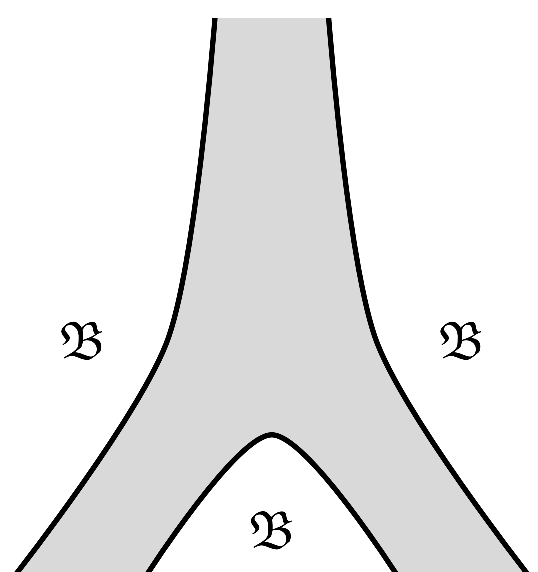
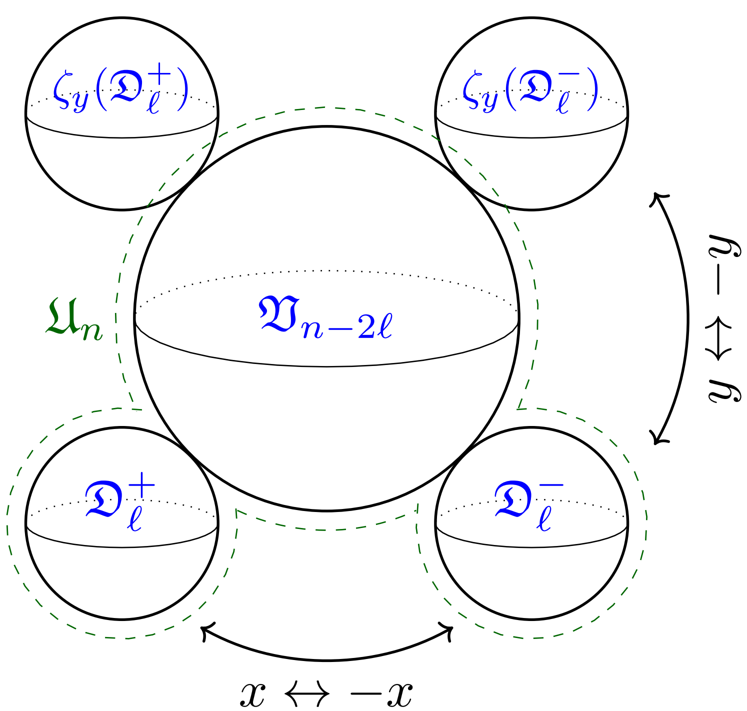
DAHA Branes & DELL

Peter Koroteev

$$\mathcal{A} \;=\; \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{cc}) \;=\; \boldsymbol{S\ddot{H}}$$



Talk at Joint AMS meeting
Denver, CO January 16th 2020



Main Conjecture

[Gukov PK Nawata Pei Saberi]

$\mathcal{M}$ - two-particle Calogero-Moser space

$\mathfrak{a}$ - spherical double affine Hecke algebra (DAHA) for sl(2)

There is a (derived) equivalence

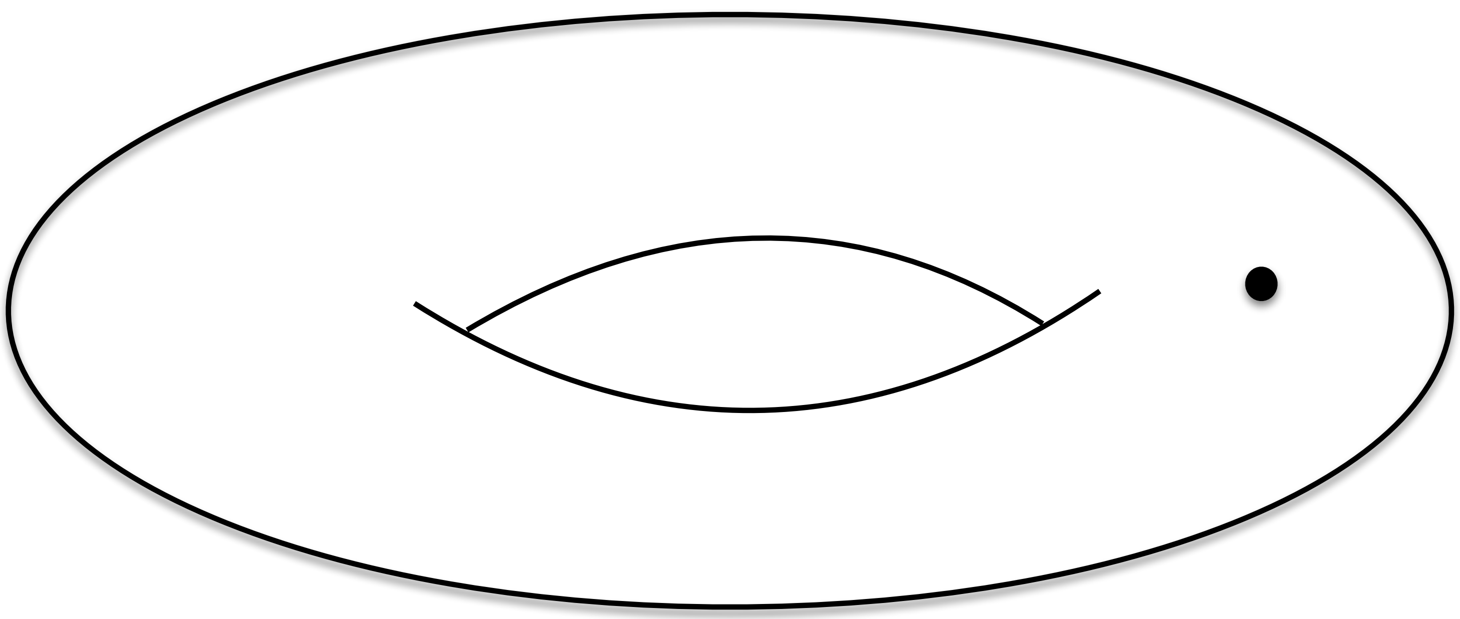
$$\begin{tikzcd} A - \mathfrak{Brane}(\mathcal{M},\omega) \arrow[r, phantom, "="] & Rep(\mathfrak{a}) \\ \mathfrak{Fuk}(\mathcal{M},\omega) \arrow[u, phantom, "\cup"] \arrow[ur, hook] & \end{tikzcd}$$

branes $\longleftrightarrow$ modules

Spherical gl(n) DAHA

[Oblomkov]

Spherical gl(n) DAHA is a **flat 1-parameter deformation (quantization)** of the space of Poisson-commuting functions on the moduli space of flat GL(n;C) connections on a torus with one simple puncture (CM space)



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \mathrm{diag}(t, \ldots, t, t^{1-n})$$

$$\mathcal{A}_n = \mathbb{C}_J\widehat{[\mathcal{M}_n]}$$

Hitchin Moduli Space

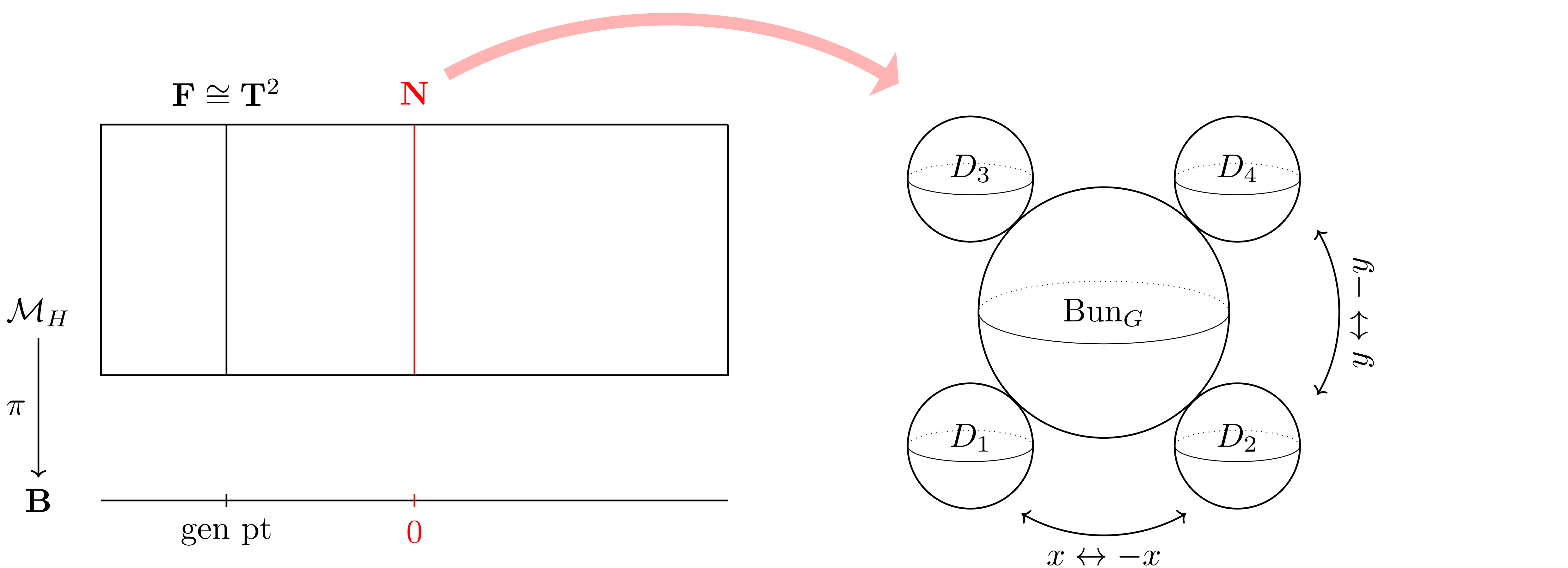
Darboux coordinates on M

$x = Tr A$ *electric* $\quad y = Tr B$ *magnetic* $\quad z = Tr AB$ *dyonic*

Nonabelian Hodge correspondence: $\mathcal{M}_{\text{flat}}(SL(2;\mathbb{C}), T^2\backslash\{\text{pt}\}) \simeq \mathcal{M}_H(SU(2), T^2\backslash\{\text{pt}\})$

$\mathcal{M}_H : \; x^2 + y^2 + z^2 + xyz - t^2 - t^{-2} - 2 = 0$ for t=1 $\mathcal{M}_n \simeq \dfrac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2}$

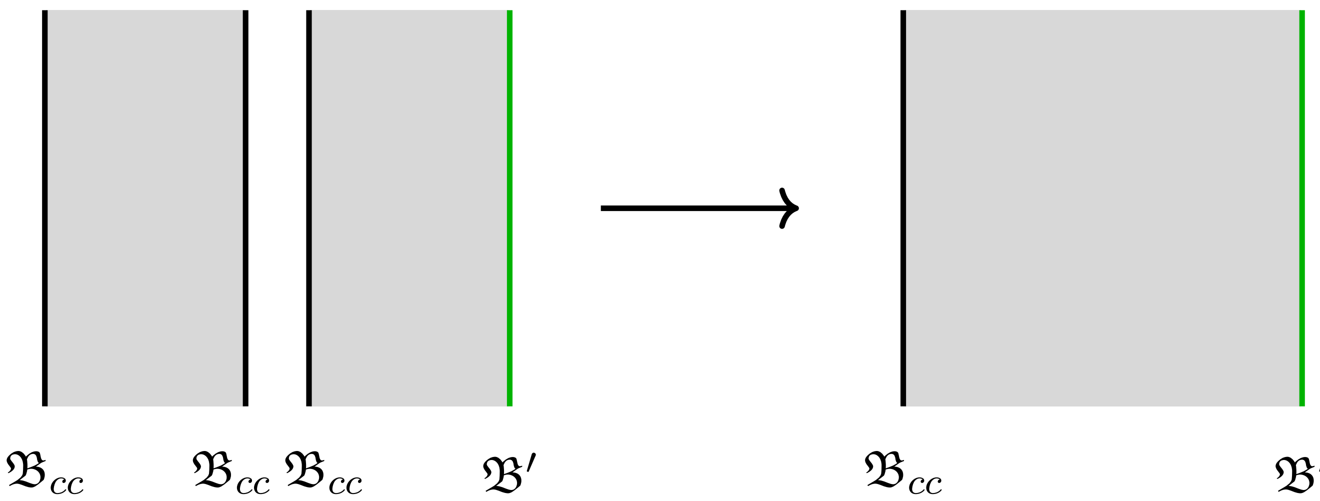
Elliptic fibration with one singular fiber of Kodaira type I_0^*



Branes and Quantization

[Gukov Witten]
[Kapustin Witten]

Algebra acts naturally by attaching open strings to closed strings



Hilbert space comes from (Bcc, B′) strings

$$\mathcal{A} = \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{cc})$$

$$\circlearrowright \qquad\qquad \circlearrowright$$

$$\mathcal{H} = \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}')$$

$\mathcal{B}' \to \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ gives a functor $\mathrm{Hom}(\mathcal{B}_{cc}, \cdot)$

$$\boxed{\mathrm{Fuk}(\mathcal{M}, \Omega) \simeq \mathrm{Rep}(\mathcal{A})}$$

$$\mathfrak{B}_{cc} : \mathcal{L} \to \mathcal{M}_H$$

$$F + B = \frac{i}{\log q}\Omega_J$$

$$\Omega_J = \frac{dx \wedge dy}{2z - xy}$$

Algebra-deformation quantization of functions on M_H

$$[x, y]_q = (q - q^{-1})z$$

$$[z, x]_q = (q - q^{-1})y$$

$$[y, z]_q = (q - q^{-1})x$$

Lagrangian A-brane → Module of DAHA

Dimension of a module

$$\mathrm{dim}V = \int_{\mathcal{M}} \mathrm{ch}(\mathcal{B}') \wedge \mathrm{ch}(\mathcal{B}_{cc}) \wedge \mathrm{Td}(\mathcal{M})$$

compact branes ↔ Finite dim reps

Highest Weight Modules

Highest weight vector

$$y\,\mathcal{Z} = (Y + Y^{-1})\mathcal{Z} = (a + a^{-1})\mathcal{Z}$$

becomes Macdonald polynomial

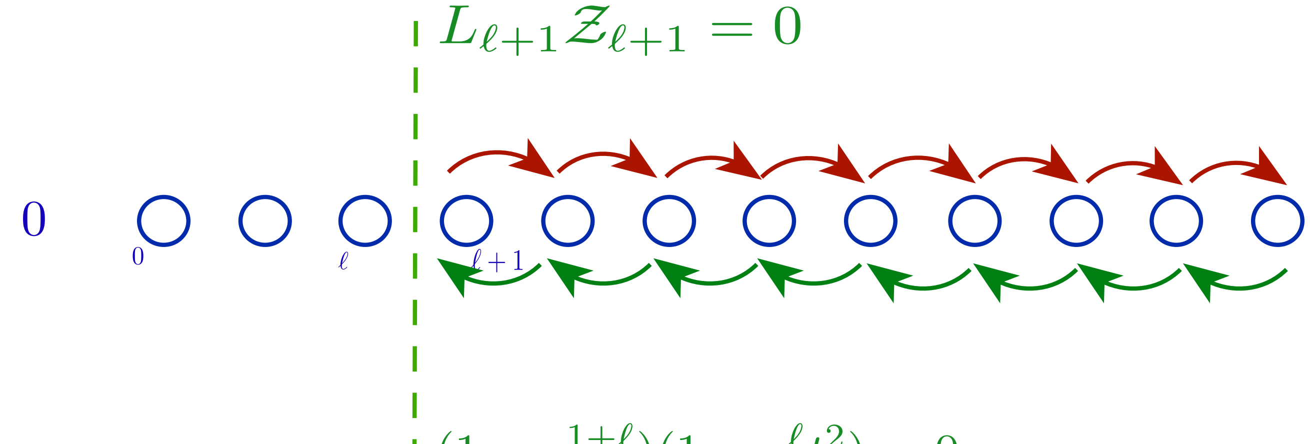
$$a^2 = q^{-2\ell} t^{-2}$$

Raising and lowering operators of sl(2) DAHA

$$R_a = x + a_k^{-1} z$$
$$L_a = x + a_k z$$

$$R_a \mathcal{Z}_a = r_a \mathcal{Z}_{a+1}$$
$$L_a \mathcal{Z}_a = l_a \mathcal{Z}_{a-1}$$

$$L_{\ell+1}\mathcal{Z}_{\ell+1} = 0$$



$$(1 - q^{1+\ell})(1 - q^{\ell} t^2) = 0$$

Shortening occurs when

$$q^{2N} = 1$$
$$t^2 = q^{-(2k-1)}$$
$$t^2 = -q^{-n}$$

$$0 \to S \to V \to V/S \to 0$$

Matching

- root of unity $q = e^{-\pi i/n}$

$$0 \to \zeta_2(\mathfrak{U}_n) \to \mathfrak{F}_{2n}^{\Delta=2} \to \mathfrak{U}_n \to 0$$

- $t^2 = q^{-(2\ell-1)}$

$$\mathfrak{D}_\ell^+\ , \quad \zeta_1(\mathfrak{D}_\ell^+)\ , \quad \zeta_2(\mathfrak{D}_\ell^+)\ , \quad \zeta_2\zeta_1(\mathfrak{D}_\ell^+)$$

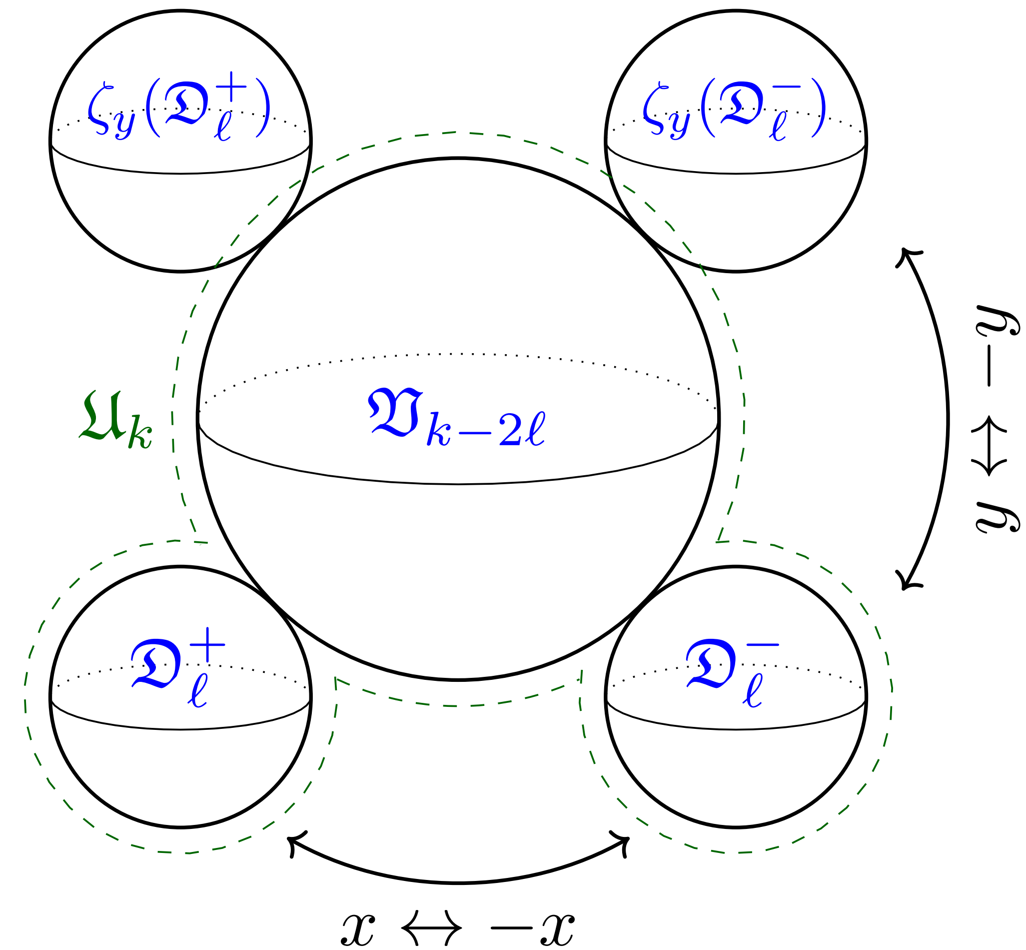
- $t^2 = -q^{-n}$
  `additional series'

- two conditions at once

$$0 \longrightarrow \iota(\mathfrak{D}_\ell^+ \oplus \mathfrak{D}_\ell^-) \longrightarrow \mathfrak{U}_n \longrightarrow \mathfrak{V}_{k+1} \longrightarrow 0$$

generic fiber

$$[\mathbf{F}] = 2[\mathrm{Bun}_G] + \sum_{i=1}^{4}[D_i]$$



overall 5 spherical objects

Double Elliptic Model

[PK Shakirov]

Hamiltonians $\widehat{\mathcal{H}}_a = \widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{O}}_a$ **in involution** $[\widehat{\mathcal{H}}_a, \widehat{\mathcal{H}}_b] = 0$

$$\widehat{\mathcal{O}}(z) = \sum_{n\in\mathbb{Z}} \widehat{\mathcal{O}}_n\, z^n = \sum_{n_1,\ldots,n_N=-\infty}^{\infty} (-z)^{\sum n_i}\, w^{\sum \frac{n_i(n_i-1)}{2}} \prod_{i<j} \theta\big(t^{n_i-n_j}\widehat{x}_i/\widehat{x}_j|p\big)\, \widehat{p}_1^{n_1}\ldots\widehat{p}_N^{n_N}$$

Spectrum

$$H_i\mathcal{Z}(p,\mathbf{x}) = E_i(p,\mathbf{a})\mathcal{Z}(p,\mathbf{x})$$

p \ q	rational	trigonometric	elliptic
r	rational CMS	$\epsilon \to 0$ trigonometric CMS	$p \to 0$ elliptic CMS *quantum cohomology*
t	$R \to 0$ rational RS (dual trig. CMS)	$\epsilon \to 0$ $R \to 0$ trigonometric RS	$R \to 0$ $p \to 0$ elliptic RS *quantum K-theory*
e	dual elliptic CMS	$w \to 0$ dual elliptic RS	$w \to 0$ $p \to 0$ DELL *Elliptic Cohomology*