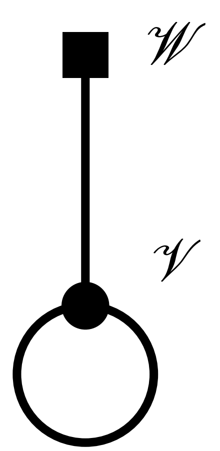
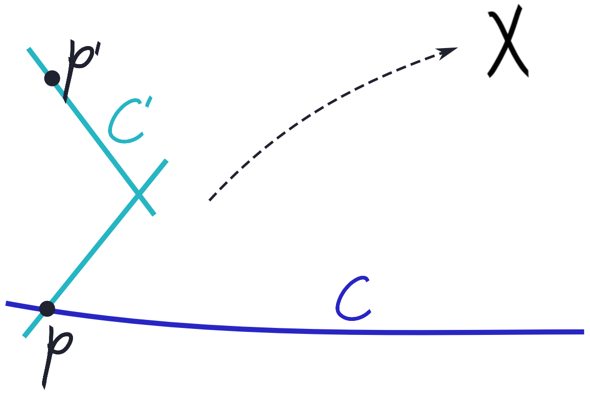
Quantum K-theory of Quiver Varieties and Many-Body Systems

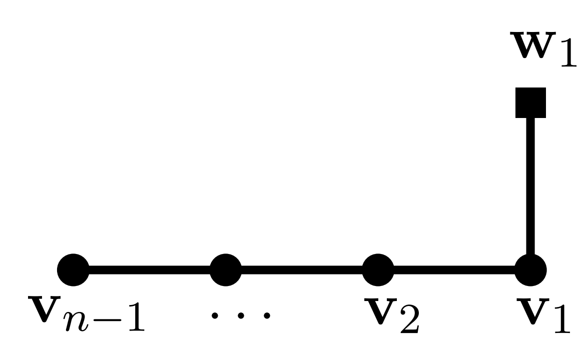
Peter Koroteev



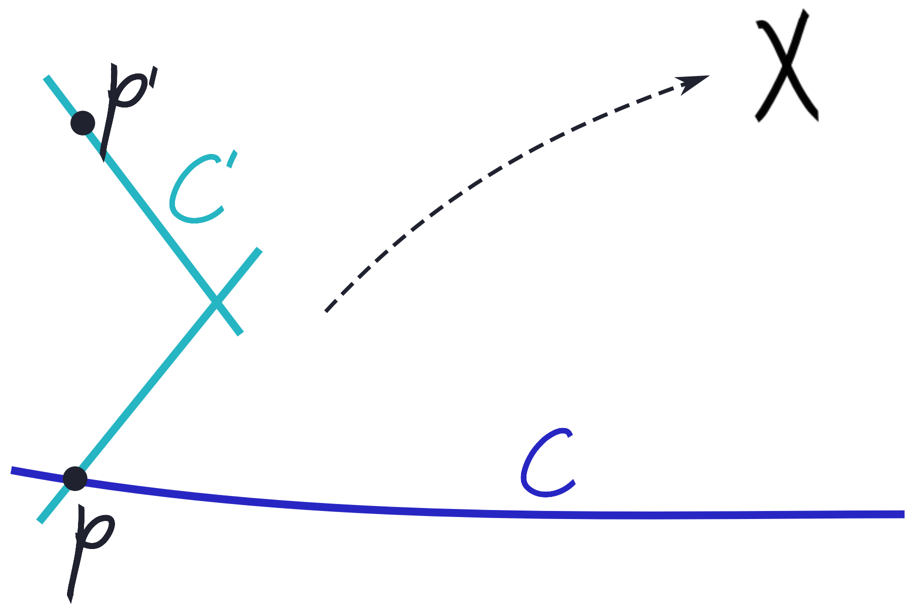


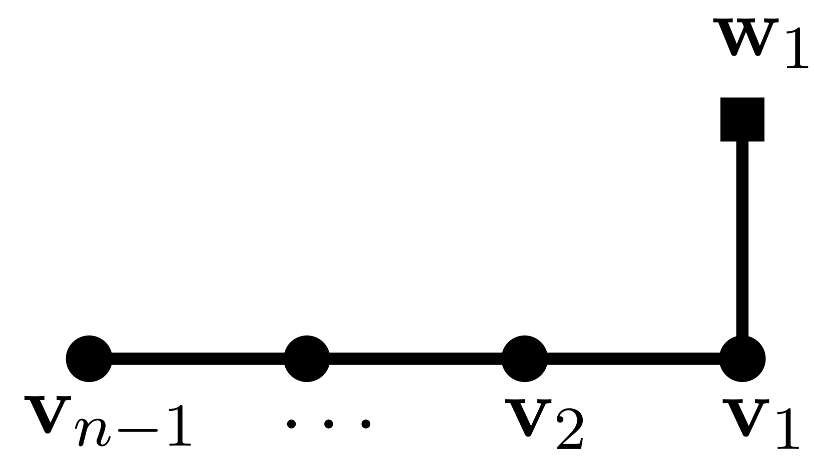
Based on 1705.10419 1802.04463 1805.00986

Talk at KIAS Geometry seminar
Seoul, Korea June 5th 2019

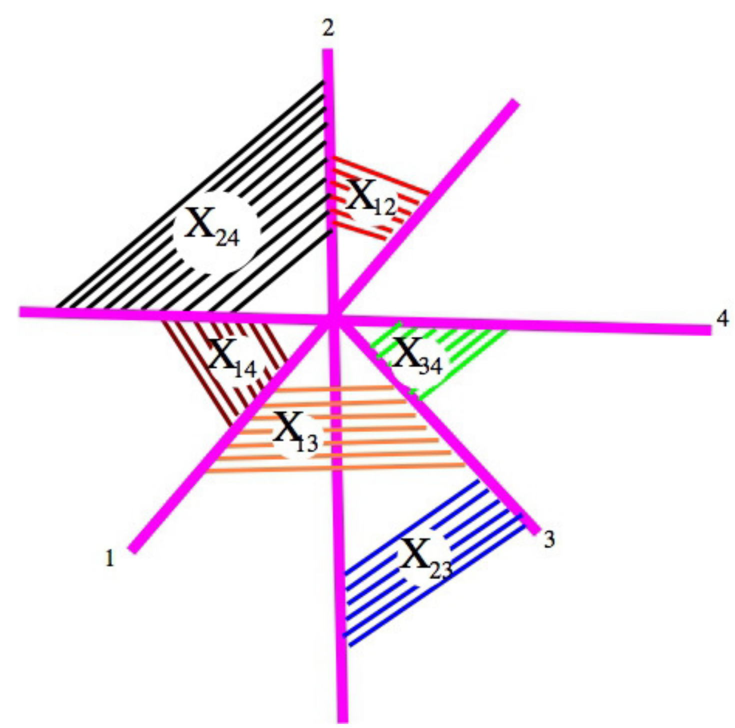


Based on new ideas, collaborations and discussions with





Aganagic
Okounkov
Zeitlin
Smirnov
Pushkar
Givental



Nekrasov
Pestun
Kimura
Sciarappa

Nekrasov

Costello, Gaiotto, Soibelman, Gukov, Nawata

Motivation

String theory have been suggesting for a long time that there is a strong connection between **geometry** and **integrability**

Study of ***Gromov-Witten*** invariants was also largely influenced by progress in string theory. For a symplectic manifold $\mathcal{X}$ GW invariants appear in the expansion of quantum multiplication in ***quantum cohomology*** ring of $\mathcal{X}$.

A particular attention is given to genus zero GW invariants.

In this talk we shall study **equivariant quantum K-theory** of large family of symplectic varieties and its connection to **integrable systems**

Physics Motivation

To see why integrability is relevant one considers supersymmetric sigma model from the base curve (P1 in our case) into $\mathcal{X}$

Witten demonstrated that relevant class of supersymmetric sigma models can be rewritten as supersymmetric gauge theories (**(2,2) GLSMs**) in two dimensions whose field content is related to geometry of $\mathcal{X}$. Sigma models thus describe infrared dynamics of GLSMs.

Nekrasov and Shatashvili showed how to obtain integrable systems from such GLSMs. It was conjectured that SUSY vacua of **2d** theories compute **quantum cohomology** ring of $\mathcal{X}$, while **3d** theories on $\mathbb{R}^2 \times S^1$ describe **quantum K-theory.**

Key References

Quantum integrability

[Bethe 1929] Solution of XXX Heisenberg chain
[Baxter, Young 60' 70'] Yang-Baxter equation
[Faddeev et al] Quantum inversed scattering
[Drinfeld, Jimbo] Yangian, Quantum Groups

Quantum geometry

[Givental Kim '95] Quantum cohomology of flag varieties
[Givental Lee 2001] Quantum K-theory of flag varieties
[Okounkov] Quantum geometry of symplectic resolutions

Geometric Representation Theory

[Braverman, Maulik, Okounkov 2011] Quantum cohomology of Springer resolution
[Maulik Okounkov 2012] Quantum groups and quantum cohomology
[Okounkov 2015] Lectures on K theoretic computations

Physics

[Witten 1993] The Verlinde algebra and the cohomology of the Grassmannian
[Nekrasov Shatashvili 2009] Supersymmetric vacua and Bethe Ansatz;
Quantum integrability and supersymmetric vacua

Quantum groups

Let $\mathfrak{g}$ Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}(t)$ loop algebra (Laurent poly valued in g)

Evaluation modules form a tensor category of $\hat{\mathfrak{g}}$

$$V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

V_i are representations of $\mathfrak{g}$

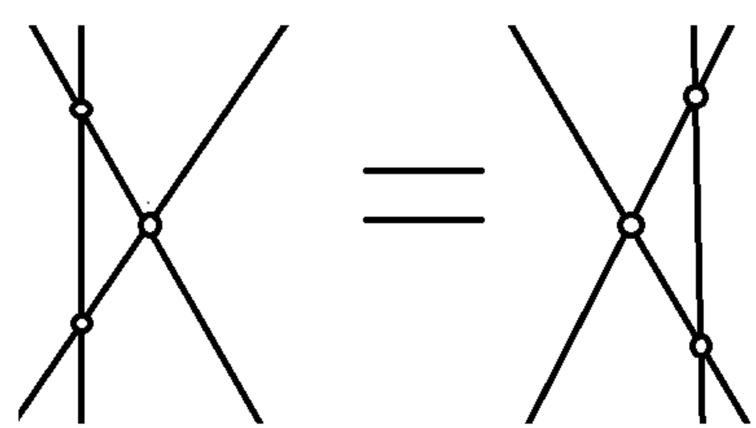
a_i are special values of spectral parameter t

Quantum group is a noncommutative deformation $U_\hbar(\hat{\mathfrak{g}})$

with a nontrivial intertwiner — R-matrix

$$R_{V_1,V_2}(a_1/a_2) : V_1(a_1) \otimes V_2(a_2) \to V_2(a_2) \otimes V_1(a_1)$$

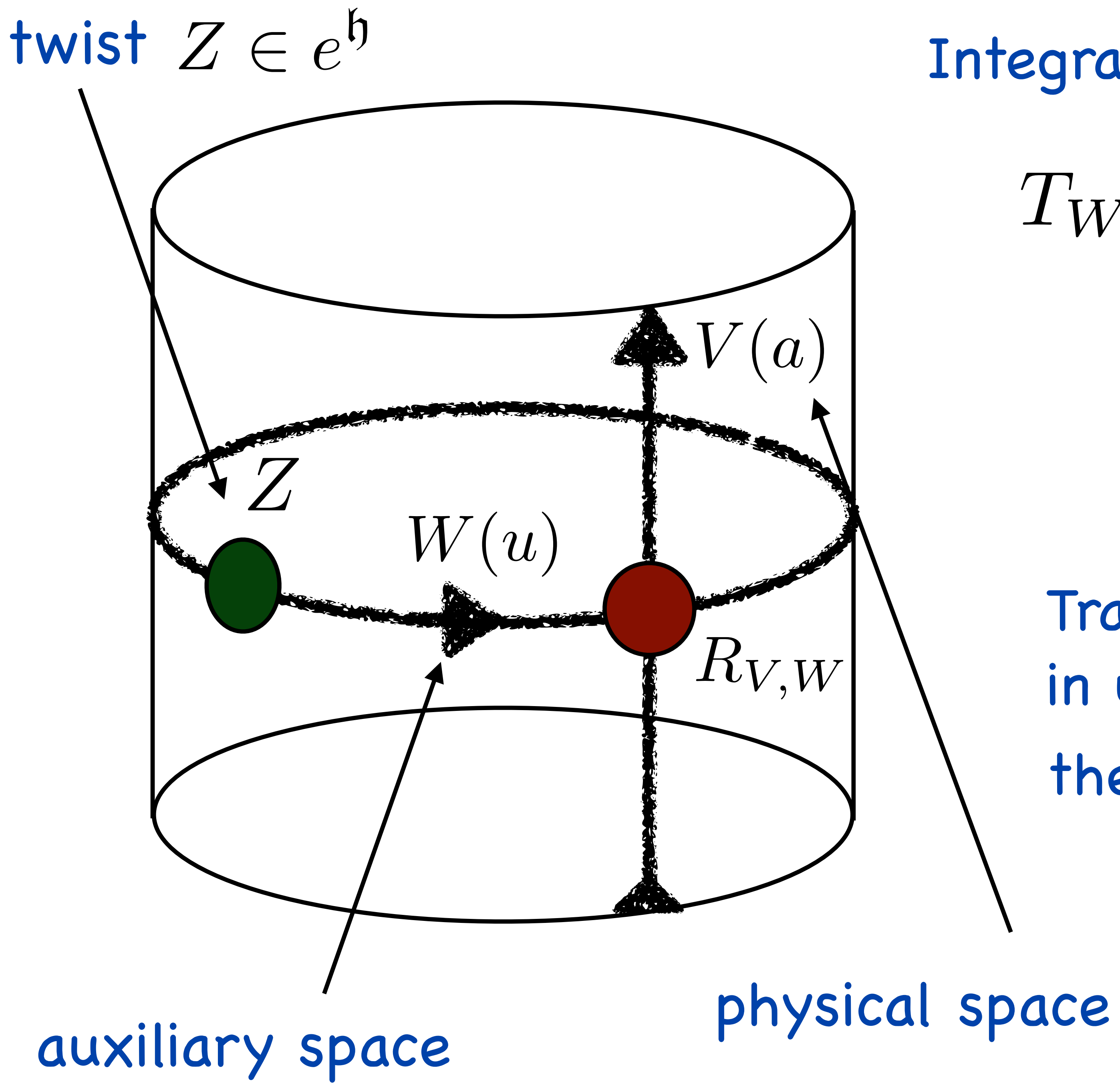
satisfying Yang-Baxter equation



Quantum Integrability

[Faddeev Reshetikhin Tachtajan]

The intertwiner represents an interaction vertex in integrable models. The quantum group is generated by matrix elements of R



Integrability comes from transfer matrix

$$T_W(u) = \mathrm{Tr}_{W(u)}\left((Z \otimes 1) T_{V,W}\right)$$

$$[T_W(u), T_W(u')] = 0$$

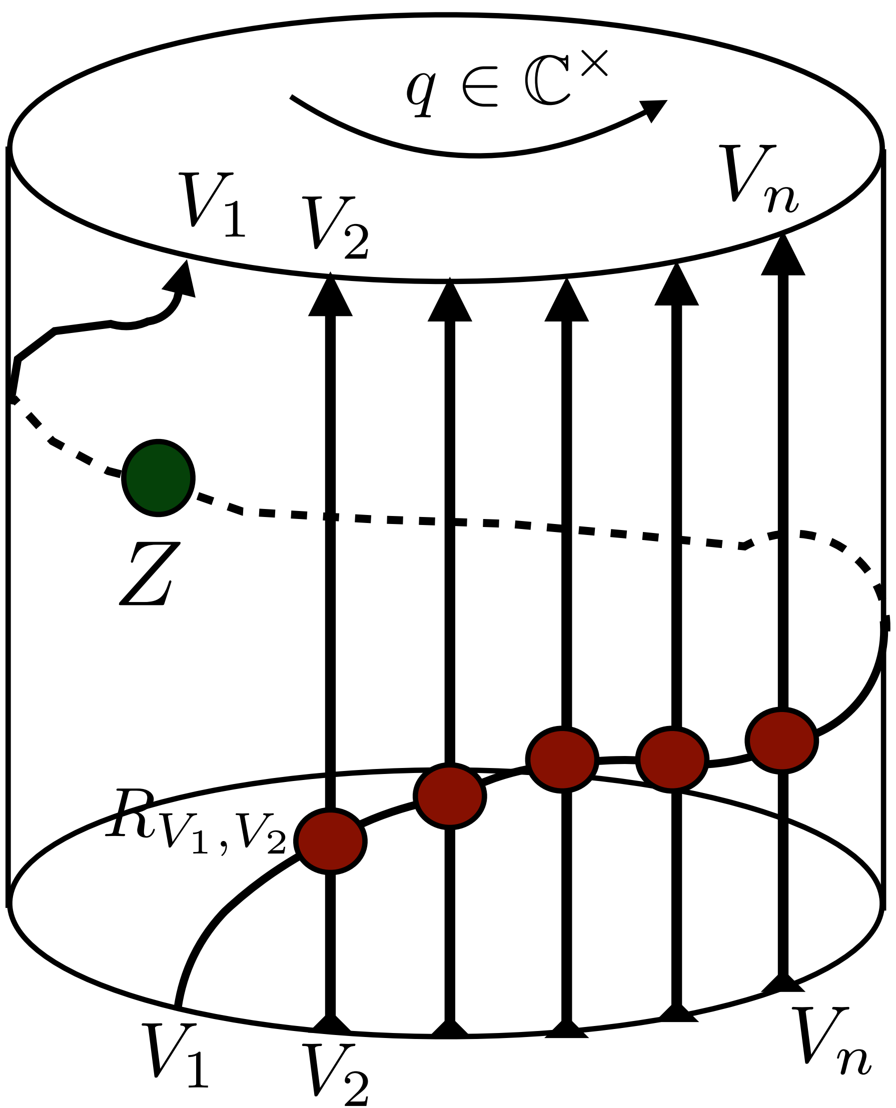
Transfer matrices are usually polynomials in u whose coefficients are the **integrals of motion**

XXZ Spin Chain

$\mathfrak{g} = \mathfrak{sl}_2$ spin-1/2 chain on n sites $V = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$

Spectrum can be found using Bethe Ansatz techniques. However, if we want to understand the problem for more general algebras we need to think of the Knizhnik-Zamolodchikov difference equation (qKZ)

$$\Psi(qa_1, \ldots a_n) = (Z \otimes 1 \otimes \cdots \otimes 1) R_{V_1,V_n} \cdots R_{V_1,V_2} \Psi(a_1, \ldots a_n)$$



where

$$\Psi(a_1, \ldots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

[I. Frenkel Reshetikhin]

In the limit $q \to 1$
qKZ becomes an eigenvalue problem

Solutions of qKZ

Schematic solution

$$\Psi_\alpha = \int \frac{d\mathbf{x}}{\mathbf{x}} f_\alpha(\mathbf{x}, a)\, \mathcal{K}(\mathbf{x}, z, a, q)$$

indexed by physical space

representation

universal kernel

$$\log \mathcal{K}(\mathbf{x}, z, a, q) \underset{q \to 1}{\sim} \frac{S(\mathbf{x}, z, a)}{\log q}$$

$$\frac{\partial S}{\partial x_i} = 0$$ Bethe equations for Bethe roots **x**

$$a_i \frac{\partial S}{\partial a_i} = \Lambda_i$$ Eigenvalues of qKZ operators

[Aganagic Okounkov]

The map $\alpha \mapsto f_\alpha(\mathbf{x}^*)$ Provides diagonalization

So we need to find `off shell' Bethe eigenfunctions $f_\alpha(\mathbf{x}, a)$

Nekrasov-Shatashvili correspondence

The answer will come from enumerative AG inspired by physics

Hilbert space of states of quantum integrable system ⟷ Equivariant K-theory of Nakajima quiver varitey (line operators in 3d SUSY gauge theory)

gauge group $G = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} U(v_i)$ (v_1,v_2,...) encode weight of rep α

Bethe roots **x** live in maximal torus of G, by integrating over **x** we project on gauge invariant functions of Bethe roots

Flavor group $G_F = \prod_i U(w_i)$ whose maximal torus give parameters **a**

Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

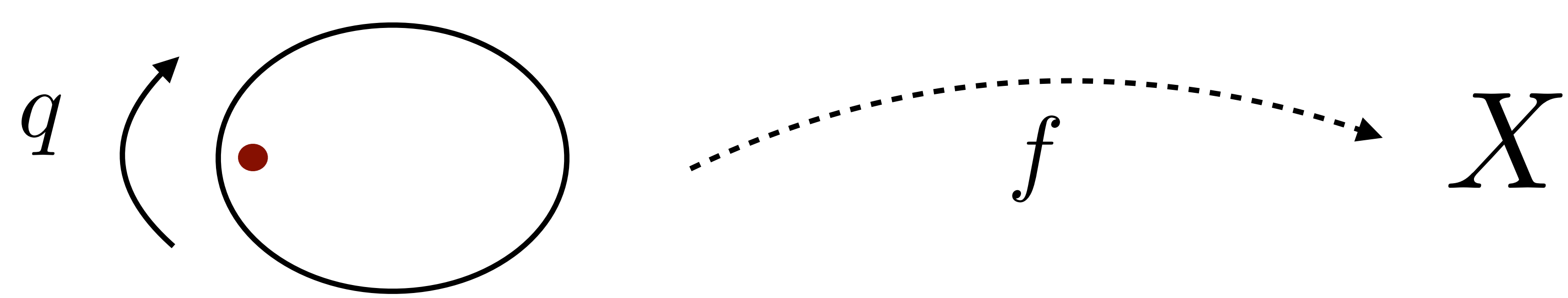
Nekrasov-Shatashvili correspondence

The quiver variety X = {Matter fields}/gauge group

X is a module of some quantum group in Nakajima correspondence construction

We will be computing integrals in K-theory of the space of quasimaps $f : \mathcal{C} - -- > X$ weighted by degree $\mathbf{z}^{\deg f}$ subject to equivariant action on the base curve $\mathbb{C}_q^\times$

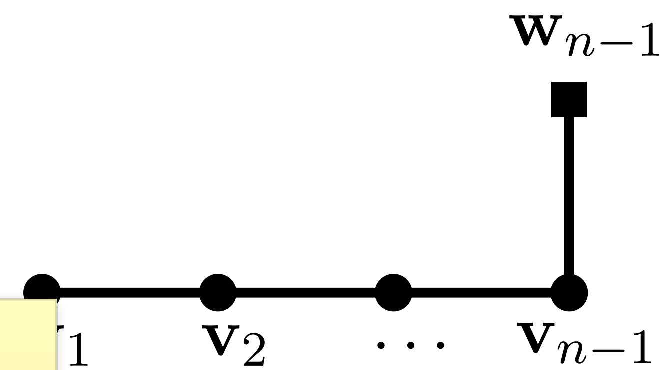
(cf Gromov-Witten invariants, top strings, etc.)



In particular we shall study quantum K-theory ring with quantum parameters z whose structure constant arise from 3 point correlators

Nakajima Quiver Varieties

Rep(**v**,**w**) — linear space of quiver reps



$\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map

Mention stability conditions here

Nakajima quiver variety $X = \mu$ $G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

Tensorial polynomials of tautological bundles Vi, Wi and their duals generate *classical T-equivariant K-theory* ring of X

Ex: T*Grassmannian

$$\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$$

$$\mathbf{v}_1 = k,\ \mathbf{w}_1 = n$$

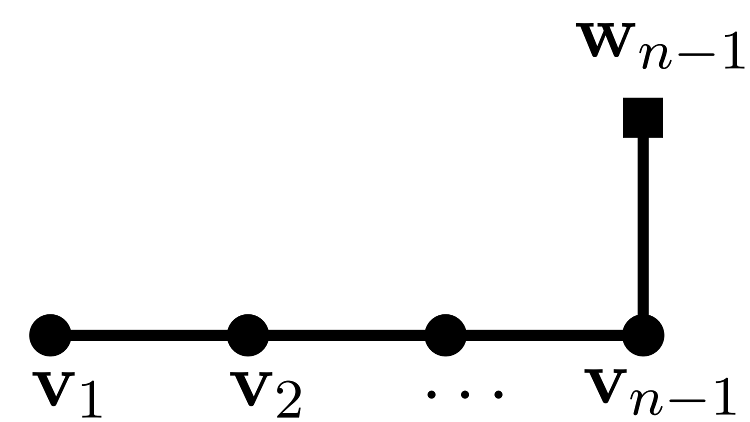
$$\tau(s_1, \cdots, s_k) = (s_1 + \cdots + s_k)^2 - \sum_{1 \le i_1 < i_2 < i_3 \le k} s_{i_1}^{-1} s_{i_2}^{-1} s_{i_3}^{-1}$$

value of a quasimap defines a map to a quotient stack which contains stable locus as an open subset

Quasimaps

Quasimap $f : \mathcal{C} \dashrightarrow X$ is described by collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ with section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$

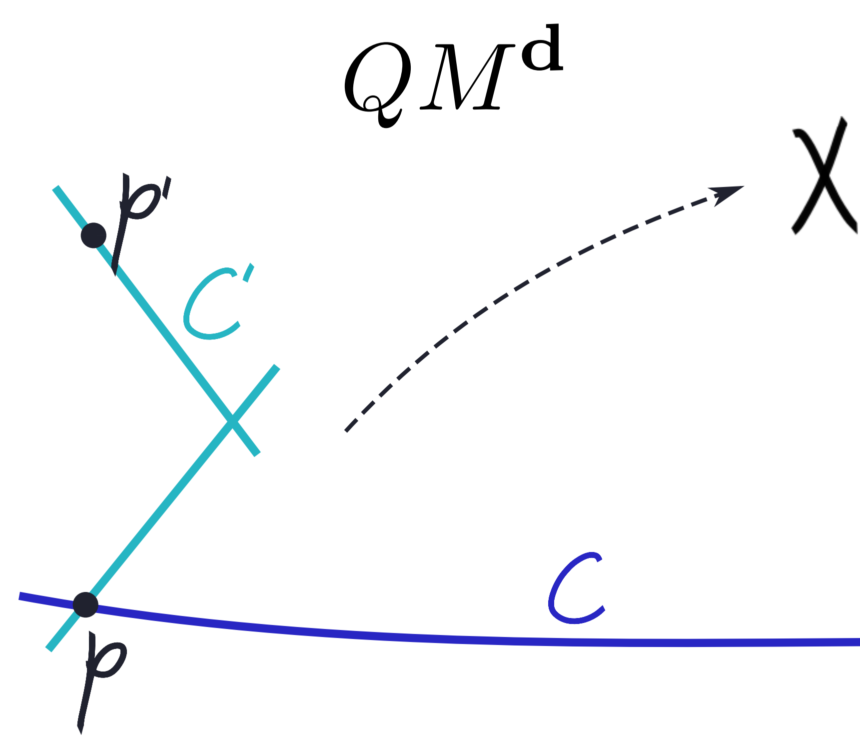
where $\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$



Degree $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$

Evaluation map

$$\mathrm{ev}_p(f) = f(p) \in [\mu^{-1}(0)/G] \supset X$$



Stable if $f(p) \in X$

for all but finitely many singular points

Resolve to make *proper* ev map

Virtual Sheaves

Deformation-obstruction theory allows one to construct virtual tangent bundle and virtual structure sheaf

[Ciocan-Fontanine, Kim, Maulik]

Fiber of the reduced virtual tangent bundle to $\mathsf{QM}^{\mathbf{d}}{}_{\mathrm{nonsing}\, p}$

$$T^{\mathrm{vir}}_{(\{\mathscr{V}_i\},\{\mathscr{W}_i\})}\mathsf{QM}^{\mathbf{d}}_{\text{nonsing p}} = H^\bullet(\mathscr{M} \oplus \hbar\, \mathscr{M}^*) - (1+\hbar)\bigoplus_i Ext^\bullet(\mathscr{V}_i, \mathscr{V}_i).$$

moment map, deformations
C* factorizations in GIT

Symmetrized virtual structure sheaf

(possible to do for quiver varieties)

$$\hat{\mathcal{O}}_{\mathrm{vir}} = \mathcal{O}_{\mathrm{vir}} \otimes \mathscr{K}_{\mathrm{vir}}^{1/2} q^{\deg(\mathscr{P})/2}$$

virtual canonical bundle

Vertex Function (g=

Say this in words: equivariant pushforward, etc.
Moduli space of quasimaps has perfect deformation-obstruction theory.

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q$ base $\mathbb{P}^1$ keeping two fixed points (0, infinity)

Define **vertex function** with quantum (Novikov) parameters $z^{\mathbf{d}} = \prod_{i \in I} z_i^{d_i}$

$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\left(QM^{\mathbf{d}}_{\text{nonsing}\, p_2}, \widehat{\mathcal{O}}_{\text{vir}} \tau(\mathscr{V}_i|_{p_1}) \right) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

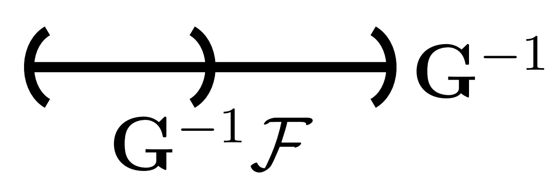
descendent

[Okounkov]
[PK Pushkar Smirnov Zeitlin]

Define **quantum K-theory** as a ring with multiplication

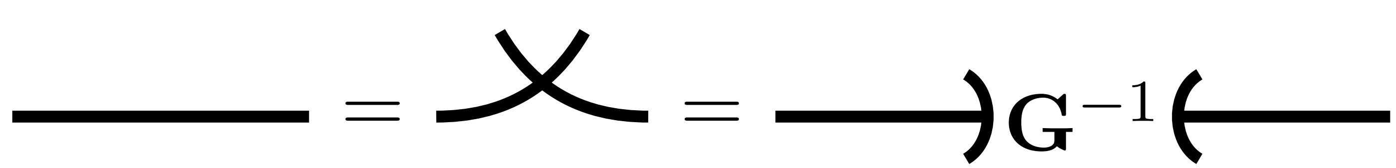
$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathcal{F} \circledast = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3 *}\left(\mathrm{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F}) \widehat{\mathcal{O}}_{\text{vir}} \right) \mathbf{G}^{-1}$$



gluing

$$\mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}$$



Vertex computation for T*Fl$_n$

At a given fixed point of extended maximal torus tangent space has

$$\mathcal{M} = \left(\mathcal{O}(d) \otimes q^{-d}\right) \oplus \left(\mathcal{O}(d) \otimes q^{-d} \otimes \frac{a_i}{a_j}\right)$$

character $\quad H^{\bullet}\left(\mathcal{O}(d) \otimes q^{-d} \otimes \frac{a_i}{a_j}\right) = \frac{a_i}{a_j}\left(1 + q^{-1} + \dots q^{-d}\right)$ $\quad$ similar to rest

Overall the contribution of $xq^{-d}\mathcal{O}(d)$ to the character is

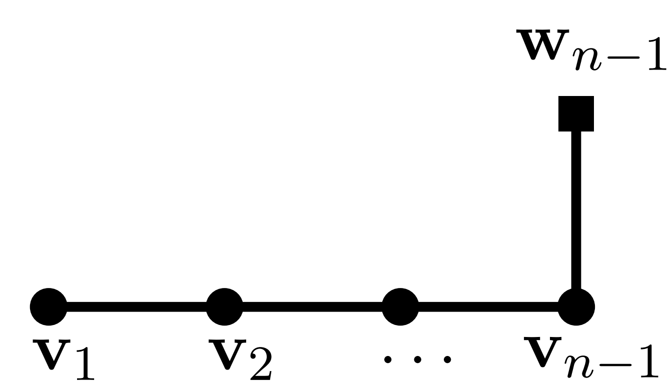
$$\{x\}_d = \frac{(\hbar/x, q)_d}{(q/x, q)_d} (-q^{1/2}\hbar^{-1/2})^d, \quad \text{where} \quad (x,q)_d = \frac{\varphi(x)}{\varphi(q^d x)} \qquad \varphi(x) = \prod_{i=0}^{\infty}(1 - q^i x)$$

Quantum class $\quad V_{\mathbf{p}}^{(\tau)}(z) = \sum\limits_{\mathbf{d}\in\mathbb{Z}^n_{\geq 0}} \sum\limits_{(\mathscr{V},\mathscr{W})\in(\mathsf{QM}^{\mathbf{d}}_{\text{nonsing}\, p_2})^{\mathsf{T}}} \hat{s}(\chi(\mathbf{d}))\, z^{\mathbf{d}} q^{\deg(\mathscr{P})/2} \tau(\mathscr{V}|_{p_1}).$

each fixed point contributes

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2} EHG \quad \tau(x_{i,j}q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1} \prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}}$$



Example for T*P₁

$$\mathbf{v}_1 = 1,\ \mathbf{w}_1 = 2$$

Vertex with trivial insertion

$$V^{(1)}_{\mathbf{p}} = \sum_{d>0} (z^\sharp)^d \prod_{i=1}^{2} \frac{\left(\frac{q}{\hbar}\frac{a_{\mathbf{p}}}{a_i};q\right)_d}{\left(\frac{a_{\mathbf{p}}}{a_i};q\right)_d} = {}_2\phi_1\left(t, t\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, \frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; z^\sharp\right)$$

two fixed points

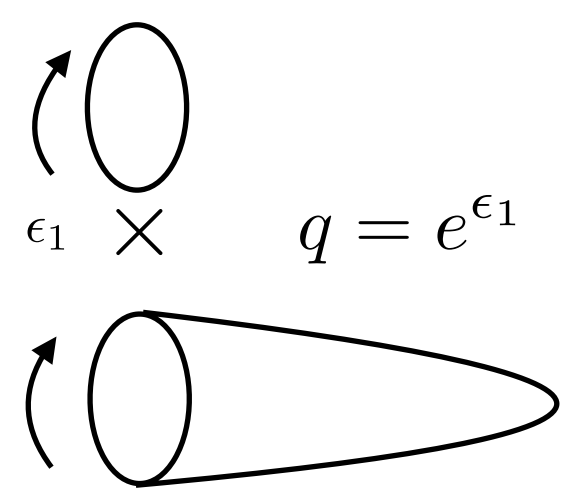
$\mathbf{p} = \{a_1\}$ and $\mathbf{p} = \{a_2\}$.

As a contour integral

$$V^{(1)}_{\mathbf{p}} = \frac{1}{2\pi i \alpha_p} \int_{C_{\mathbf{p}}} \frac{ds}{s} (z^\sharp)^{-\frac{\log s}{\log q}} \prod_{i=1}^{2} \frac{\varphi\left(t\frac{s}{a_i}\right)}{\varphi\left(\frac{s}{a_i}\right)}$$

Physics: Vortex partition function

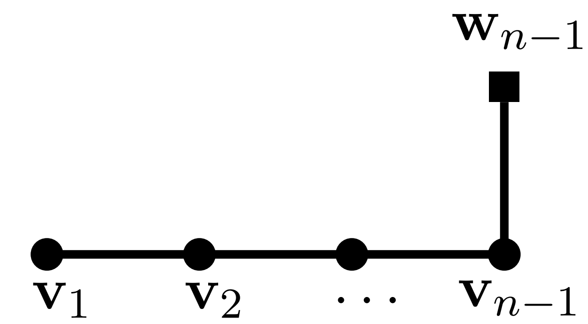
$\mathcal{N} = 2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma$



Lagrangian depends on twisted masses a_1, a_2

FI parameter z and U(1) R-symmetry $\log \hbar$

Bethe Equations



Saddle point approximation provides the operator of quantum multiplication

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V_{\boldsymbol{p}}^{(\tau)}(z)}{V_{\boldsymbol{p}}^{(1)}(z)}$$

For the cotangent bundle to partial flag variety we get

[PK Pushkar Smirnov Zeitlin]

Theorem 3.4. *The eigenvalues of $\hat{\tau}(z)\circledast$ is given by $\tau(s_{i,k})$, where $s_{i,k}$ satify Bethe equations:*

$$\prod_{j=1}^{\mathbf{v}_2} \frac{s_{1,k}-s_{2,j}}{s_{1,k}-\hbar s_{2,j}} = z_1(-\hbar^{1/2})^{-\mathbf{v}_1'} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_1} \frac{s_{1,j}-s_{1,k}\hbar}{s_{1,j}\hbar - s_{1,k}}\,,$$

$$\prod_{j=1}^{\mathbf{v}_{i+1}} \frac{s_{i,k}-s_{i+1,j}}{s_{i,k}-\hbar s_{i+1,j}} \prod_{j=1}^{\mathbf{v}_{i-1}} \frac{s_{i-1,j}-\hbar s_{i,k}}{s_{i-1,j}-s_{i,k}} = z_i(-\hbar^{1/2})^{-\mathbf{v}_i'} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_i} \frac{s_{i,j}-s_{i,k}\hbar}{s_{i,j}\hbar - s_{i,k}}\,, \tag{23}$$

$$\prod_{j=1}^{\mathbf{w}_{n-1}} \frac{s_{n-1,k}-a_j}{s_{n-1,k}-\hbar a_j} \prod_{j=1}^{\mathbf{v}_{n-2}} \frac{s_{n-2,j}-\hbar s_{n-1,k}}{s_{n-2,j}-s_{n-1,k}} = z_{n-1}(-\hbar^{1/2})^{-\mathbf{v}_{n-1}'} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_{n-1}} \frac{s_{n-1,j}-s_{n-1,k}\hbar}{s_{n-1,j}\hbar - s_{n-1,k}}\,,$$

where $k=1,\dots,\boldsymbol{v}_i$ for $i=1,\dots,\boldsymbol{v}_{n-1}$.

which are Bethe Ansatz Equations for gl(n) XXZ spin chain

K theory and Many-Body systems

Now we would like to connect quantum K-theory of X with *integrable many-body systems*

Consider vertex for T*Fln with trivial insertion V(**z**,**a**,h,q)

Theorem 1 [PK]:

Given integrals of motion of <u>trigonometric Ruijsenaars-Schneider model</u>

$$T_r(\vec{\zeta}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\hbar\,\zeta_i-\zeta_j}{\zeta_i-\zeta_j} \prod_{i\in\mathcal{I}} p_k \qquad p_k f(\zeta_k) = f(q\zeta_k) \qquad z_i = \zeta_{i+1}/\zeta_i$$

then vertex is their mutual eigenfunction $T_r(\vec{\zeta})\mathsf{V}^{(1)}_{\boldsymbol{p}} = e_r(\mathbf{a})\mathsf{V}^{(1)}_{\boldsymbol{p}}\,, \qquad r=1,\dots,n$

Theorem 2 [PK Zeitlin]:

Given integrals of motion of dual <u>trigonometric Ruijsenaars-Schneider model</u>

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{t\,a_i-a_j}{a_i-a_j} \prod_{i\in\mathcal{I}} p_k \qquad p_k f(a_k) = f(qa_k) \qquad t = \frac{q}{\hbar}$$

then vertex is their mutual eigenfunction

$$T_r(\boldsymbol{a})\mathrm{V}(\boldsymbol{a},\vec{\zeta}) = S_r(\vec{\zeta},t)\mathrm{V}(\boldsymbol{a},\vec{\zeta})\,, \qquad r=1,\dots,\mathbf{w_{n-1}}$$

Baxter Operator

Consider quantum tautological bundles $\widehat{\Lambda^k V_i}(z),\ k=1,\dots,\mathbf{v}_i$

and their generating function —
Baxter Q-operator

$$\mathbf{Q}_i(u)=\sum_{k=0}^{\mathbf{v}_i}(-1)^k u^{\mathbf{v}_i-k}\hbar^{\frac{ik}{2}}\widehat{\Lambda^k V_i}(z)$$

Proposition: The eigenvalue of quantum multiplication by **Qi** is

$$Q_i(u)=\prod_{k=1}^{\mathbf{v_i}}(u-\hbar^{\frac{i}{2}}s_{i,k})$$

Using results from integrability we can write XXZ Bethe equations in term of polynomials

$$Q_i(u)=\prod_{\alpha=1}^{\mathbf{v}_i}(u-\sigma_{i,\alpha})\ ,\qquad P(u)=Q_n(u)=\prod_{a=1}^{\mathbf{w}_{n-1}}(u-\alpha_a)$$

Theorem [PK]: Given Lax matrix of tRS model

$$L_{ij}=\frac{\prod\limits_{k\neq j}^{n}\left(\hbar^{-1/2}\zeta_i-\hbar^{1/2}\zeta_k\right)}{\prod\limits_{k\neq i}^{n}(\zeta_i-\zeta_k)}p_j\qquad p_j=-\frac{\mathbf{Q}_j(0)}{\mathbf{Q}_{j-1}(0)}=\hbar^{j-\frac{1}{2}}\widehat{\Lambda^j V_j}(z)\circledast\widehat{\Lambda^{j-1}V^*}_{j-1}(z)$$

$$p_i=\frac{s_{i+1,1}\cdot\dots\cdot s_{i+1,i+1}}{s_{i,1}\cdot\dots\cdot s_{i,i}},\qquad i=1,\dots,n-1$$

we get $$P(u)=det\Big(u-L\Big)$$

Example for T*P1

Vertex

$$V = \frac{e^{\frac{\log \zeta_2 \cdot \log a_1 \cdots a_n}{\log q}}}{2\pi i} \int_C \frac{ds}{s}\, e^{\frac{\log \zeta_1/\zeta_2 \cdot \log s}{\log q}} \frac{\varphi\left(\hbar \frac{s}{a_1}\right)}{\varphi\left(\frac{s}{a_1}\right)} \frac{\varphi\left(\hbar \frac{s}{a_2}\right)}{\varphi\left(\frac{s}{a_2}\right)}$$

tRS Hamiltonians

$$T_1(\vec{\zeta}) = \frac{\hbar\zeta_1 - \zeta_2}{\zeta_1 - \zeta_2} p_1 + \frac{\hbar\zeta_2 - \zeta_1}{\zeta_2 - \zeta_1} p_2$$

$$T_2(\vec{\zeta}) = p_1 p_2 \,.$$

Energy equations

$$T_1(\vec{\zeta})\mathrm{V} = \mathrm{V}^{(T_1(s))} = (a_1 + a_2)\mathrm{V}$$

$$T_2(\vec{\zeta})\mathrm{V} = a_1 a_2 \mathrm{V}\,,$$

K theory via tRS

Classical limit (q ->1) implies the following

Theorem 4.7. *Quantum equivariant K-theory of the cotangent bundle to complete n-flag is given by*

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[\zeta_1^{\pm 1},\dots,\zeta_n^{\pm 1};a_1^{\pm 1},\dots,a_n^{\pm 1},\hbar^{\pm 1};p_1^{\pm 1},\dots,p_n^{\pm 1}]}{\{H_r(\zeta_i,p_i,\hbar)=e_r(\alpha_1,\dots,\alpha_n)\}}, \tag{63}$$

where the ideal is generated by equations of motion of all Hamiltonians of **tRS model**

$\zeta_1,\dots,\zeta_n$ are coordinates $p_1,\dots,p_n$ are momenta

symplectic form $\Omega = \sum_{i=1}^{n} \frac{dp_i}{p_i} \wedge \frac{d\zeta_i}{\zeta_i}$

Momenta can be determined from derivatives of Yang-Yang function XXZ for Bethe equations. They define Lagrangian $\mathcal{L} \subset T^*\left(\mathbb{C}^\times\right)^n$ whose generating function is given by the Yang-Yang function.

[Gaiotto PK]
[Bullimore Kim PK]

Five-Vertex model and qToda

In previous formulae we can take $\hbar \to \infty$

(70) $$Q_{i+1}(u) - \frac{\mathfrak{z}_{i+1}}{\mathfrak{z}_i} Q_{i-1}(u) \cdot u \cdot \mathfrak{p}_{i+1} = Q_i(u)\widetilde{Q}_i(u)\,, \quad i = 1, \ldots, n$$

where $z_i^{\#} = \frac{\mathfrak{z}_i}{\mathfrak{z}_{i+1}}$, $\widetilde{Q}_i(u)$, $i = 1, \ldots, n-1$ *are monic polynomials of degree one and*

(71) $$\mathfrak{p}_i = -\frac{Q_i(0)}{Q_{i-1}(0)}\,.$$

Analogously to **XXZ/tRS** duality we can formulate **5-vert/qToda** duality

Theorem 5.3. *System of equations (70) is equivalent to*

(72) $$M(u) = \det A(u)\,,$$

where $A(u)$ *is the Lax matrix of the difference Toda chain. It has the following nonzero elements*

(73) $$A_{i+1,i} = 1\,, \quad A_{i,i} = u - \mathfrak{p}_i\,, \quad A_{i,i+1} = -u\frac{\mathfrak{z}_{i+1}}{\mathfrak{z}_i}\mathfrak{p}_{i+1}\,.$$

We recover the statement by Givental and Lee

Theorem 5.4. *Quantum equivariant K-theory of the complete n-dimensional flag variety is given by*

(78) $$QK_{T'}(\mathbb{F}l_n) = \frac{\mathbb{C}[\mathfrak{z}_1^{\pm 1}, \ldots, \mathfrak{z}_n^{\pm 1}; \mathfrak{a}_1^{\pm 1}, \ldots, \mathfrak{a}_n^{\pm 1}; \mathfrak{p}_1^{\pm 1}, \ldots, \mathfrak{p}_n^{\pm 1}]}{\{H_r^{q\text{-}Toda}(\mathfrak{z}_i, \mathfrak{p}_i) = e_r(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)\}}\,,$$

qLanglands

Come to my talk on Monday and I'll tell you how to prove quantum/ classical duality and beyond using **qOpers**