How Supersymmetry Helps to Understand Hydrodynamics

Peter Koroteev

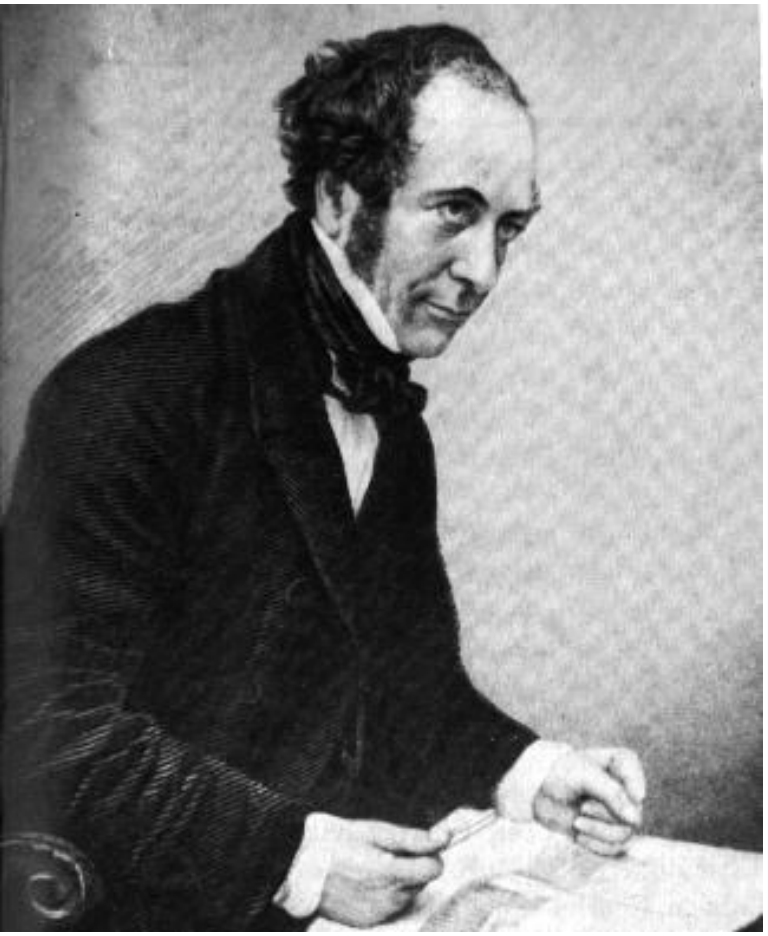
PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

THE UNIVERSITY OF CALIFORNIA DAVIS

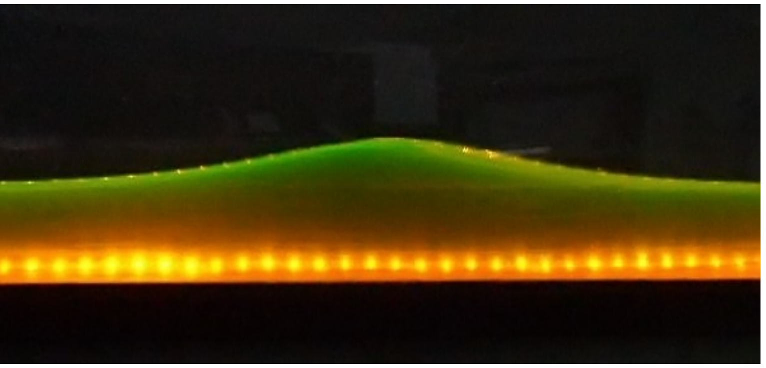
1510.00972 1601.08238 with A. Sciarappa and in progress with S. Gukov

Talk at conference CAQCD 2016
May 15th 2016

John Scott Russell and the solitary wave [1844]



….. I followed it [wave] on ***horseback***, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height.



Union Canal at Hermiston
Edinburgh, Schotland

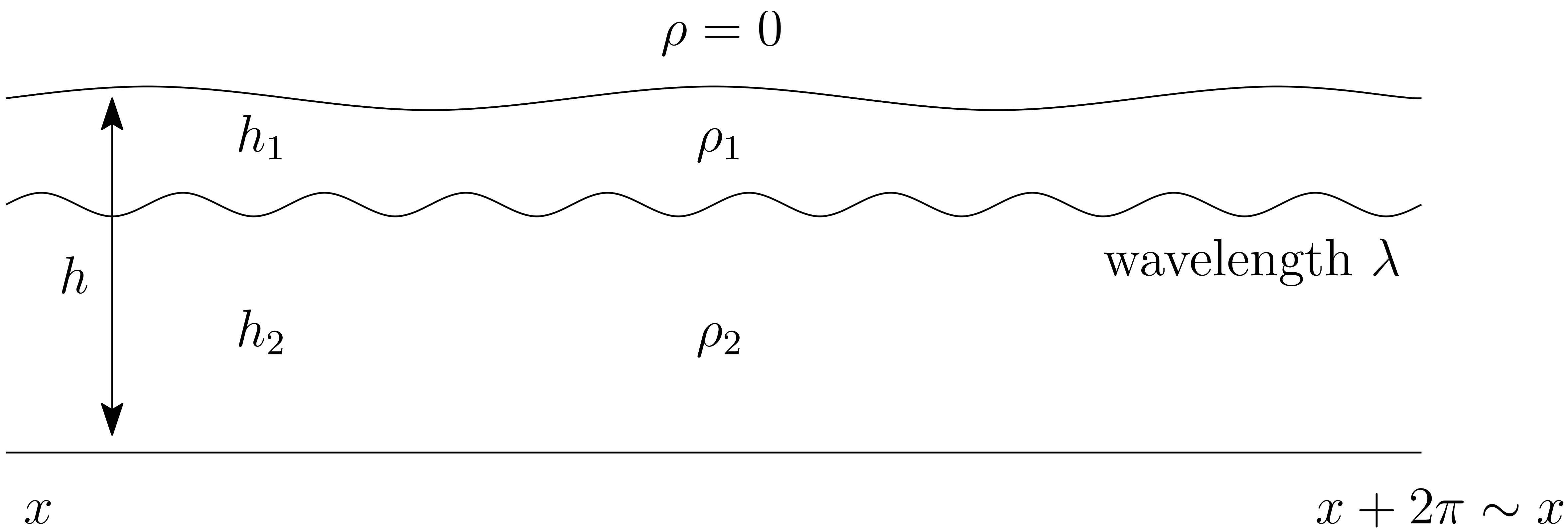
Russell had a *horse* and was able to follow the soliton to observe it and study it

In other words, if you don't have a *horse* - the soliton won't come and you won't learn anything

In this talk I shall describe how to ride a `*horse*' - supersymmetric gauge theories (or topological strings) to get some better understanding of physics of solitons in one-dimensional channel

Intermediate Long Wave model

Describes propagation of waves along the interface of two 1d fluids



- $h \ll \lambda$, long wave: Korteweg-de Vries (KdV) regime for $\delta \to 0$
- $h \gg \lambda$, short wave: Benjamin-Ono (BO) regime for $\delta \to \infty$
- $h \sim \lambda$, intermediate wave: Intermediate Long Wave (ILW) regime for $\delta \sim 1$

Integrable ILW equation

$$u_t = 2u_{xx} - i\beta\partial_x^2 u^H \qquad u^H = \frac{1}{2\pi} P.V. \int_0^{2\pi} \zeta(y-x;\widetilde{p})u(y)dy$$

Kernel — Weierstrass zeta function, simplifies in Korteweg de-Vries and Benjamin-Ono limits

KdV equation $\quad u_t = 2uu_x + \frac{\beta}{3}u_{xxx}$

Poisson bracket $\quad \{u(x), u(y)\} = \delta'(x-y)$

Rewrite ILW as evolution equation $\quad u_t = \{u, I_2\}$

Integrals of motion

$$I_1 = \int \left[\frac{1}{2}u^2\right] dx\,, \quad I_2 = \int \left[\frac{1}{3}u^3 + i\frac{\beta}{2}uu_x^H\right] dx\,,$$

$$\{I_l, I_m\} = 0$$

Soliton Solutions

n-Solitonic Ansatz

$$u(x,t) = \sum_{j=1}^{n} \left(\frac{i\beta}{x - a_j(t)} - \frac{i\beta}{x - a_j^*(t)} \right)$$

For non-periodic Benjamin-Ono we get equations of motion for Calogero-Moser-Sutherland (CMS) model

$$\ddot{a}_j = \sum_{l \neq j}^{n} \frac{2\beta^2}{(a_j - a_l)^3}$$

Poles describe propagation of solitons

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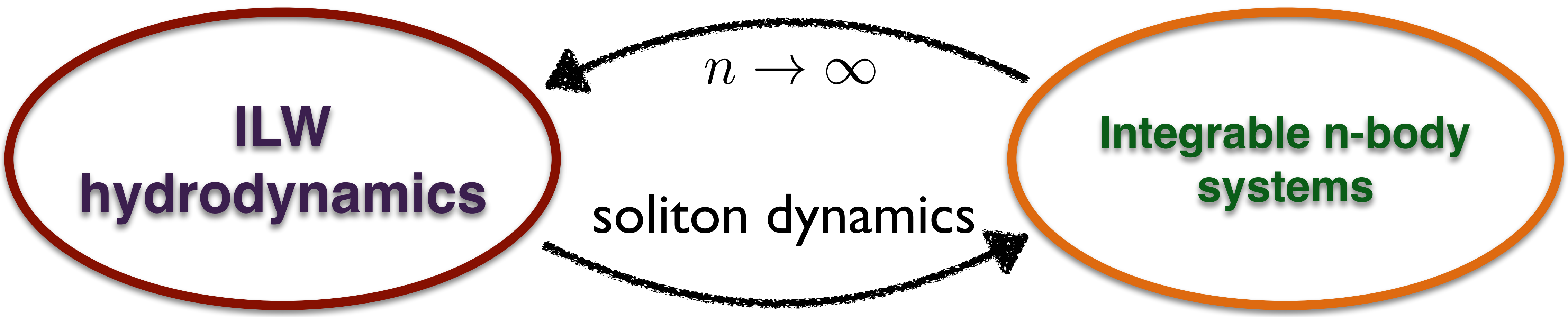
There exist a `hydro' version for most of known integrable many-body systems

Duality

Starting with an integrable many-body system we can take a thermodynamical limit by sending the number of its particles to infinity

EOM become hydrodynamical equations [Abanov Bettelheim Wiegmann]



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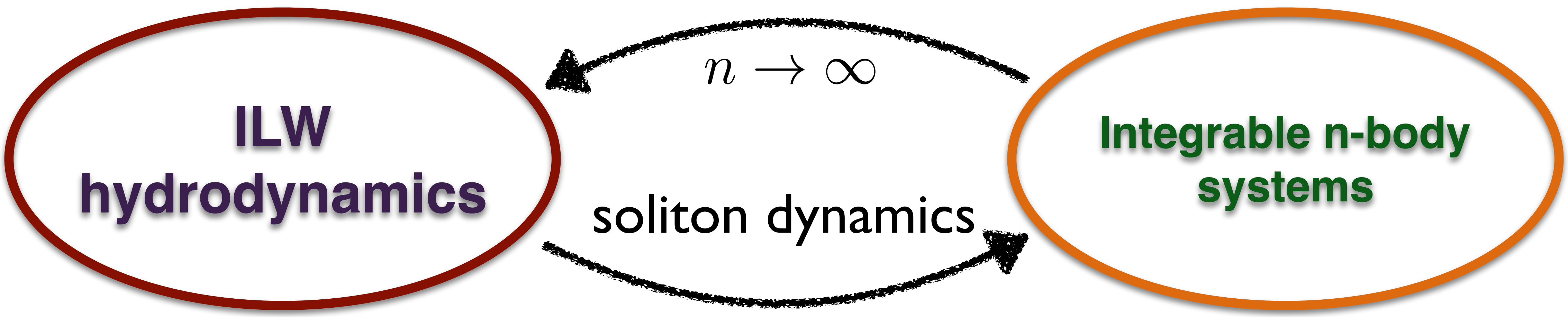

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Our task it to understand quantum spectrum!

Quantization

Expand in Fourier modes

$$u(x) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} a_m e^{imx}$$

Promote Poisson brackets to commutators

$$[a_m, a_{-n}] = \hbar m \delta_{m,n}$$

Quantum Hamiltonians need to be corrected to ensure commutativity

$$\widehat{I}_l \; =: I_l : + o(\hbar) \quad \text{such that} \quad [\widehat{I}_l, \widehat{I}_m] = 0$$

$$\widehat{I}_2 = \sum_{m>0} a_{-m} a_m$$

$$\widehat{I}_3 = i\frac{\beta+\beta^{-1}}{2}\sum_{m>0} m\frac{1+(-\widetilde{p})^m}{1-(-\widetilde{p})^m}\, a_{-m}a_m + \frac{1}{2}\sum_{m,n>0}(a_{-m-n}a_m a_n + a_{-m}a_{-n}a_{m+n})$$

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$$\widehat{I}_3 (c_1 \overline{a}_{-1}^3 + c_2 \overline{a}_{-2} \overline{a}_{-1} + c_3 \overline{a}_{-3}) |0\rangle = E_3 (c_1 \overline{a}_{-1}^3 + c_2 \overline{a}_{-2} \overline{a}_{-1} + c_3 \overline{a}_{-3}) |0\rangle$$

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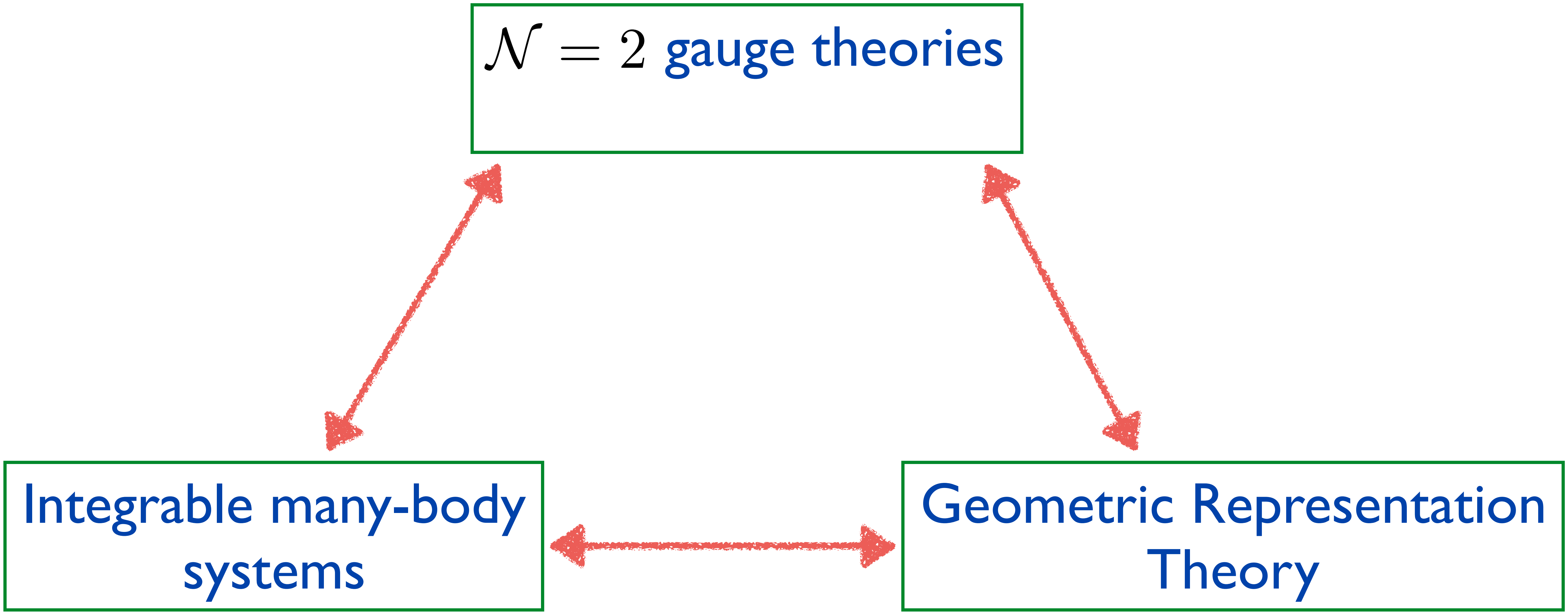
Finding quantum spectrum is hard- need more effective tools

Three-way approach

our `horses'



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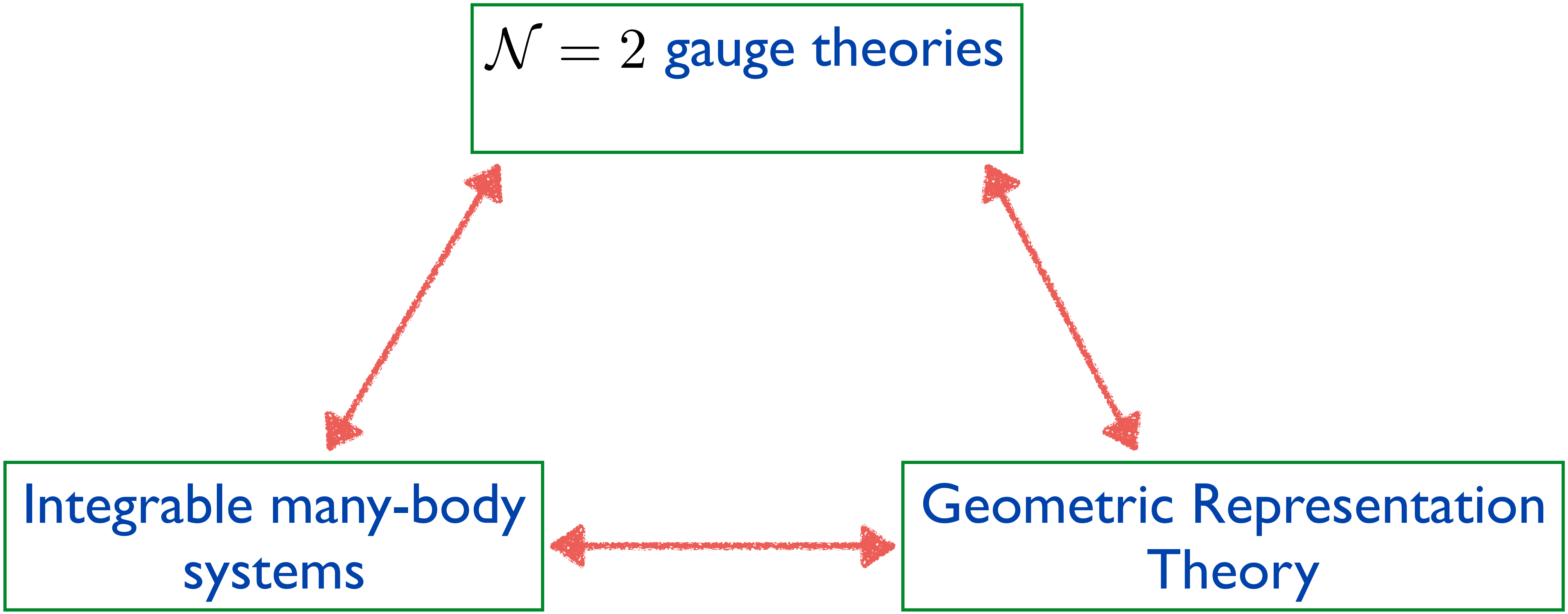

Large-n limits are manifest in each description!

N=2 Gauge Theories

We focus on N=2 gauge theories which have Seiberg-Witten description in IR

At the moment we have plethora of exact results for those theories thanks to Nekrasov's computation of instanton partition functions

Nekrasov's original woks has been greatly extended in to:
- various supergravity backgrounds (e.g. spheres)
- quiver gauge theories
- five and six-dimensional theories on $X_D = \mathbb{R}^4 \times \Sigma$
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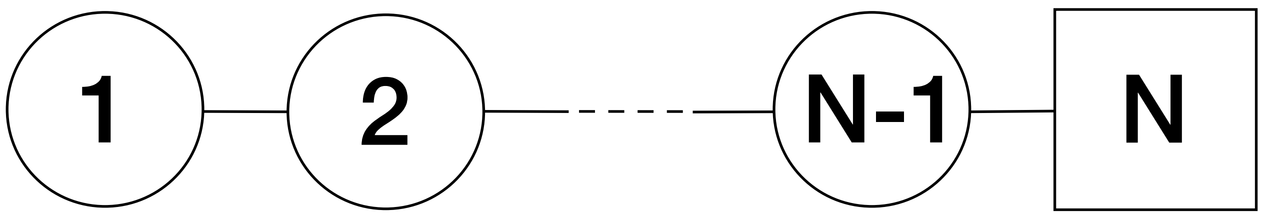
We shall study theories with adjoint matter on

$$X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma \qquad X_5 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1_\gamma$$

3d Theory

$\mathcal{N} = 2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma$ T[U(N)]



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Theory depends on twisted masses μ_i and FI parameters τ_i

and N=2* mass $t = e^m$

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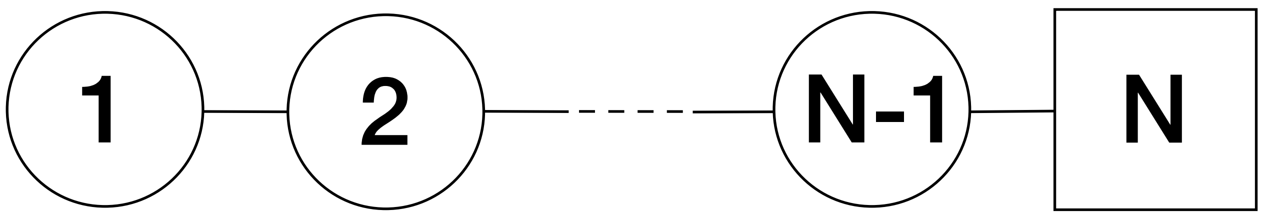

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$$\mathcal{B}\sim\ {}_2\phi_1\left(t,t\frac{\mu_1}{\mu_2},q\frac{\mu_1}{\mu_2};q;\frac{\tau_1}{\tau_2}\right)$$

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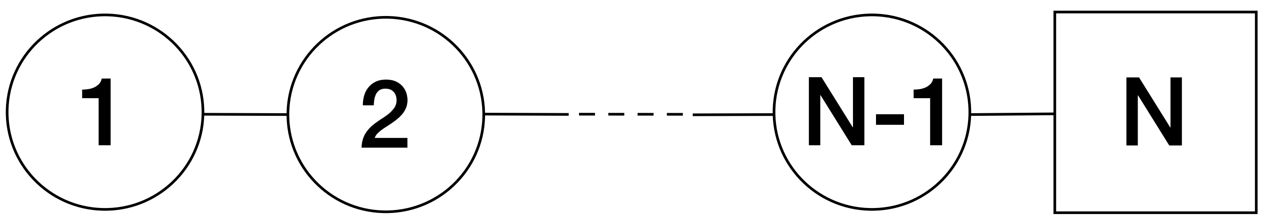

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$$\mathcal{B}\sim\ {}_2\phi_1\left(t,t\frac{\mu_1}{\mu_2},q\frac{\mu_1}{\mu_2};q;\frac{\tau_1}{\tau_2}\right)$$

is the eigenstate of the relativistic Calogero system!

$$D^{(1)}\mathcal{B}=(\mu_1+\mu_2)\mathcal{B} \qquad D^{(1)}\sim\sum_{i\neq j}\frac{t\tau_i-\tau_j}{\tau_i-\tau_j}e^{\hbar\partial_{\log\tau_i}}$$

Generic 3d quiver

For generic T[U(N)] quiver

$$D^{(k)}\mathcal{B} = \left\langle W_k^{U(n)} \right\rangle \mathcal{B}$$

In other words, the eigenvalue of tRS Hamiltonian is a VEV of background Wilson loop around the compact circle

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[Gaiotto Witten] [Bullimore Kim PK]
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We have just constructed a (complex) representation of the double affine Hecke algebra (DAHA)

[PK Gukov in prog]
[Cherednik]
[Oblomkov]

Elliptic Generalization

[Bullimore Kim PK]

3d theory describes trigonometric model, so we need a continuous parameter which interpolates between two regimes

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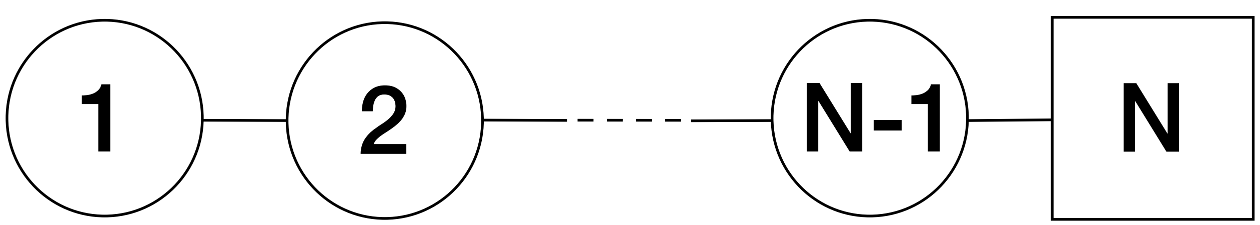
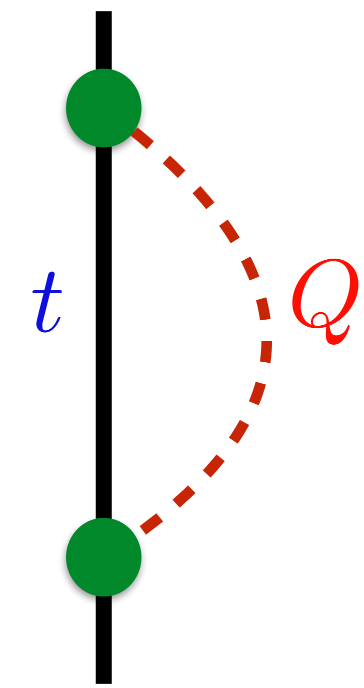
Gauging global symmetry of 3d theory
by the gauge group of the bulk 5d theory



5d U(N)
N=1*



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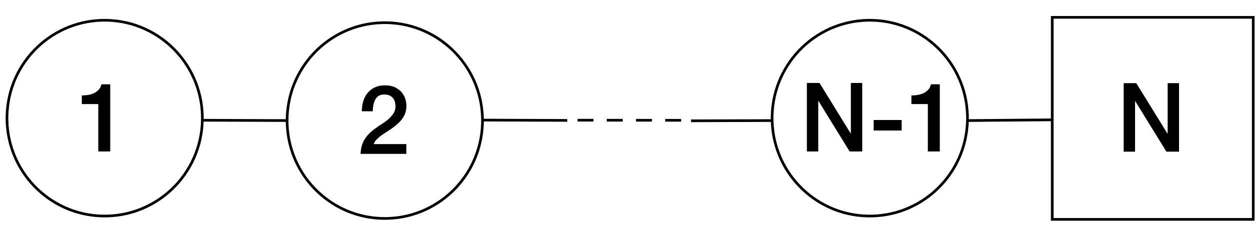

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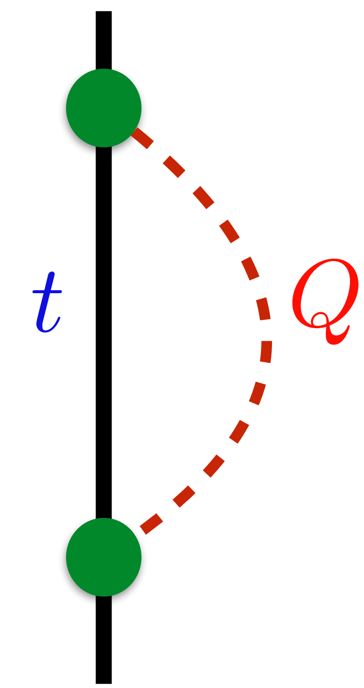

$$D^{(1)}_{p,q,t} \sim \sum \frac{\theta(t\frac{\tau_i}{\tau_j}|p)}{\theta(q\frac{\tau_i}{\tau_j}|p)} e^{\hbar\partial_{\log \tau_i}} \qquad D^{(k)}_{p,q,t}\mathcal{Z}^{5\mathrm{d}/3\mathrm{d}} = \left\langle W^{U(n)}_{\Lambda^k} \right\rangle \mathcal{Z}^{5\mathrm{d}/3\mathrm{d}}$$

[cf. resurgence stuff]

Gauge/Integrability duality

quantum eRS model	**5d/3d theory**
number of particles n	rank 3d flavor group / 5d gauge group
particle positions τ_j	3d Fayet-Iliopoulos parameters
interaction coupling t	3d $\mathcal{N}=2^*$ / 5d $\mathcal{N}=1^*$ deformation $e^{-i\gamma m}$
shift parameter q	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic deformation p	5d instanton parameter $Q=e^{-8\pi^2\gamma/g^2_{YM}}$
eigenvalues $E^{(\lambda;n)}_{tRS}$	$\langle W^{U(n)}_{\square}\rangle$ for 5d $U(n)$ in NS limit at fixed μ_a
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Now we study large-n behavior of the eigenvalues and the eigenfunctions

Mapping States

Consider partition λ of $k < n$ (assume p=0)

Specify $\mu_a = q^{\lambda_a} t^{n-a} \quad , \quad a = 1, \ldots, n$ for T[U(n)] theory

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Partition function series truncates to Macdonald polynomials!

$$D^{(1)}_{n,\vec{\tau}}(q,t) P_\lambda(\vec{\tau}; q,t) = E^{(\lambda;n)}_{tRS} P_\lambda(\vec{\tau}; q,t)$$

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E.g. k=2

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2} q, t^{1/2} q) = P_{\square\square}(\tau_1, \tau_2; q, t)$$

$$\mathcal{B}(\tau_1, \tau_2; t^{-1/2}, t^{-1/2} q^2) = P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\tau_1, \tau_2 | q, t)\,.$$

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Their exact form depends on n

$$P_{(2,0)}(\tau_1, \tau_2; q, t) = \tau_1 \tau_2 + \frac{1 - qt}{(1+q)(1-t)} (\tau_1^2 + \tau_2^2)$$

Change of Variables

However, after change of variables $$p_m = \sum_{l=1}^{n} \tau_l^m$$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

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Starting with Fock vacuum $|0\rangle$

Construct Hilbert space $a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $a_{-\lambda}|0\rangle = a_{-\lambda_1} \cdots a_{-\lambda_l}|0\rangle$

Free boson realization

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}$$

(more involved with p)

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Vortex series encodes all states! Now need to describe eigenvalues

U(1) Instantons [cf. Sasha's talk]

Mathematicians know this space already. They found similar structure on the moduli space of U(1) (non-commutative) instantons

[Nakjima]
[Schiffmann Vaserot]

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Instanton - KK monopole propagating along the compact circle

KK modes give different topological sectors

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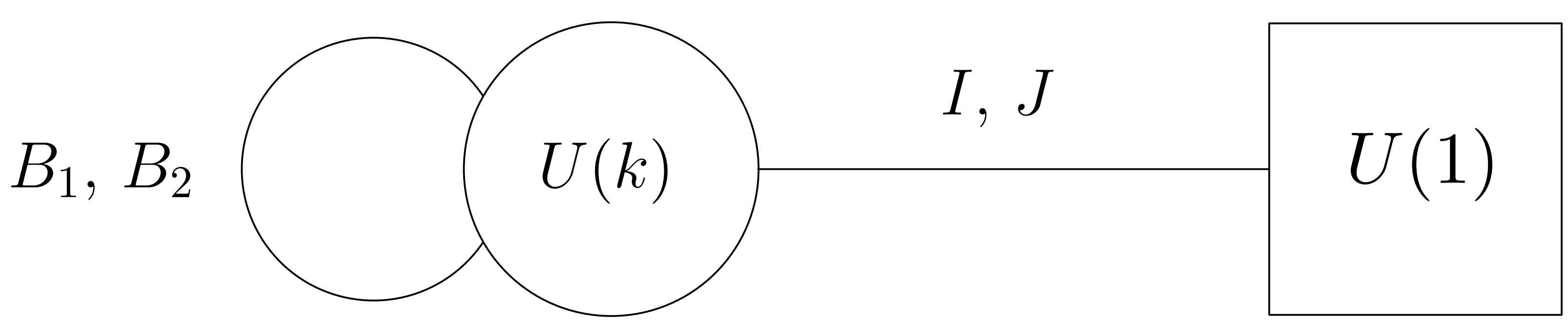
Higgs branch of the 3d N=4 ADHM quiver $\mathcal{M}_{k,1}$



superpotential $$W = \mathrm{Tr}_k \{\chi([B_1, B_2] + IJ)\}$$

ADHM quiver

Using supersymmetry we can effectively describe $\mathcal{M}_{k,1}$

We need to find the twisted chiral ring of the ADHM gauge theory

[Nekrasov Shatashvili]

$$(\sigma_s - 1)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\sigma_s - q\sigma_t)(\sigma_s - t^{-1}\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - qt^{-1}\sigma_t)} = \frac{\widetilde{p}}{\sqrt{qt^{-1}}}(1 - qt^{-1}\sigma_s)\prod_{\substack{t=1\\t\neq s}}^{k}\frac{(\sigma_s - q^{-1}\sigma_t)(\sigma_s - t\sigma_t)}{(\sigma_s - \sigma_t)(\sigma_s - q^{-1}t\sigma_t)}$$

where $\sigma_s = e^{i\gamma\Sigma_s},\ q = e^{i\gamma\epsilon_1},\ t = e^{-i\gamma\epsilon_2}$ $\widetilde{p} = e^{-2\pi\xi}$ FI coupling

The Duality

[PK Sciarappa]

Eigenvalues at large-n

$$\left\langle W_{\square}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

Wilson line VEV becomes an equivariant Chern character for $\mathcal{M}_{k,1}$

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Spaces of states is acted upon elliptic Heisenberg algebra

[Feigin et.al.]

$$[\lambda_m, \lambda_n] = -\frac{1}{m} \frac{(1-q^m)(1-t^{-m})(1-(pq^{-1}t)^m)}{1-p^m} \delta_{m+n,0}$$

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elliptic RS	**3d ADHM theory**	**3d/5d coupled theory,** $n \to \infty$
coupling t	twisted mass $e^{-i\gamma\epsilon_2}$	5d $\mathcal{N}=1^*$ mass deformation $e^{-i\gamma m}$
quantum shift q	twisted mass $e^{i\gamma\epsilon_1}$	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic parameter p	FI parameter $\widetilde{p} = -p/\sqrt{qt^{-1}}$	5d instanton parameter Q
eigenstates λ	ADHM Coulomb vacua	5d Coulomb branch parameters
eigenvalues	$\langle \mathrm{Tr}\, \sigma \rangle$	$\langle W_{\square}^{U(\infty)} \rangle$ in NS limit $\widetilde{\epsilon}_2 \to 0$

Mathematical Results

Spherical Hall algebra as large-n limit of DAHA

Trigonometric RS to BO

$$\lim_{n\to\infty} K_T(T^*\mathbb{F}_n) \simeq K^{\mathrm{cl}}_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$

$$\widetilde{\mathcal{M}_1} = \bigoplus_{k=0}^{\infty} \mathcal{M}_{1,k}$$ Instanton moduli space

No mathematical object is known to describe spectrum of elliptic RS

Our proposal

$$\mathcal{E}^Q_T(T^*\mathbb{F}_n) := \mathbb{C}[p_i^{\pm 1}, \tau_i^{\pm 1}, Q, t, \mu_i^{\pm 1}]/\mathcal{I}_{\mathrm{eRS}}$$

Large-n limit

$$\lim_{n\to\infty} \mathcal{E}^Q_T(T^*\mathbb{F}_n) \simeq K_{q,t}\left(\widetilde{\mathcal{M}_1}\right)$$

Open questions

Quantum KdV

Knot homology

What happens for 6d theories at large n?   Holography?

Physics construction for elliptic cohomology

Thanks to the organizers for **fun** and **productive** conference !!!