \sum_{i=1}^{N-1} (\sigma - m_i) \log \frac{|\sigma - m_i|^2 + D}{|\sigma - m_i|^2} + \frac{1}{4\pi} \sum_{j=1}^{\tilde{N}-1} (\sigma - \mu_j) \log \frac{|\sigma - \mu_j|^2 - D}{|\sigma - \mu_j|^2} =$$

$$= (\sigma - m_0)\, |n_0|^2 + (\sigma - \mu_0)\, |\rho_0|^2 + \frac{uN}{4\pi}\, \sigma\,.$$

Solution of (2,2) model

Phase transitions -- artifact of large-N

$$\left(|\sigma - m_0|^2 + D\right) n_0 = 0\,, \quad \left(|\sigma - \mu_0|^2 - D\right) \rho_0 = 0$$

Higgs in n (Hn)

$$\rho_0 = 0 \quad D = -|\sigma - m|^2$$

$$r = \begin{cases} \frac{N-\tilde{N}}{2\pi} \log \frac{m}{\Lambda}\,, & \mu < m \\ \frac{N}{2\pi} \log \frac{m}{\Lambda} - \frac{\tilde{N}}{2\pi} \log \frac{\mu}{\Lambda}\,, & \mu > m\,. \end{cases}$$

Higgs in rho (Hρ)

$$n_0 = 0 \quad D = |\sigma - \mu|^2$$

$$r = \begin{cases} \frac{N-\tilde{N}}{2\pi} \log \frac{\mu}{\Lambda}\,, & \mu > m \\ \frac{N}{2\pi} \log \frac{m}{\Lambda} - \frac{\tilde{N}}{2\pi} \log \frac{\mu}{\Lambda}\,, & \mu < m \end{cases}$$

Coulomb (C)

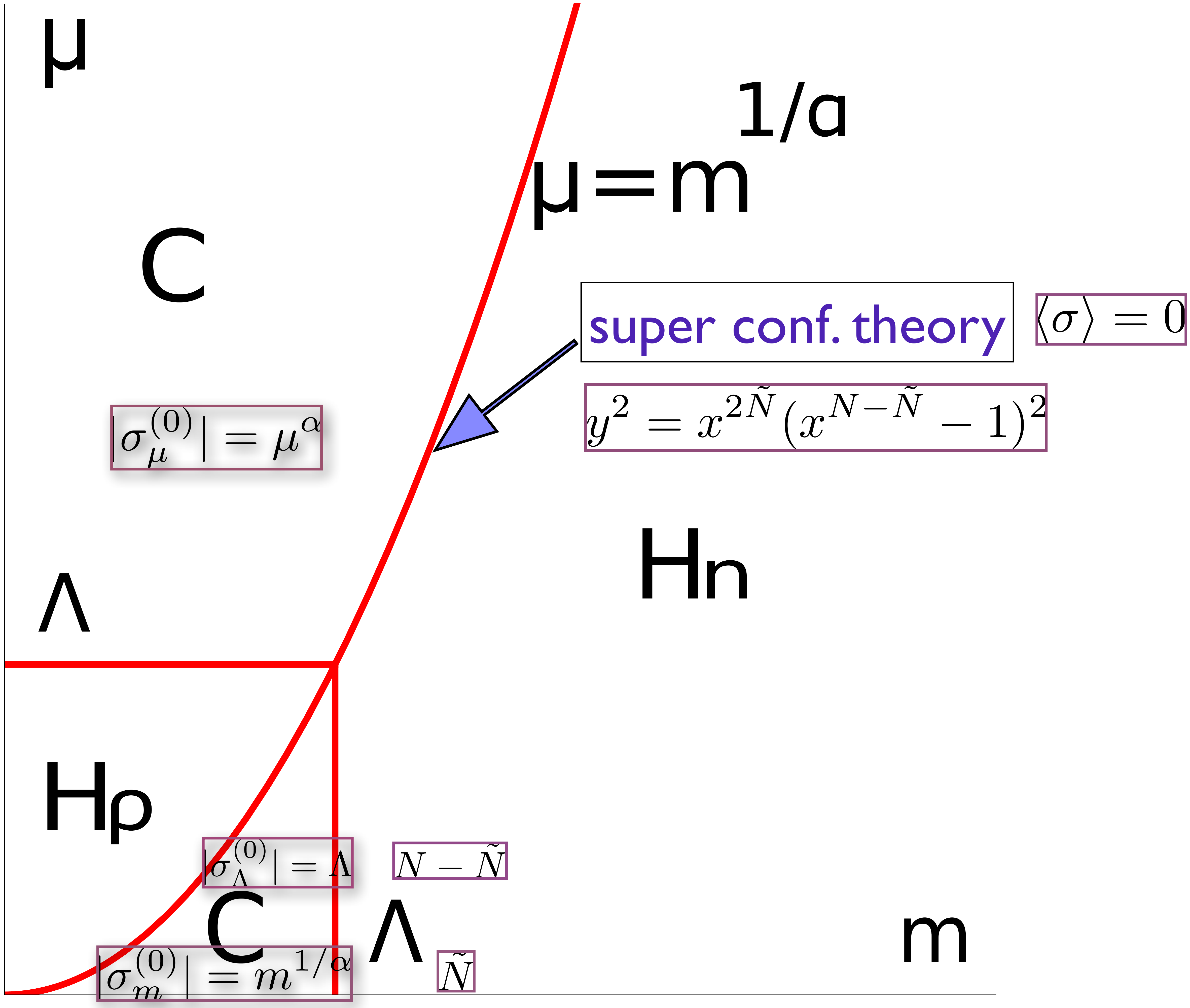
$$n_0 = \rho_0 = 0$$

renormalized FI term vanishes in C phase

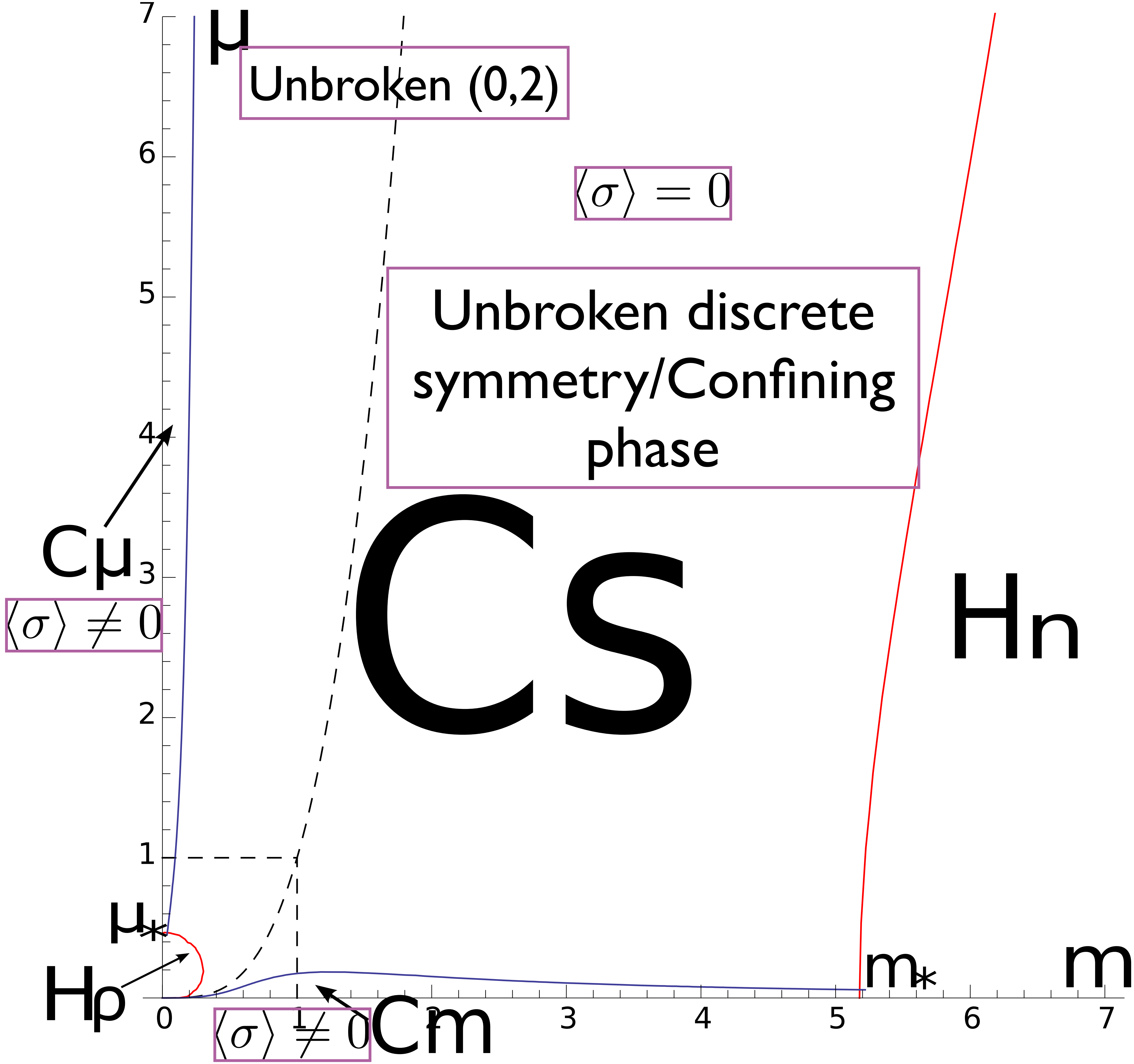
in (2,2) from exact superpotential

$$\frac{\prod_i (\sigma - m_i)}{\prod_i (\sigma - \mu_j)} = \Lambda^{N-\tilde{N}} \qquad \sigma = 0$$

is one of the solutions...



μ
1/α
μ=m
C
super conf. theory
$\langle\sigma\rangle = 0$
$y^2 = x^{2\tilde{N}}(x^{N-\tilde{N}} - 1)^2$
$|\sigma_\mu^{(0)}| = \mu^\alpha$
Hn
Λ
Hp
$|\sigma_\Lambda^{(0)}| = \Lambda$
$N - \tilde{N}$
C
Λ
$\tilde{N}$
m
$|\sigma_m^{(0)}| = m^{1/\alpha}$

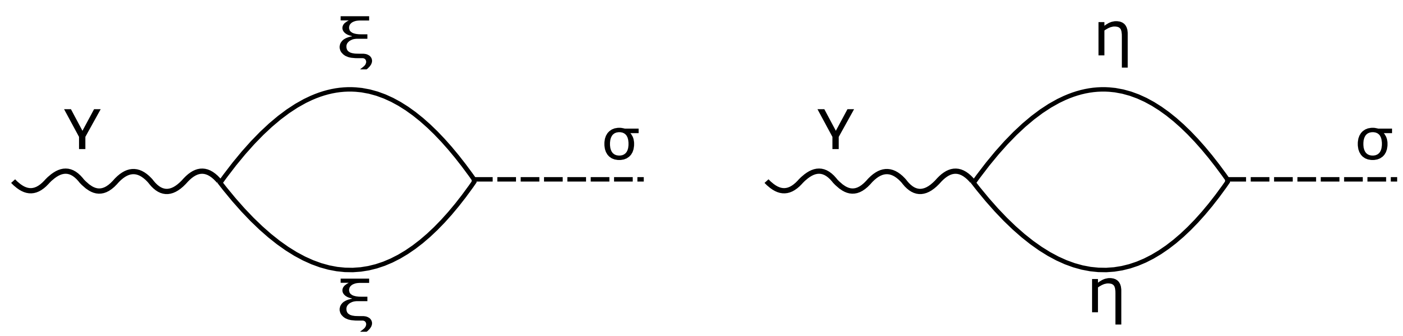

μ
Unbroken (0,2)
⟨σ⟩ = 0
Unbroken discrete symmetry/Confining phase
Cs
Hn
Cμ
⟨σ⟩ ≠ 0
μ*
Hp
m*
m
⟨σ⟩ ≠ 0
Cm
0
1
2
3
4
5
6
7

Spectrum

[Bolokhov Shifman Yung]
[PK Monin Vinci]

$$\mathcal{L} = -\frac{1}{4e_\gamma^2}F_{\mu\nu}^2 + \frac{1}{e_{\sigma\,1}^2}(\partial_\mu \mathfrak{Re}\,\sigma)^2 + \frac{1}{e_{\sigma\,2}^2}(\partial_\mu \mathfrak{Im}\,\sigma)^2 + i\mathfrak{Im}(\bar{b}\,\delta\sigma)\epsilon_{\mu\nu}F^{\mu\nu} - V_{\text{eff}}(\sigma) + \text{Fermions}$$

Anomaly



$$b = \frac{N}{4\pi}\left(\frac{1}{N}\sum_{i=1}^{N-1}\frac{1}{\bar{\sigma}_0 - \bar{m}_i} - \alpha\frac{1}{\tilde{N}}\sum_{i=1}^{\tilde{N}-1}\frac{1}{\bar{\sigma}_0 - \bar{\mu}_i}\right)$$

$$m_\gamma = e_{\sigma\,2}e_\gamma|b|$$

Photon becomes massless in Cs phase!!

Confinement!

Note that Lambda vacua disappear at large deformations

Need to sit in zero-vacua

e.g. in Cm phase

$$m_\gamma = \sqrt{6}\,\Lambda\left(\frac{\Lambda}{m}\right)^{1/\alpha}\left(\left(\frac{m}{\Lambda}\right)^{2/\alpha} - \left(\frac{\mu}{\Lambda}\right)^2 e^{u/\alpha}\right)\mathrm{e}^{-\frac{u}{2\alpha}}$$

Massless goldstino in fermionic sector

NSVZ in (0,2) sigma model

[Cui Shifman]

$\mathbb{P}^N$ sigma models exhibit instanton solutions

Let us now remove half of the fermions

An instanton has four bosonic zero modes but only two fermionic ones

$$A_{\rm inst} = \frac{y}{z - z_0}\,, \qquad A^\dagger_{\rm inst} = \frac{\bar{y}(1 + 4i\theta^\dagger \beta^\dagger)}{\bar{z}_{\rm ch} - \bar{z}_0 - 4i\theta^\dagger \alpha}$$

One loop corrections in the instanton background do not cancel completely

$$d\mu = \left(\frac{M^2}{g^2}\right)^{n_b} \left(\frac{g^2}{M}\right)^{n_f} (M)^{-1} e^{-\frac{4\pi}{g^2}}\ d\log(y) d\log(\bar{y})\ dz_0 d\bar{z}_0\ d\alpha d\beta^\dagger$$

One loop modification

Exact beta function

$$\beta(g^2) = -\frac{g^4}{2\pi}\frac{1}{1 - \frac{g^2}{4\pi}}$$

What does it mean for 4d/2d?

Conclusions and open questions

- Study of SQCD BPS (and beyond) spectrum can effectively be done using 2d NLSM (and GLSM)
- 4d/2d duality helps to understand AGT in NS limit by reducing it to bispectral duality
- Relationship w/ another 4d/2d duality [Vafa et al]
- Generalize to other AGT pairs
- Holography for Non-Abelian vortices
- A lot is unknown about (0,2) theory... how far can we push 4d/2d?