DAHA talk KSU 10/04/18 w/ Gukov, Nawata, Saberi

- DAHA & Reps

Geometric Representation category of DAHA: in terms of Hitchin moduli space of once-punctured torus

- Geometry of $\mathcal{M}_H$ + Categorification
- Brane quantization

Category of A-branes on $\mathcal{M}_H$ = Representation category of (spherical) DAHA

A-Brane$(\mathcal{M}_H)$ = DAHA Rep$(\mathcal{A})$

checked it in rank 1

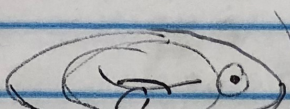
$\cup$

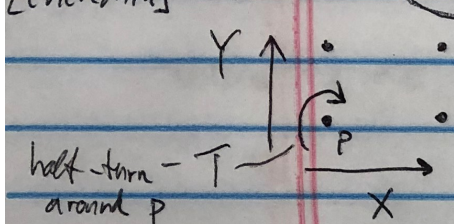
Fuk $(\mathcal{M}_H(\underbrace{T^2 \setminus pt}_{C}, SU(2)))$

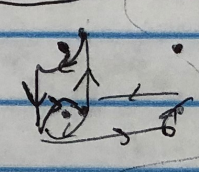
subcategory

① DAHA of $A_1$

[Cherednik] $sl_2$





Orbifold fundamental group of $C/\mathbb{Z}_2$

$$\pi_1^{orb}(C/\mathbb{Z}_2) = \frac{\langle T, X, Y\rangle}{\{TXT = X^{-1},\ TY^{-1}T = Y,\ Y^{-1}X^{-1}YXT^2 = 1\}}$$

- Deforming to $Y^{-1}X^{-1}YXT^2 = q^{-1}$ - double affine braid group $\hat{\hat{B}}_r$

$$\hat{\hat{H}}[sl_2]_{q,t} = \frac{\mathbb{C}[q,t]\otimes\mathbb{C}[T, X^{\pm1}, Y^{\pm1}]}{\left\{\begin{array}{l} TXT = X^{-1},\ Y^{-1}X^{-1}YXT^2 = q^{-1} \\ T^{-1}YT = Y,\ (T-t)(T+t^{-1}) = 0 \end{array}\right\}}$$

- Deforming by adding quadratic Hecke relation w/ another parameter $t$ $(T-t)(T+t^{-1}) =$

Alternatively:
- Hecke - 1 parameter deformation of group alg. of Weyl group
  for $sl_2$ braid relation is trivial
  quadratic relation $(T-t)(T-t^{-1}) = 0$
- Affine Hecke: $\langle X^{\pm1}, T\rangle / \{TXT = X^{-1}\}$
- DAHA - 2 copies of AHA connected together via $Y^{-1}X^{-1}YXT^2 = q$

**1.1** ~~Sph~~ Spherical DAHA ~~$\mathcal{A}$~~ $\mathcal{A}$ $\mathcal{O}\!\iota$ is generated by symmetric poly of $X, Y$ $^{\pm1}$.

$e = \frac{T + t^{-1}}{t + t^{-1}}$, s.t. $e^2 = e$ $\quad \mathcal{O}\!\iota = e\hat{\hat{H}}e$

$\mathcal{O}\!\iota$ is generated by

$x = X + X^{-1}$

$y = Y + Y^{-1}$

$z = q^{-\frac{1}{2}}\frac{X}{Y} + q^{\frac{1}{2}}\frac{Y}{X} = \frac{[x,y]_q}{(q^{-1}-q)}$

$[x,y]_q = q^{-\frac{1}{2}}xy - q^{\frac{1}{2}}yx$

relations $[x,y]_q = (q^{-1}-q)z$

$[y,z]_q = (q^{-1}-q)x$

$[z,x]_q = (q^{-1}-q)y$

$+\ \Omega = q^{-1}x^2 + qy^2 + q^{-1}z^2 - q^{-1/2}xyz - (q^{-1/2}t - q^{1/2}t^{-1})^2 + (q^{1/2} + q^{-1/2})^2 = 0$

'quadratic Casimir'

go to last page!

② Geometry of $\mathcal{M}_H$

$S_p$ DAHA$_{sl_n}$ – deformation quantization of ~~sl~~ $sl_n$ character variety of once-punctured torus [Oblomkov]

Non-abelian Hodge correspondence

$$\mathcal{M}_{flat}(T^2\setminus pt, SL(2;\mathbb{C})) \cong \mathcal{M}_H(\underbrace{T^2\setminus pt}_{C}; SU(2))$$

Ramified Higgs bundle $(E,\varphi)$

flat connection

$E$ - hol bundle over $C$

$A + \phi + \phi^*$

+ harm. metric

Hitchin equations

$\varphi \in H^0(K_C \otimes ad(E) \otimes \mathcal{O}(p))$

$$\begin{cases} F - [\varphi, \bar\varphi] = 0 \\ \bar D_A \varphi = 0 \end{cases}$$

line bundle ~~whose sections~~ are holom. away from p

due to ramification

$A = \alpha_p \, d\theta$

$\varphi = \frac{1}{2}(\beta_p + i\gamma_p)\frac{dz}{z}$

Hyper kähler space $(I, J, K)$

induced by $C$

induced by character variety

"$IJ$"

$x^2 + y^2 + z^2 - xyz = 2 + t^2 - t^{-2}$

$x, y, z$ are holomorphic w.r.t. $J$

$$\Omega_J = \omega_K + i\omega_I = \frac{dx \wedge dy}{2x - 2y}$$

~~Assume no ramification $(t=1)$~~

$\mathcal{M}_H$:

$A \sim \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$

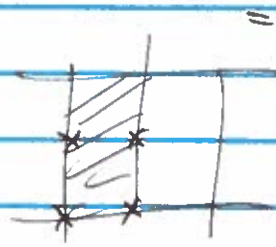
$B \sim \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}$

$\mathcal{M}_H(T^2, SL(2;\mathbb{C}))$

$$\mathcal{M}_H = \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2}$$ Weyl of $SL_2$

real locus $\mathcal{M}_H(T^2, SU(2))$

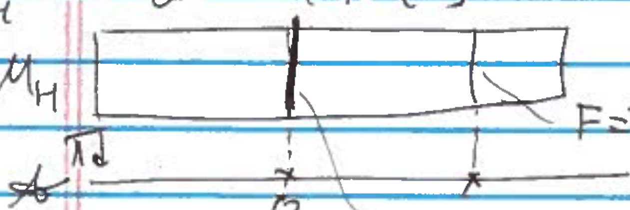
$$= \frac{S^1 \times S^1}{\mathbb{Z}_2} =: Bun_G$$



– 'pillow case'

Use Hitchin fibration

$\mathcal{M}_H \xrightarrow{\pi} \mathcal{B} = \{Tr\,\varphi^2\}$





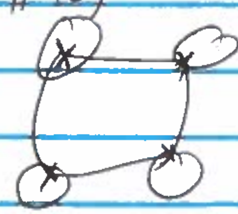
$F = T^2$ – generic fiber

Kodaira $I_0^*$

$\widehat{D}_4$

$\pi^{-1}(0) = \mathcal{N}$ – global nilpotent cone

w/o ramification $\mathcal{N}$ =



– $\mathbb{P}^1$ arising from ramification

| | $I$ | $J$ | $K$ |
|---|---|---|---|
| fibers are: | $B$ | $A$ | $A$ |

$(B, A, A)$ -brane

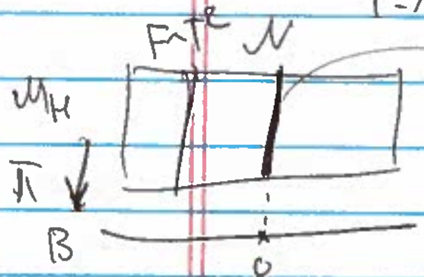
hol

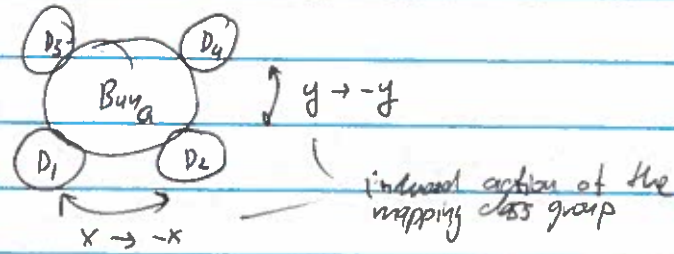
Lagrangian

This will change for $\mathcal{N}$ once we turn on $t$

( $I$ varies holomorphically in $\beta_p + i\gamma_p$ and independent on $\alpha_p$ )

$$\mathcal{N} = \mathrm{Bun}_G \cup \bigcup_{i=1}^{4} D_i \in H_2(\mathcal{M}_H(C_p, G), \mathbb{Z})$$




Intersection form of cycles is Cartan matrix of $\widehat{D}_4$. The matrix has one null-vector in basis $([\mathrm{Bun}_G], [D_1], \dots, [D_4])$, which we need to identify w/ a generic fiber $F = 2[\mathrm{Bun}_G] + \sum_{i=1}^{4} [D_i]$

Cohomology classes of Kähler forms $\omega_I, \omega_J, \omega_K$ in terms of ramification data

$$\alpha_p = \int_{D_i} \frac{\omega_I}{2\pi}, \quad \beta_p = \int_{D_i} \frac{\omega_J}{2\pi}, \quad \gamma_p = \int_{D_i} \frac{\omega_K}{2\pi}$$

$$(\cancel{2\alpha_p}\ 2\alpha_p) \sim \alpha_p \in \mathbb{Z}$$

$\mathcal{N}$ stops being holomorphic in $I$ once $\beta_p + i\gamma_p \neq 0$

Generic fiber is holomorphic in $I$

$$\int_F \frac{\omega_I}{2\pi} = 1, \quad \int_F \omega_J = \int_F \omega_K = 0$$

Cohomology classes dual to $\mathrm{Bun}_G$

$$\int_{\mathrm{Bun}_G} \frac{\omega_I}{2\pi} = \frac{1}{2} - 2\alpha_p$$

$$\int_{\mathrm{Bun}_G} \frac{\omega_J}{2\pi} = -2\beta_p$$

$$\int_{\mathrm{Bun}_G} \frac{\omega_K}{2\pi} = -2\gamma$$

(3) Brane quantization [Gukov–Witten], [Kapustin, Gaiotto]

2d $\Sigma$ model $\quad \Sigma \longrightarrow \mathcal{M}_{flat}(T^2 \setminus p^?, SL(2;\mathbb{C}))$

A-model on $(\mathcal{X}, \omega_K)$

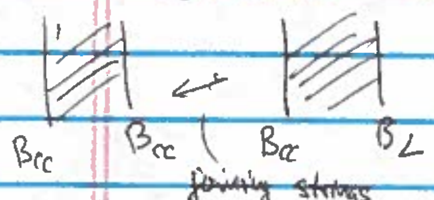
A-brane in $Fuk(\mathcal{X})$ $\quad \begin{matrix} E \\ \downarrow \\ \mathcal{L} \end{matrix} \quad \omega_K|_{\mathcal{L}} = 0 \quad c_1(E) = F_L$

[40] coisotropic submanifold defined by ~~zero locus of~~ Poisson commuting functions w/ $\omega_K$

~~canonical~~ coisotropic $B_{cc} = \begin{matrix} \mathcal{L} \\ \downarrow \\ \mathcal{X} \end{matrix}$ deformed line bundle

w/ curvature $c_1(\mathcal{L}) = F = \omega_J, \quad \omega_K^{-1} F = J$ $\quad$ will give quantization + B field

$\mathrm{Hom}(B_{cc}, B_{cc}) = A \ (SH, \mathrm{DAHA})$




$\mathrm{Hom}(B_{cc}, B_{cc}) = A$

$\mathrm{Hom}(B_{cc}, B_L) = V$ - rep. of $A$

$Fuk(\mathcal{X}, \omega_K) \longleftrightarrow \mathrm{Rep}(\mathrm{DAHA})$

$$\Omega = -\frac{1}{\hbar} \frac{dz \wedge dy}{2z - xy}, \quad q = e^{2\pi i \hbar}$$

$\mathrm{Re}(\Omega) = F, \quad \mathrm{Im}(\Omega) = \omega_K$

$F, \mathcal{N}$ are $Ob(Fuk(\mathcal{X}))$

GHRR

$$\dim V = \int ch(\mathcal{L} \otimes E) \wedge Td(B_{cc}) = \dim H^0(L, B_{cc} \otimes B_L \otimes K_L^{1/2})$$

comes from B-field

Apply GHRR to branes and check the formula.

$\hbar = |\hbar| e^{i\theta}$

finite-dim modules $\updownarrow$ compact Lagrangians

$\int \frac{F}{2\pi}$ ~~$= 2m$~~

$$m + \int_F \frac{B}{2\pi} = -\frac{\cos\theta}{|\hbar|}$$

(4) Matching. Compute

- $B_F$ $\quad \int_{\mathbb{F}} e^{\frac{1}{\hbar}\omega_I^2} \wedge Td(\mathbb{F}) = \frac{1}{|\hbar|} \xleftrightarrow{\text{WKB quantization}}$ $\hbar = 2N$, $\underline{\mathcal{D}_N \oplus \oint(\mathcal{D}_N)}$

- $B_{Bun_G}$ $\quad \int_{Bun_G} e^{\frac{1}{\hbar}\omega_I} \wedge Td(Bun_G) = \frac{1}{\hbar} - 2c + 1 \longleftrightarrow \mathcal{D}_{N-2K}$

$c = \frac{2k-1}{2}$

$$0 \to \mathcal{D}_{N-2k} \to \mathcal{D}_N \to \mathcal{D}_k^+ \oplus \mathcal{D}_k^- \to 0$$

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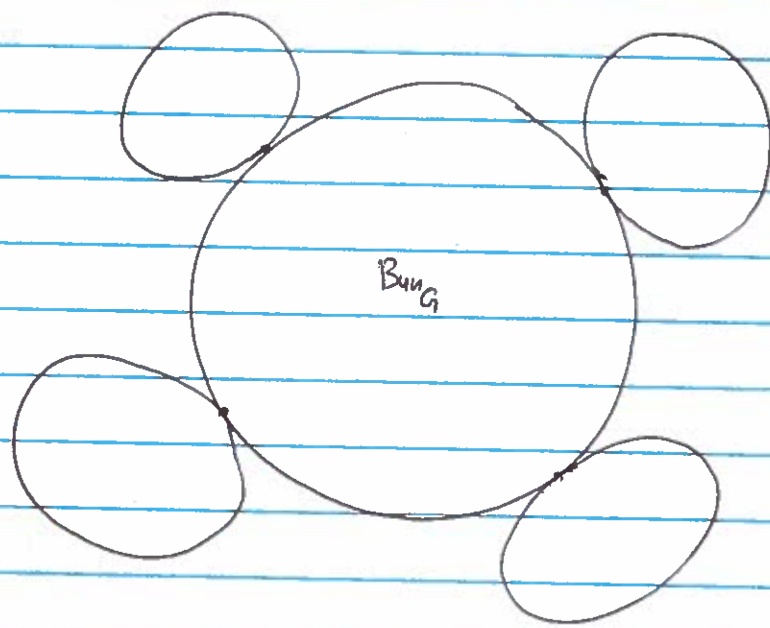


## 1.2 Highest weight vectors

study highest weight reps of $\mathfrak{A}$

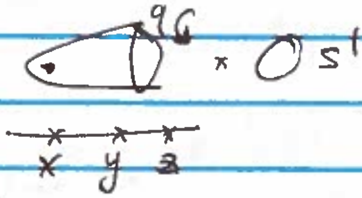
$$y = Y + Y^{-1}, \qquad YZ = aZ \qquad a \in \mathbb{C}^\times$$

$$yZ = (Y+Y^{-1})Z = (a+a^{-1})Z$$

$X$ acts by multiplication, $y$ – $q$-shift operator

$$PX = qXP, \qquad y = \frac{tX - t^{-1}X^{-1}}{X - X^{-1}}P + \frac{t^{-1}X - tX^{-1}}{X - X^{-1}}P^{-1}$$ – Macdonald (Ruijsenaars–Schneider operator)

( DAHA arises as K-theory of line operators in $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^3 \times S^1$, $\mathbb{R}^3 = \mathbb{R}^2_q \times \mathbb{R}$ )




Find $Z \in \mathbb{C}[q,t] \otimes \mathbb{C}[[X]]$

$$yZ = (a+a^{-1})Z \Rightarrow Z^{(1)} = {}_2\phi_1(t^2, t^2a^2; qa^2; q; qt^2X^2)$$

$$Z^{(2)} = Z^{(1)}(a \to a^{-1})$$ – Weyl reflection

For generic values of $a$ we have an infinite-dim representation

However, $a^2 = q^{-2\ell}t^{-2}$, $\ell \in \mathbb{Z}_+$

$$Z^{(1)}\Big|_{(sym)} = P_\ell(X,q,t)$$ – Macdonald polynomial of 1 variable indexed by $\ell$

$$P_0 = 1$$

$$P_1 = X + \frac{1}{X} = x$$

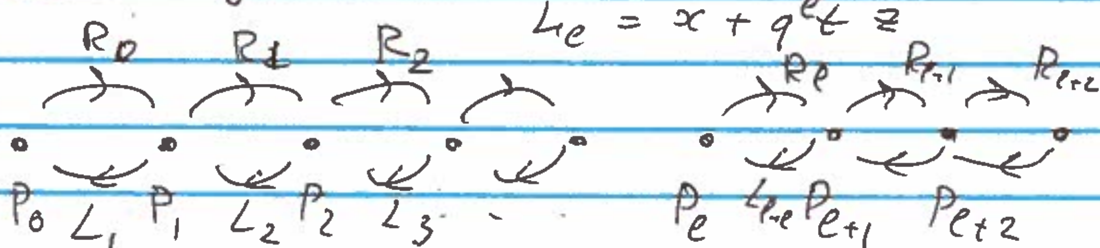
$$P_2 = X^2 + \frac{(1+q^2)(1-t^2)}{1-q^2t^2} + \frac{1}{X^2} = x^2 - \frac{(1-q^2)(1+t^2)}{1-q^2t^2}$$

## 1.3. Polynomial representation

$$\mathrm{pol}: \mathfrak{A} \to \mathrm{End}(\mathbb{C}[q,t] \otimes \mathbb{C}[X^{\pm 1}]^{\mathbb{Z}_2})$$

Raising & lowering operators: $R_\ell = x + q^{-\ell}t^{-1}z$

$L_\ell = x + q^{\ell}t\, z$




$$R_\ell P_\ell = r_\ell P_{\ell+1} \qquad r_\ell = 1 - q^{-2\ell}t^{-2}$$

$$L_\ell P_\ell = \ell_\ell P_{\ell-1} \qquad \ell_\ell = \frac{(1-q^{2\ell})(1-q^{2(\ell-1)}t^4)}{1-q^{2(\ell-1)}t^2}$$

[Oblomkov] showed that $\mathrm{Rep}(\mathfrak{A}) \cong \mathrm{Rep}(\hat{A}_1)$

$q = e^{2\pi i \hbar}$

$\hbar$ – $\Omega$ – background parameter

$x = \mathrm{tr}\, A$
$y = \mathrm{tr}\, B$
$z = \mathrm{Tr}\, AB$

$$\left( \pi_1(C_p) = \frac{\langle A, B, C\rangle}{ABA^{-1}B^{-1} = C} \right)$$

We can show that $\mathcal{M}_{flat} = \dfrac{\langle A, B, C\rangle}{(ABA^{-1}B^{-1} = C)} \Big/ SL(2;\mathbb{C})$

$$\mathcal{M}_{flat} = \{ (\underbrace{x}_{\text{Wilson}}, \underbrace{y}_{\text{t'Hooft}}, \underbrace{z}_{\text{dyonic}}) \in (\mathbb{C}^*)^3 \mid x^2 + y^2 + z^2 - xyz - 2 - \underbrace{\mathrm{Tr}\, C}_{t^2 + t^{-2}} = 0 \}$$

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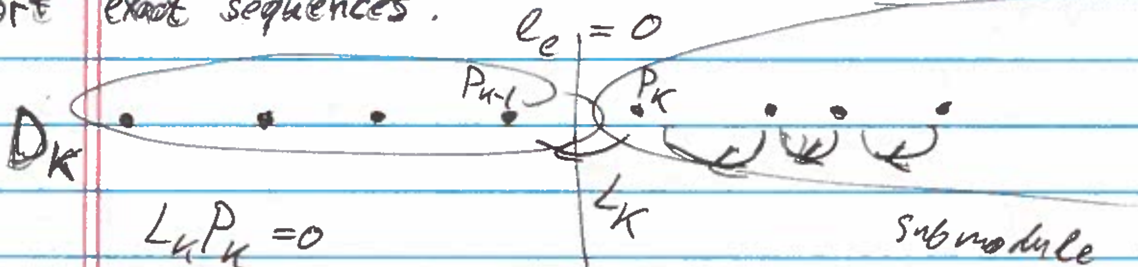
Short exact sequences:




$L_K P_K = 0$

submodule $S_K$

$$1 \to S_K \hookrightarrow V_K \twoheadrightarrow D_K \to 1$$

finite modules. We will classify them and count their dimension

Miura oper for each oper
dropping the condition that seed stay.