Quantum K-theory of Quiver Varieties and Many-Body Systems

Peter Koroteev

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

THE UNIVERSITY OF CALIFORNIA DAVIS

based on 1705.10419 with P. Pushkar, A. Smirnov, A. Zeitlin

Talk at ISQS25
Prague June 9th 2017

Motivation

Supersymmetric gauge theories/String theory have been suggesting for a long time that there is a strong connection between **geometry** and **integrability**

Study of ***Gromov-Witten*** invariants was partly influenced by progress in string theory. For a symplectic manifold X GW invariants appear in the expansion of quantum multiplication in ***quantum cohomology*** ring of X.

A particular attention is given to genus zero GW invariants.

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Supersymmetric gauge theories/String theory have been suggesting for a long time that there is a strong connection between **geometry** and **integrability**

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In this talk we shall study **equivariant quantum K-theory** of large family of symplectic varieties and its connection to **integrable systems**

Sigma Model

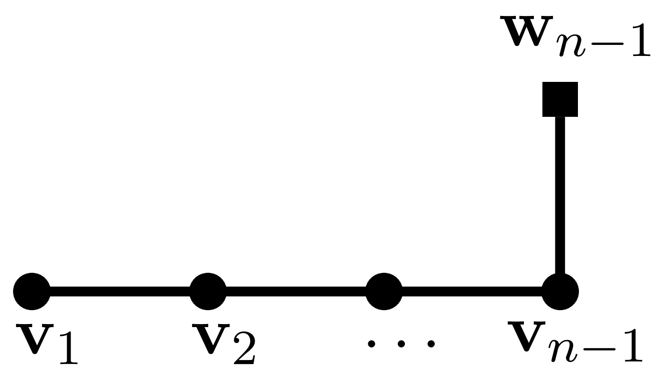
To see how integrability arises one considers supersymmetric sigma model from the base curve (P1 in our case) into $\mathcal{X}$

Witten demonstrated that relevant class of supersymmetric sigma models can be rewritten as supersymmetric gauge theories (**(2,2) GLSMs**) in two dimensions whose field content is related to geometry of $\mathcal{X}$. Sigma models thus describe infrared dynamics of GLSMs.

Nekrasov and Shatashvili showed how to obtain integrable systems from such GLSMs. It was conjectured that SUSY vacua of **2d** theories compute **quantum cohomology** ring of $\mathcal{X}$, while **3d** theories on $\mathbb{R}^2 \times S^1$ describe **quantum K-theory.**

Classical K-theory

Rep(**v**,**w**) — linear space of quiver reps



$\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map

Nakajima quiver variety $X = \mu^{-1}(0)/\!\!/G$ $\quad G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

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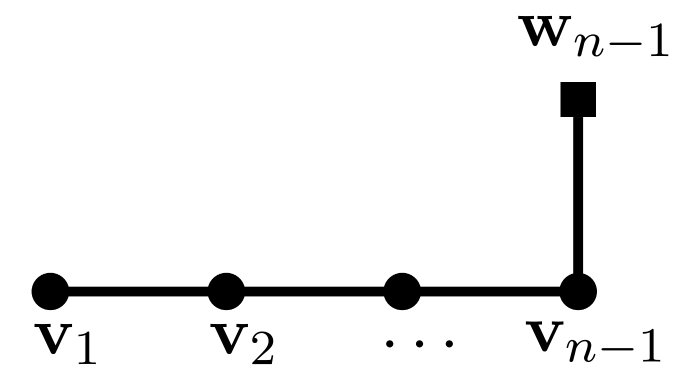

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Tensorial polynomials of tautological bundles **V**i, **W**i and their duals generate classical T-equivariant K-theory ring of X

Bilinear form $(\mathcal{F},\mathcal{G}) = \chi(\mathcal{F}\otimes\mathcal{G}\otimes K^{-1/2})$

We shall study its quantum deformation

Quasimaps

[Okounkov]

Definition 2.1. *A quasimap f from $\mathcal{C}$ to X*

(2) $$f : \mathcal{C} \dashrightarrow X$$

Is a collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ together with a section of the bundle

(3) $$f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar),$$

satisfying $\mu = 0$, where

$$\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j),$$

so that $\mathscr{W}_i$ are trivial bundles of rank $\mathbf{w}_i$ and μ is the moment map. Here $\hbar$ is a trivial line bundle with weight $\hbar$ introduced to have the action of T on the space of quasimaps. The degree of a quasimap is a the vector of degrees of bundles $\mathscr{V}_i$.

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Open subset of nonsingular quasimaps is endowed with evaluation map to X

Relative quasimaps are resolutions with proper ev map

$$\begin{array}{ccc} & \mathsf{QM}^{\mathbf{d}}{}_{\text{relative}\, p} & \\ \nearrow & & \searrow{\scriptstyle \widetilde{\text{ev}}_p} \\ \mathsf{QM}^{\mathbf{d}}{}_{\text{nonsing}\, p} & \xrightarrow{\text{ev}_p} & X \end{array}$$

Relative point is replaced by a chain of non-rigid projective lines, such that the endpoint and all the nodes are not singular

Vir Structure and Gluing

Deformation-obstruction theory allows one to construct virtual tangent bundle and virtual structure sheaf [Ciocan-Fontanine, Kim, Maulik]

Fiber of the reduced virtual tangent bundle to $\mathsf{QM}^{\mathbf{d}}{}_{\text{nonsing}\, p}$

$$T^{\text{vir}}_{(\{\mathscr{V}_i\},\{\mathscr{W}_i\})}\mathsf{QM}^{\mathbf{d}}_{\text{nonsing p}} = H^{\bullet}(\mathscr{M} \oplus \hbar\, \mathscr{M}^*) - (1+\hbar)\bigoplus_i Ext^{\bullet}(\mathscr{V}_i, \mathscr{V}_i).$$

Symmetrized virtual structure sheaf $\hat{\mathcal{O}}_{\text{vir}} = \mathcal{O}_{\text{vir}} \otimes \mathscr{K}_{\text{vir}}^{1/2} q^{\deg(\mathscr{P})/2}$

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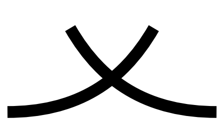
Symmetrized virtual structure sheaf $\hat{\mathcal{O}}_{\text{vir}} = \mathcal{O}_{\text{vir}} \otimes \mathscr{K}_{\text{vir}}^{1/2} q^{\deg(\mathscr{P})/2}$

 denotes the base curve $\mathbb{P}^1$,

 denotes a marked point (absolute point),

 denotes a relative point,

 denotes a nonsingular point.

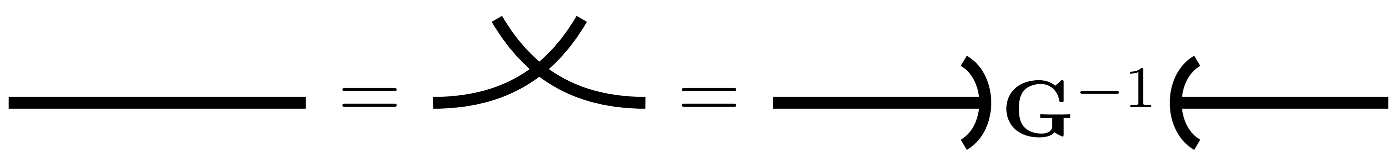
 denotes a node on the base curve.

Gluing operator

Define gluing operator $\mathbf{G} \in \mathrm{End}(K_{\mathsf{T}}(X))[[z]]$

as 

Used in degeneration formula when smooth $\mathcal{C}_\varepsilon$ curve degenerates to nodal curve $\mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}$. We can replace quasimap counts.



$$\chi(\mathsf{QM}(\mathcal{C}_0 \to X), \hat{\mathcal{O}}_{\mathrm{vir}} z^{\mathbf{d}}) = \left(\mathbf{G}^{-1}\mathrm{ev}_{1,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^{\mathbf{d}}), \mathrm{ev}_{2,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^{\mathbf{d}})\right)$$

$$\mathrm{ev}_i : \mathsf{QM}(\mathcal{C}_{0,i} \to X)_{\text{relative gluing point}} \to X$$

Quantum K-ring

Quantum parameters $z^{\mathbf{d}} = \prod_{i \in I} z_i^{d_i}$

Definition 2.2. *An element of the quantum K-theory*

$$\hat{\tau}(z) = \sum_{\mathbf{d}=\overrightarrow{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\Big(QM^{\mathbf{d}}_{\mathrm{relative}\, p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathscr{V}_i|_{p_1})\Big) \in QK_{\mathsf{T}}(X) \tag{8}$$

is called quantum tautological class corresponding to τ*. In picture notation it will be represented by*

For any $\mathcal{F} \in K_{\mathsf{T}}(X)$ define operator of quantum multiplication

$$\mathcal{F} \circledast = \sum_{\mathbf{d}=\overrightarrow{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3*}\Big(\mathrm{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F})\widehat{\mathcal{O}}_{\mathrm{vir}}\Big)\, \mathbf{G}^{-1}$$

Define the quantum equivariant K-theory ring of X

$QK_{\mathsf{T}}(X) = K_{\mathsf{T}}(X)[[z]]$ with multiplication $A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$

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unit element $\hat{\mathbf{1}}(z) =$

Vertex Function

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q$ which scales the original $\mathbb{P}^1$ keeping two points fixed

Define **bare vertex function** with descendent

$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\Big(QM^{\mathbf{d}}_{\mathrm{nonsing}\, p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathscr{V}_i|_{p_1}) \Big) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

Localization computation gives

$$V^{(\tau)}_{\mathbf{p}}(z) = \sum_{\mathbf{d}\in\mathbb{Z}^n_{\geq 0}} \sum_{(\mathscr{V},\mathscr{W})\in(\mathsf{QM}^{\mathbf{d}}_{\mathrm{nonsing}\, p_2})^{\mathsf{T}}} \hat{s}(\chi(\mathbf{d}))\, z^{\mathbf{d}} q^{\deg(\mathscr{P})/2} \tau(\mathscr{V}|_{p_1})$$

where

$$\hat{s}(x) = \frac{1}{x^{1/2} - x^{-1/2}}, \quad \hat{s}(x+y) = \hat{s}(x)\hat{s}(y)$$

applied to virtual tangent character $\chi(\mathbf{d}) = \mathrm{char}_{\mathsf{T}}\Big(T^{vir}_{\{(\mathscr{V}_i\},\ \mathscr{W}_{n-1})} \mathsf{QM}^{\mathbf{d}} \Big)$

Vertex for A-type Quivers

In localization formula each equivariant line bundle contributes with

$$\{x\}_d = \frac{(\hbar/x,q)_d}{(q/x,q)_d}\,(-q^{1/2}\hbar^{-1/2})^d, \quad \text{where} \quad (x,q)_d = \frac{\varphi(x)}{\varphi(q^d x)}$$

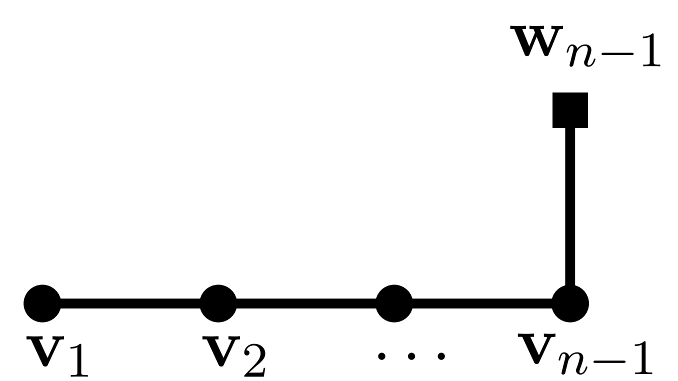
After classifying fixed points of space of nonsingular quasimaps we can compute the vertex

Proposition 3.2. *Let* $\boldsymbol{p} = \mathbf{V}_1 \subset \ldots \subset \mathbf{V}_{n-1} \subset \{a_1, \cdots, a_{\mathbf{w}_{n-1}}\}$ $(\mathbf{V}_i = \{x_{i,1}, \ldots x_{i,\mathbf{v}_i}\})$ *be a chain of subsets defining a torus fixed point* $\boldsymbol{p} \in X^{\mathsf{T}}$. *Then the coefficient of the vertex function for this point is given by:*

$$V^{(\tau)}_{\boldsymbol{p}}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j} q^{-d_{i,j}}),$$

where $\mathbf{d} = (d_1, \ldots, d_{n-1}), d_i = \sum_{j=1}^{\mathbf{v}_i} d_{i,j}, N(\mathbf{d}) = \mathbf{v}'_i d_i,$

$$E = \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}},$$



Bethe Equations

Saddle point approximation provides the operator of quantum multiplication

$$\tau_{\boldsymbol{p}}(z) = \lim_{q \to 1} \frac{V_{\boldsymbol{p}}^{(\tau)}(z)}{V_{\boldsymbol{p}}^{(1)}(z)}$$

For the cotangent bundle to partial flag variety we get

Theorem 3.4. *The eigenvalues of $\hat{\tau}(z)\circledast$ is given by $\tau(s_{i,k})$, where $s_{i,k}$ satify Bethe equations:*

$$\prod_{j=1}^{\mathbf{v}_2} \frac{s_{1,k} - s_{2,j}}{s_{1,k} - \hbar s_{2,j}} = z_1 (-\hbar^{1/2})^{-\mathbf{v}_1'} \prod_{\substack{j=1 \\ j \neq k}}^{\mathbf{v}_1} \frac{s_{1,j} - s_{1,k}\hbar}{s_{1,j}\hbar - s_{1,k}},$$

$$\prod_{j=1}^{\mathbf{v}_{i+1}} \frac{s_{i,k} - s_{i+1,j}}{s_{i,k} - \hbar s_{i+1,j}} \prod_{j=1}^{\mathbf{v}_{i-1}} \frac{s_{i-1,j} - \hbar s_{i,k}}{s_{i-1,j} - s_{i,k}} = z_i (-\hbar^{1/2})^{-\mathbf{v}_i'} \prod_{\substack{j=1 \\ j \neq k}}^{\mathbf{v}_i} \frac{s_{i,j} - s_{i,k}\hbar}{s_{i,j}\hbar - s_{i,k}}, \tag{23}$$

$$\prod_{j=1}^{\mathbf{w}_{n-1}} \frac{s_{n-1,k} - a_j}{s_{n-1,k} - \hbar a_j} \prod_{j=1}^{\mathbf{v}_{n-2}} \frac{s_{n-2,j} - \hbar s_{n-1,k}}{s_{n-2,j} - s_{n-1,k}} = z_{n-1} (-\hbar^{1/2})^{-\mathbf{v}_{n-1}'} \prod_{\substack{j=1 \\ j \neq k}}^{\mathbf{v}_{n-1}} \frac{s_{n-1,j} - s_{n-1,k}\hbar}{s_{n-1,j}\hbar - s_{n-1,k}},$$

where $k = 1, \ldots, \boldsymbol{v}_i$ *for* $i = 1, \ldots, \boldsymbol{v}_{n-1}$.

which are Bethe Ansatz Equations for gl(n) XXZ spin chain

Bethe Equations

Baxter Q-operator generates quantum tautological bundles

$$\mathbf{Q}_i(u) = \sum_{k=0}^{\mathbf{v}_i} (-1)^k u^{\mathbf{v}_i - k} \hbar^{\frac{ik}{2}} \widehat{\Lambda^k V_i}(z)$$

Proposition 3.5. *The eigenvalues of the operator $\mathbf{Q}_i(u)$ are the following polynomials in u:*

$$Q_i(u) = \prod_{k=1}^{\mathbf{v_i}} (u - \hbar^{\frac{i}{2}} s_{i,k}), \tag{26}$$

so that the coefficients are elementary symmetric functions in $s_{i,k}$ for fixed i.

To obtain the full Hilbert space of gl(n) XXZ chain one takes a disjoint union of all partial flag varieties with fixed framing, so that in the basis of fixed points the classical equivariant K-theory expressed as $\mathbb{C}^n(a_1) \otimes \mathbb{C}^n(a_2) \otimes \ldots \mathbb{C}^n(a_{\mathbf{w}_{n-1}})$ of evaluation reps of $U_\hbar(\widehat{\mathfrak{gl}}(n))$

XXZ in Baxter form

After slight change of variables we can write Bethe equations in Baxter form

Lemma 4.1. *The equation for Bethe root* $\sigma_{i,\alpha}$ *arises as* $u = \sigma_{i,\alpha}$ *locus of the following equation*

$$\hbar^{\frac{\Delta_i}{2}} \frac{\zeta_i}{\zeta_{i+1}} \frac{Q_{i-1}^{(1)} Q_i^{(-2)} Q_{i+1}^{(1)}}{Q_{i-1}^{(-1)} Q_i^{(2)} Q_{i+1}^{(-1)}} = -1\,, \tag{33}$$

where $\Delta_i = \boldsymbol{v}_{i+1} + \boldsymbol{v}_{i-1} - 2\boldsymbol{v}_i$.

$$Q_i(u) = \prod_{\alpha=1}^{\mathbf{v}_i} (u - \sigma_{i,\alpha})\,, \qquad P(u) = Q_n(u) = \prod_{a=1}^{\mathbf{w}_{n-1}} (u - \alpha_a) \qquad Q^{(n)}(u) = Q_i(\hbar^{-\frac{n}{2}} u)$$

For cotangent bundles to complete flags $\mathbf{v}_i = i,\ \mathbf{w}_{n-1} = n$

$$\zeta_{i+1} Q_i^{(1)} \widetilde{Q}_i^{(-1)} - \zeta_i Q_i^{(-1)} \widetilde{Q}_i^{(1)} = (\zeta_{i+1} - \zeta_i) Q_{i-1} Q_{i+1} \tag{35}$$

$\widetilde{Q}_i$ correspond generate quantum tautological classes of exterior powers of the flop flag variety.

XXZ/tRS duality

In the remaining time we shall represent **XXZ** Bethe equations as equations of motion of **trigonometric Ruijsenaars-Schneide**r model

We continue working with complete flags

Proposition 4.4. *Solutions of (35) are given by*

(36)
$$Q_j(u) = \frac{det\Big(M_{1,\dots,j}\Big)}{det\Big(V_{1,\dots,j}\Big)}\,, \qquad \widetilde{Q}_j(u) = \frac{det\Big(M_{1,\dots,j-1,j+1}\Big)}{det\Big(V_{1,\dots,j-1,j+1}\Big)}\,,$$

where

(37)
$$M_{i_1,\dots,i_j} = \begin{bmatrix} q_{i_1}^{(j-1)} & \zeta_{i_1} q_{i_1}^{(j-3)} & \cdots & \zeta_{i_1}^{j-1} q_{i_1}^{(1-j)} \\ \vdots & \vdots & \ddots & \vdots \\ q_{i_j}^{(j-1)} & \zeta_{i_j} q_{i_j}^{(j-3)} & \cdots & \zeta_{i_j}^{j-1} q_{i_j}^{(1-j)} \end{bmatrix}\,, \qquad V_{i_1,\dots,i_j} = \begin{bmatrix} 1 & \zeta_{i_1} & \cdots & \zeta_{i_1}^{j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_{i_j} & \cdots & \zeta_{i_j}^{j-1} \end{bmatrix}\,,$$

and where we define polynomials $q_i = u - p_i$ and numbers in the parentheses in the superscripts denote multiplicative shifts of the argument of q_i.

The proof uses Desnanot-Jacobi determinant formula

XXZ/tRS duality

Lax matrix of tRS model can be written explicitly

Theorem 4.5. *Let L be the following matrix*

(49)
$$L_{ij} = \frac{\prod\limits_{k\neq j}^{n}\left(\hbar^{-1/2}\zeta_i - \hbar^{1/2}\zeta_k\right)}{\prod\limits_{k\neq i}^{n}(\zeta_i - \zeta_k)} p_j \,,$$

where

(50)
$$p_j = -\frac{\mathbf{Q}_j(0)}{\mathbf{Q}_{j-1}(0)} = \hbar^{-j+\frac{1}{2}}\widehat{\Lambda^j V_j}(z) \circledast \Lambda^{j-1}\widehat{V^*}_{j-1}(z)\,, \quad j = 1,\dots,n$$

Then polynomial $P(u)$ from (32) can be represented as

(51)
$$P(u) = det\Big(u - L\Big)\,.$$

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It generates tRS(Macdonald) Hamiltonians

$$\det\left(u \cdot 1 - L(\zeta_i, p_i, \hbar)\right) = \sum_{r=0}^{n} (-1)^r H_r(\zeta_i, p_i, \hbar) u^{n-r}$$

in particular

$$H_1 = Tr\, L = \sum_{i=1}^{n} \prod_{j\neq i}^{n} \frac{\zeta_i\, \hbar^{-1/2} - \zeta_j\, \hbar^{1/2}}{\zeta_i - \zeta_j} p_i \,, \qquad H_n = det\, L = \prod_{k=1}^{n} p_k$$

Main Theorem

Combining the results together

Theorem 4.7. *Quantum equivariant K-theory of the cotangent bundle to complete n-flag is given by*

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[\zeta_1^{\pm 1},\dots,\zeta_n^{\pm 1}; a_1^{\pm 1},\dots,a_n^{\pm 1},\hbar^{\pm 1}; p_1^{\pm 1},\dots,p_n^{\pm 1}]}{\{H_r(\zeta_i,p_i,\hbar)=e_r(\alpha_1,\dots,\alpha_n)\}}, \tag{63}$$

where the ideal is generated by equations of motion of all Hamiltonians of tRS model

$\zeta_1,\dots,\zeta_n$ are coordinates $\quad$ $p_1,\dots,p_n$ are momenta

symplectic form $\quad \Omega = \sum_{i=1}^{n} \frac{dp_i}{p_i} \wedge \frac{d\zeta_i}{\zeta_i}$

Momenta can be determined from derivatives of Yang-Yang function XXZ for Bethe equations. They define Lagrangian $\mathcal{L} \subset T^*\left(\mathbb{C}^\times\right)^n$ whose generating function is given by the Yang-Yang function.

[Gaiotto PK]
[Bullimore Kim PK]

Five-Vertex model and qToda

In previous formulae we can take $\hbar \to \infty$

$$Q_{i+1}(u) - \frac{\mathfrak{z}_{i+1}}{\mathfrak{z}_i} Q_{i-1}(u) \cdot u \cdot \mathfrak{p}_{i+1} = Q_i(u)\widetilde{Q}_i(u)\,, \quad i = 1, \ldots, n \tag{70}$$

where $z_i^{\#} = \frac{\mathfrak{z}_i}{\mathfrak{z}_{i+1}}$, $\widetilde{Q}_i(u)$, $i = 1, \ldots, n-1$ *are monic polynomials of degree one and*

$$\mathfrak{p}_i = -\frac{Q_i(0)}{Q_{i-1}(0)}\,. \tag{71}$$

Analogously to **XXZ/tRS** duality we can formulate **5-vert/qToda** duality

Theorem 5.3. *System of equations (70) is equivalent to*

$$M(u) = \det A(u)\,, \tag{72}$$

where $A(u)$ *is the Lax matrix of the difference Toda chain. It has the following nonzero elements*

$$A_{i+1,i} = 1\,, \quad A_{i,i} = u - \mathfrak{p}_i\,, \quad A_{i,i+1} = -u\frac{\mathfrak{z}_{i+1}}{\mathfrak{z}_i}\mathfrak{p}_{i+1}\,. \tag{73}$$

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We recover the statement by Givental and Lee

Theorem 5.4. *Quantum equivariant K-theory of the complete n-dimensional flag variety is given by*

(78) $$QK_{T'}(\mathbb{F}l_n) = \frac{\mathbb{C}[\mathfrak{z}_1^{\pm 1}, \dots, \mathfrak{z}_n^{\pm 1}; \mathfrak{a}_1^{\pm 1}, \dots, \mathfrak{a}_n^{\pm 1}; \mathfrak{p}_1^{\pm 1}, \dots, \mathfrak{p}_n^{\pm 1}]}{\{H_r^{q\text{-}Toda}(\mathfrak{z}_i, \mathfrak{p}_i) = e_r(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\}}\,,$$

Conclusions

Elliptic cohomology

DAHA and K-theory spaces of quasimaps

Elliptic generalizations of DAHA

…. and other ideas some of which are already discussed in physics literature…..