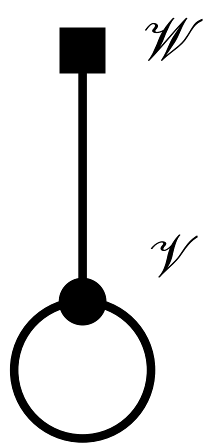
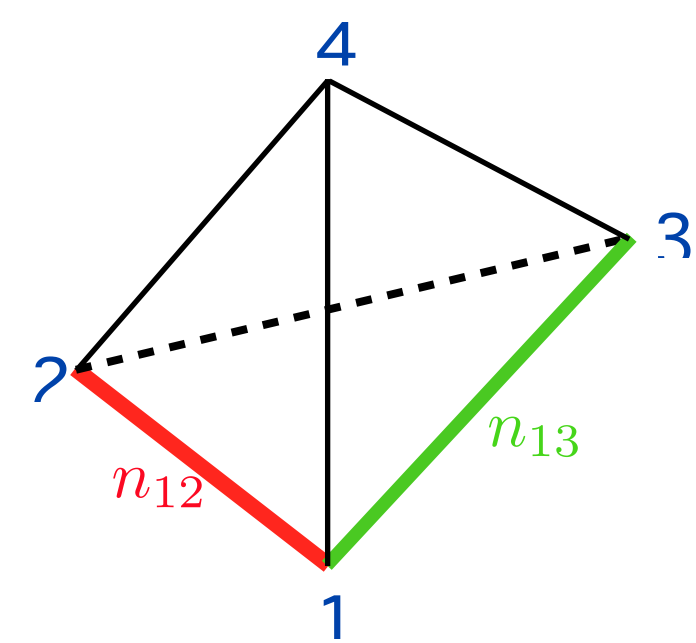
Quiver W-algebras & Defects from Gauge Origami

Peter Koroteev





Based on 1908.04394

Talk at CMSA Matter Seminar
Boston, MA October 9th 2019

Seiberg-Witten Solution

[Seiberg Witten 1994]

Provides mass spectrum of BPS particles of $\mathcal{N}$=2 gauge theory in 4d in the infrared

In IR spectrum given by period integrals of the curve

$$2u = p^2 - \left(z + \frac{1}{z}\right) \qquad \lambda = p\frac{dz}{z}$$

Masses of BPS particles $\quad S_j(u) = \oint_{\gamma_j} \lambda$

Potential $\quad V \sim \mathrm{Tr}|[\phi, \phi]|^2$

UV vacuum $\quad \langle \phi \rangle = a\sigma_3/2$

Coordinate on the moduli space $\quad u = \langle \mathrm{tr}\phi^2 \rangle$

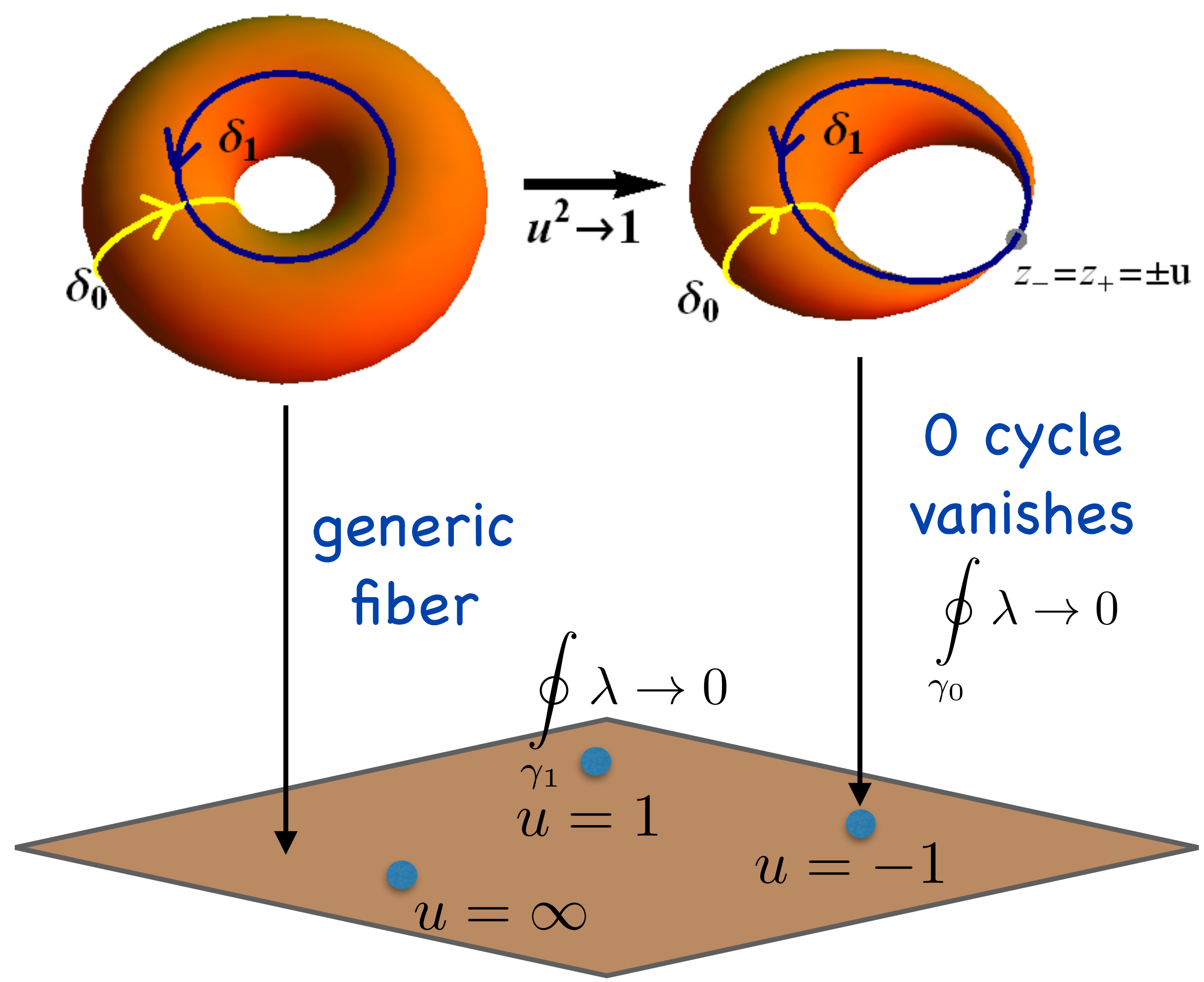
Using S-duality define dual magnetic variables (a, a_D) $\quad a_D = \dfrac{\partial \mathcal{F}(a)}{\partial a}$

One-loop correction in the semi-classical region

$$a_D \sim \frac{ia}{\pi}\left(1 + \ln\frac{a^2}{\Lambda^2}\right)$$

$$a \sim \sqrt{u}$$

Monodromy around infinity $\quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

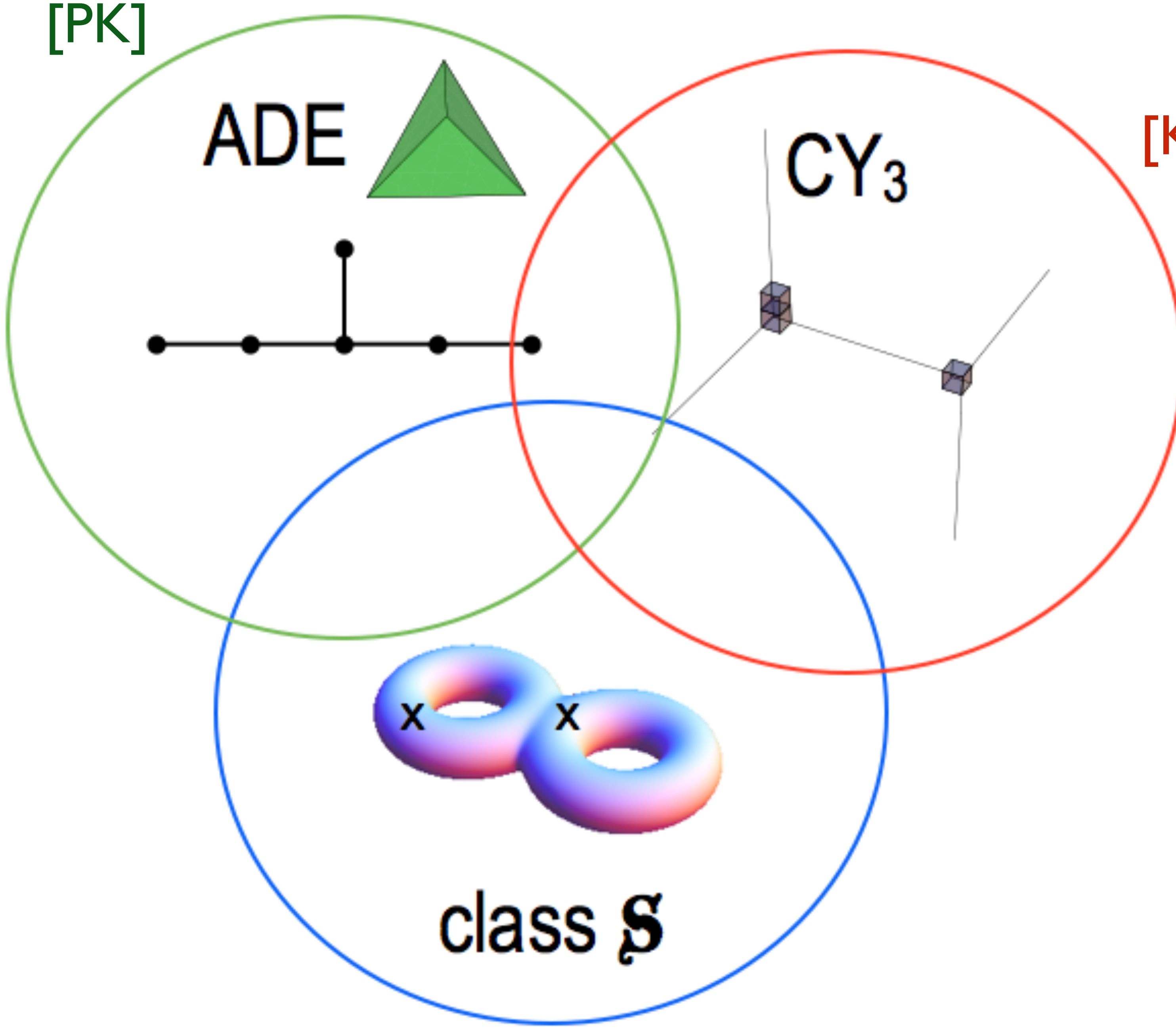


Appendix of [Gulden Janas Kamenev PK]

Landscape of $\mathcal{N}$=2 theories

[Nekrasov Pestun Shatashvili]
[Kimura Pestun]
[PK]

[Katz Klemm Vafa]



[Gaiotto]
[Alday Gaiotto Tachikawa]

BPS/CFT

- *Physically*: Connects BPS observables of $\mathcal{N}=2$ supersymmetric gauge theories with CFT correlators

- *Mathematically*: Relates structures arising on moduli spaces of sheaves (instantons) with vertex operator algebras

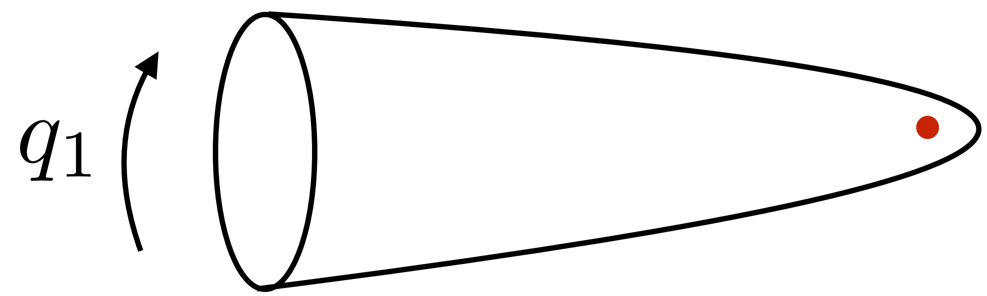
- Canonical example: [**A**lday **G**aiotto **T**achikawa]
  Partition functions vs. CFT conformal blocks
  Symmetries of the instanton moduli spaces vs.
  Vertex operator algebras

BPS/CFT (AGT)

Gauge theory in Omega background

$$\mathbb{R}^2_{q_1} \times \mathbb{R}^2_{q_2} \times S^1$$

$q_{1,2} = e^{\epsilon_{1,2}}$



complex scalar gets shifted $\quad \phi \to \phi + \epsilon_1 \dfrac{\partial}{\partial \varphi_{12}} + \epsilon_2 \dfrac{\partial}{\partial \varphi_{34}}$

Nekrasov used it to count instantons which localize on the tip

AGT states that $\quad \mathcal{Z}_{\mathrm{Nek}} = \mathcal{F}_{\mathrm{CFT}}$

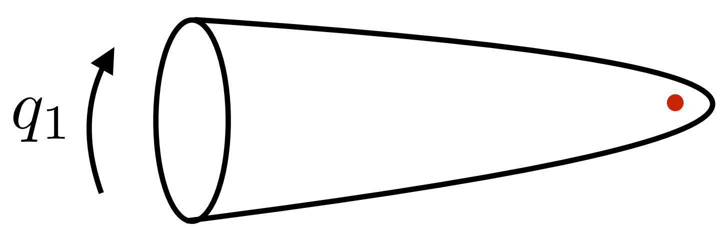
Omega background data is matched with the CFT central charge and (q)VOA (i.e. W-algebra) data

AGT Correspondence

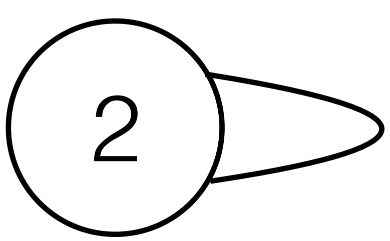
Class-*S* theories are constructed in M-theory with M5 branes wrapping $\mathcal{M}_4 \times \mathcal{C}$ [Gaiotto]

Twisted compactification of the theory on M5 branes — (2,0) 6d theory on $\mathcal{C}$ leads to $\mathcal{N}{=}2$ theory on $\mathcal{M}_4$

$\mathcal{N}{=}2^*$ SU(2) 4d gauge theory on $\mathbb{R}^2_{q_1} \times \mathbb{R}^2_{q_2}$

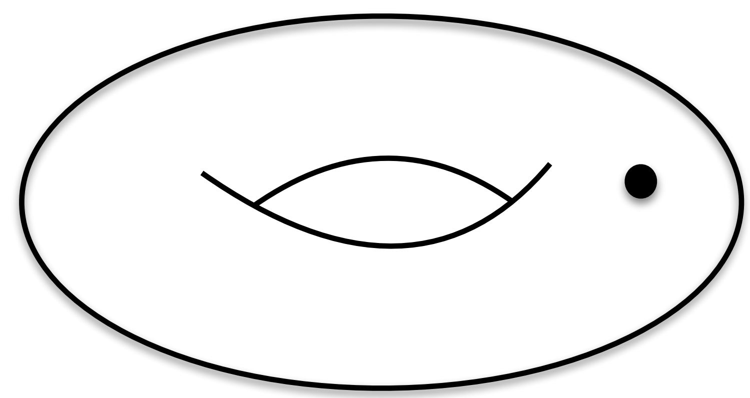


with adj hyper of mass ***m***
gauge coupling τ



Liouville CFT on a torus with one puncture

thin neck with sewing parameter $q = e^{2\pi i\tau}$



AGT: $\mathcal{Z}_{\mathrm{Nek}} = \mathcal{F}_{\mathrm{CFT}}$

BPS/CFT and Geometry

Mathematicians have now several ***proofs*** of BPS/CFT (AGT) in limiting cases (no fundamental matter), those proofs do not use the original class-S construction [Schiffmann Vaserot] [Negut]

Physics **proof*** by Kimura and Pestun uses direct localization computations

One of our goals is to understand BPS/CFT **geometrically**

Namely we want describe instanton counting and vertex operator algebras in terms of **quantum geometry** (quantum cohomology or quantum K-theory) of some family of spaces

In other words we want (q)VOAs to **emerge** from quantum geometry

$$\mathfrak{E} \simeq U_{q_1,q_2}\left(\widehat{\widehat{\mathfrak{gl}_1}}\right) \simeq \mathscr{E}_{q_1,q_2} \simeq \mathfrak{gl}_\infty \mathrm{DAHA}^S_{q_1,q_2} \simeq \mathrm{DIM}_{q_1,q_2} \simeq D(\mathscr{A}_{\mathrm{shuffle}})$$

Recent Developments

Vertex Algebras at the Corner [Gaiotto Rapcak]

VOAs at junctions of supersymmetric intersections in N=4 SYM

COHA and VOAs [Rapcak Soibelman Yang Zhao]

Action of COHA on the moduli space of *spiked* instantons

Quiver W-algebras [Kimura Pestun]

4,5,6d quiver gauge theories on R^4 x S in Omega background

The Magnificent Four [Nekrasov]

D8 brane probed by D0 branes in B field

$U(1)^4 \subset \mathrm{Spin}(8)$ + additional *nongeometric* U(1) symmetry

q_1, q_2, q_3, q_4

Large-n Limit

String theory enjoys **large-n** dualities

AdS/CFT, Gopakumar-Vafa

Gauge theories are known to have effective description when the rank of the gauge group becomes large $U(n)$ $n \to \infty$

Similar ideas work in mathematics — stable limits

We shall see that BPS/CFT can be viewed as a large-n duality!

Can skip this slide in talks shorter than 1 hrs

U(n) theory+defect

$\downarrow n \to \infty$

U(1) theory

$\mathcal{N}$=2 gauge theories

[PK Sciarappa]
[Li, Costello]

Triality

Integrable many-body systems

n-particle Calogero model

$\downarrow n \to \infty$

ILW hydrodynamics

[Maulik Okounkov]
[PK Sciarappa]

Representation theory
Algebraic geometry

gl(n) DAHA

$\downarrow n \to \infty$

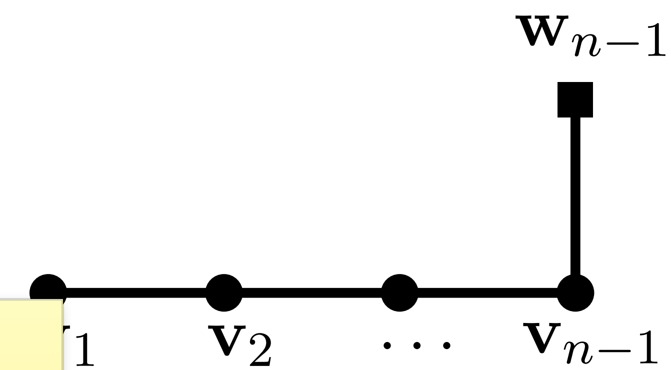
DI, Hall algebra, qW, etc.

[Schiffmann Vaserot][Negut]

Large-n limits are manifest in each description!

Nakajima Quiver Varieties

Rep(**v**,**w**) — linear space of quiver reps



$\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map

Mention stability conditions here

Nakajima quiver variety $X = \mu$ $\quad G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

Tensorial polynomials of tautological bundles Vi, Wi and their duals generate *classical T-equivariant K-theory* ring of X

Ex: T*Grassmannian

$\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$

$\mathbf{v}_1 = k, \; \mathbf{w}_1 = n$

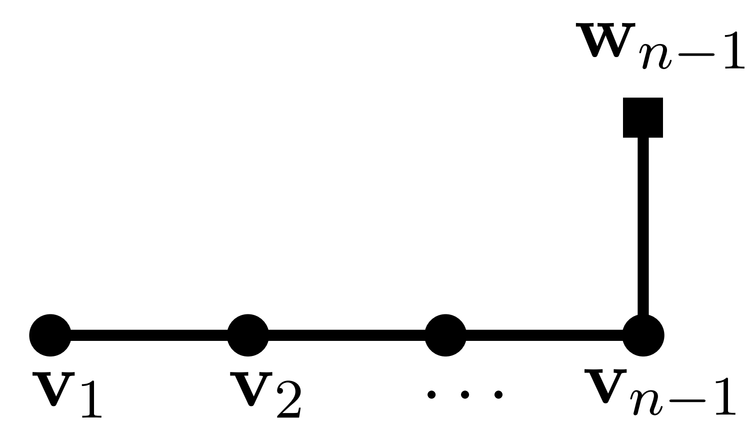
$$\tau(s_1, \cdots, s_k) = (s_1 + \cdots + s_k)^2 - \sum_{1 \le i_1 < i_2 < i_3 \le k} s_{i_1}^{-1} s_{i_2}^{-1} s_{i_3}^{-1}$$

Quasimaps

value of a quasimap defines a map to a quotient stack which contains stable locus as an open subset

Quasimap $f : \mathcal{C} \dashrightarrow X$ is described by collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ with section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$

where $\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$

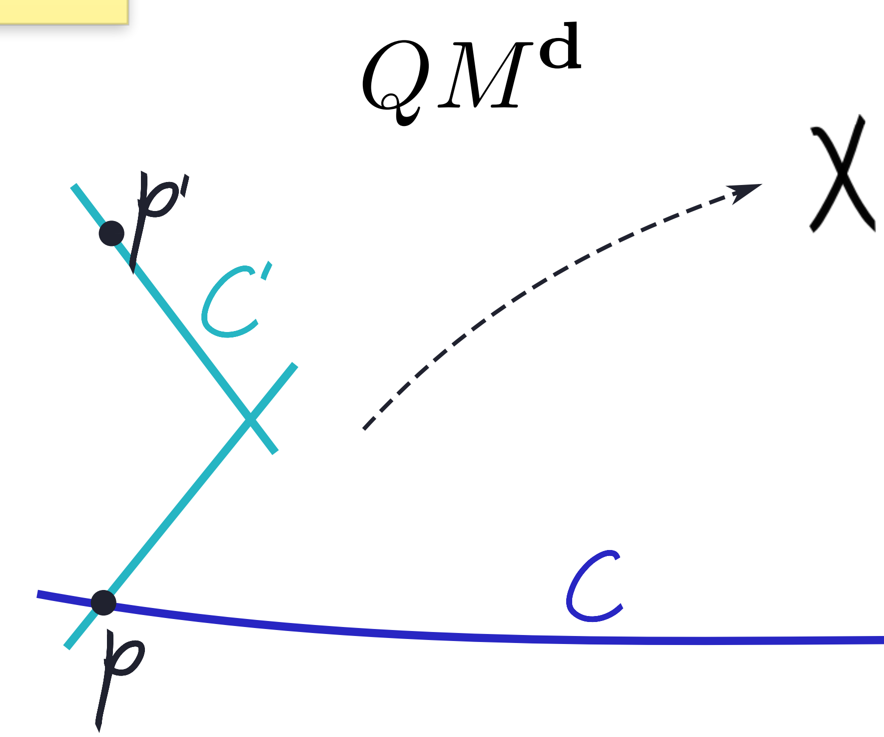


Degree $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$

Skip if a talk is short

Evaluation map

$$\mathrm{ev}_p(f) = f(p) \in [\mu^{-1}(0)/G] \supset X$$



Stable if $f(p) \in X$

for all but finitely many singular points

Resolve to make *proper* ev map

Vertex Function (g

Say this in words: equivariant pushforward, etc.
Moduli space of quasimaps has perfect deformation-obstruction theory.

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q$
base $\mathbb{P}^1$ keeping two fixed points (0, infinity)

Define **vertex function** with quantum (Novikov) parameters $z^{\mathbf{d}} = \prod_{i\in I} z_i^{d_i}$

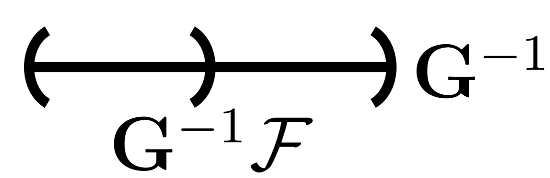
$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\left(QM^{\mathbf{d}}_{\mathrm{nonsing}\, p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathscr{V}_i|_{p_1}) \right) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

[Okounkov]
[Pushkar Smirnov Zeitlin]
[PK Pushkar Smirnov Zeitlin]

Define **quantum K-theory** as a ring with multiplication

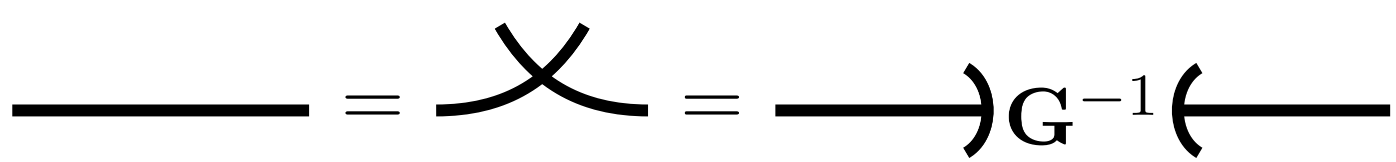
$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathcal{F} \circledast = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3 *} \left(\mathrm{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F}) \widehat{\mathcal{O}}_{\mathrm{vir}} \right) \mathbf{G}^{-1}$$



gluing

$$\mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}$$



Vertex for A-type Quivers

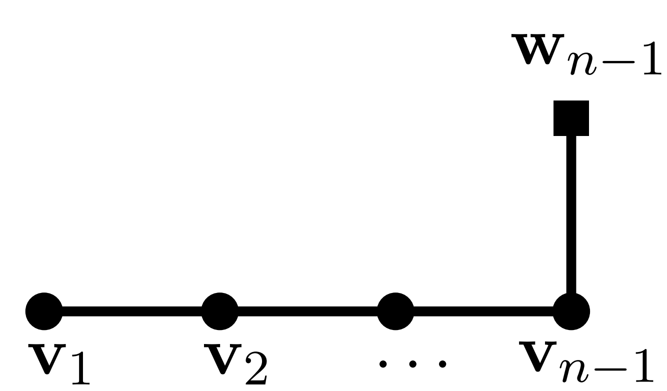
Each equivariant line bundle contributes with

$$\{x\}_d = \frac{(\hbar/x,q)_d}{(q/x,q)_d}\,(-q^{1/2}\hbar^{-1/2})^d, \quad \text{where} \quad (x,q)_d = \frac{\varphi(x)}{\varphi(q^d x)}$$

After classifying fixed points of space of nonsingular quasimaps we can compute the vertex

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j}q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1}\prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}}$$



Bethe Equations

[Nekrasov Shatashvili]

Saddle point approximation provides the operator of quantum multiplication

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V^{(\tau)}_{\boldsymbol{p}}(z)}{V^{(1)}_{\boldsymbol{p}}(z)}$$

For the cotangent bundle to partial flag variety we get

[PK Pushkar Smirnov Zeitlin]

Theorem 3.4. *The eigenvalues of $\hat{\tau}(z)\circledast$ is given by $\tau(s_{i,k})$, where $s_{i,k}$ satify Bethe equations:*

$$\prod_{j=1}^{\mathbf{v}_2} \frac{s_{1,k}-s_{2,j}}{s_{1,k}-\hbar s_{2,j}} = z_1\left(-\hbar^{1/2}\right)^{-\mathbf{v}'_1} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_1} \frac{s_{1,j}-s_{1,k}\hbar}{s_{1,j}\hbar-s_{1,k}}\,,$$

$$\prod_{j=1}^{\mathbf{v}_{i+1}} \frac{s_{i,k}-s_{i+1,j}}{s_{i,k}-\hbar s_{i+1,j}} \prod_{j=1}^{\mathbf{v}_{i-1}} \frac{s_{i-1,j}-\hbar s_{i,k}}{s_{i-1,j}-s_{i,k}} = z_i\left(-\hbar^{1/2}\right)^{-\mathbf{v}'_i} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_i} \frac{s_{i,j}-s_{i,k}\hbar}{s_{i,j}\hbar-s_{i,k}}\,, \tag{23}$$

$$\prod_{j=1}^{\mathbf{w}_{n-1}} \frac{s_{n-1,k}-a_j}{s_{n-1,k}-\hbar a_j} \prod_{j=1}^{\mathbf{v}_{n-2}} \frac{s_{n-2,j}-\hbar s_{n-1,k}}{s_{n-2,j}-s_{n-1,k}} = z_{n-1}\left(-\hbar^{1/2}\right)^{-\mathbf{v}'_{n-1}} \prod_{\substack{j=1\\ j\neq k}}^{\mathbf{v}_{n-1}} \frac{s_{n-1,j}-s_{n-1,k}\hbar}{s_{n-1,j}\hbar-s_{n-1,k}}\,,$$

where $k=1,\dots,\boldsymbol{v}_i$ for $i=1,\dots,\boldsymbol{v}_{n-1}$.

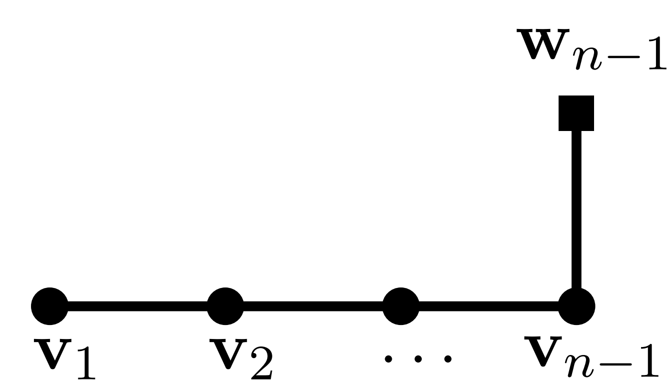
which are Bethe Ansatz Equations for gl(n) XXZ spin chain

K-theory Vertex Functions

After classifying fixed points of space of nonsingular quasimaps we can compute the vertex using the localization theorem

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j}q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1}\prod_{j,k=1}^{\mathbf{v}_i}\{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}} \qquad x_{i,j}\in\{a_1,\dots a_{\mathbf{w}_n}\}$$



Vertex (trivial class)

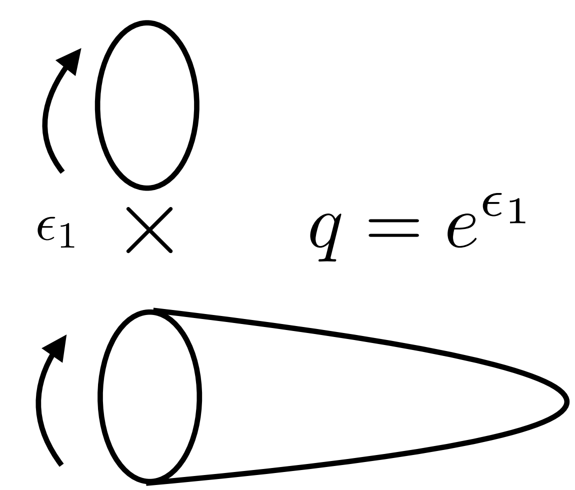
$$V = {}_2\phi_1\left(\hbar, \hbar\frac{a_1}{a_2}, q\frac{a_1}{a_2}; q; z\right) \qquad \mathbf{v}_1 = 1,\ \mathbf{w}_1 = 2$$

Vortex (defet partition function)

$\mathcal{N} = 2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1}\times S^1_\gamma$

Lagrangian depends on twisted masses a_1, a_2

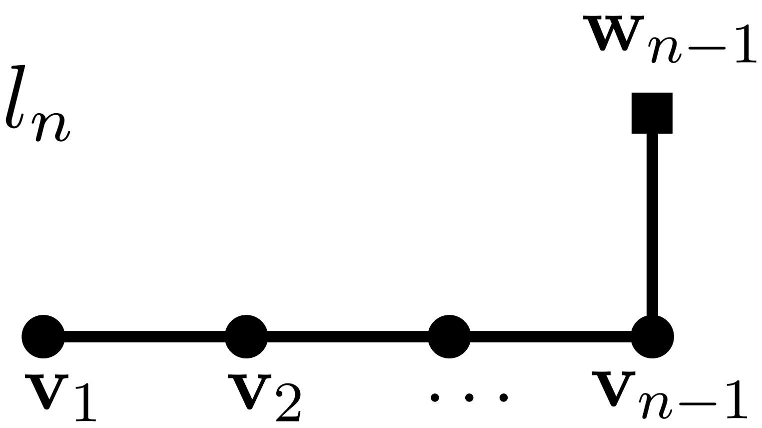
FI parameter z and U(1) R-symmetry $\log\hbar$



Difference Equations

[PK Pushkar Smirnov Zeitlin]
[PK, PK Zeitlin]

$$X = T^*\mathbb{F}l_n$$



Quantum K-theory Ring $q \to 0$

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[z_i^{\pm 1}, a_i^{\pm 1}, T_q^{\pm 1}\hbar, q]}{\mathcal{I}_{\mathrm{tRS}}}$$

The K-theory vertex function satisfies equation of motion of *trigonometric Ruijsenaars-Schneider* model

$$\hat{H}_d V = e_d(z_1, \ldots, z_{n-1}) V$$

$$\hat{H}_d = \sum_{I \subset \{1,\ldots,n\}, |I|=d} \Big(\prod_{i \in I, j \notin I} \frac{a_i \hbar^{\frac{1}{2}} - a_j \hbar^{-\frac{1}{2}}}{a_i - a_j} \Big) \prod_{i \in I} T_i^q$$

3d Mirror version (a.k.a. bispectral dual)

$$\hat{H}_d^! V = e_d(a_1, \ldots, a_{n-1}) V$$

$$\hat{H}_d^!(a_i, \hbar, T_a^q) = \hat{H}_d(z_i/z_{i+1}, \hbar^{-1}, T_z^q)$$

If time will be tight then say in words that A and B are holonomies of electric and magnetic operators

Should be around 50% of time here!!!!

Spherical DAHA

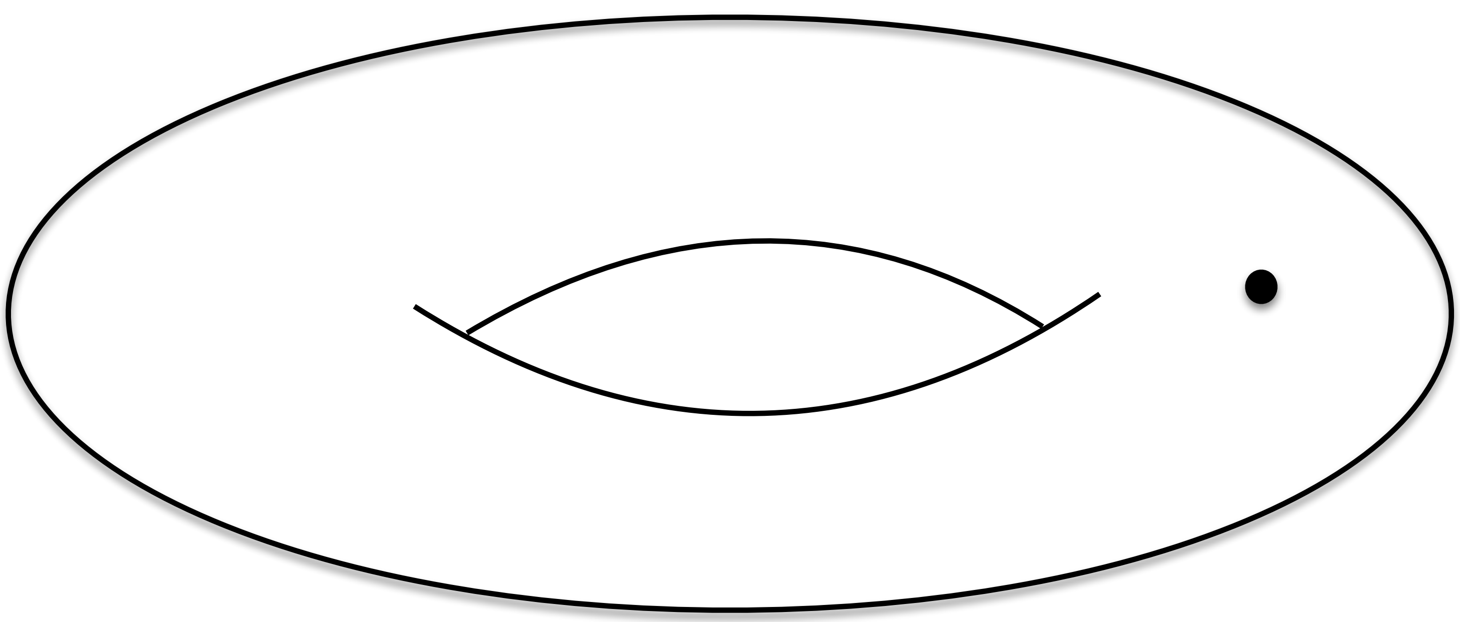
ijsenaars-Schneider Hamiltonians form a maximal commuting subalgebra inside **spherical double affine Hecke algebra for gl(n)**

$$\{\hat{H}_1, \ldots, \hat{H}_n\} \subset \mathrm{DAHA}^{\mathfrak{S}_n}_{q,\hbar}(\mathfrak{gl}_n) =: \mathcal{A}_n$$

$\hat{H}_d$ are also known as Macdonald operators

[Oblomkov]

Spherical gl(n) DAHA is a **deformation quantization** of the moduli space of flat GL(n;C) connections on a torus with one simple puncture



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \mathrm{diag}(\hbar, \ldots, \hbar, \hbar^{1-n})$$

$$\mathcal{A}_n = \widehat{\mathbb{C}_J[\mathcal{M}_n]}$$

Line Operators and Branes

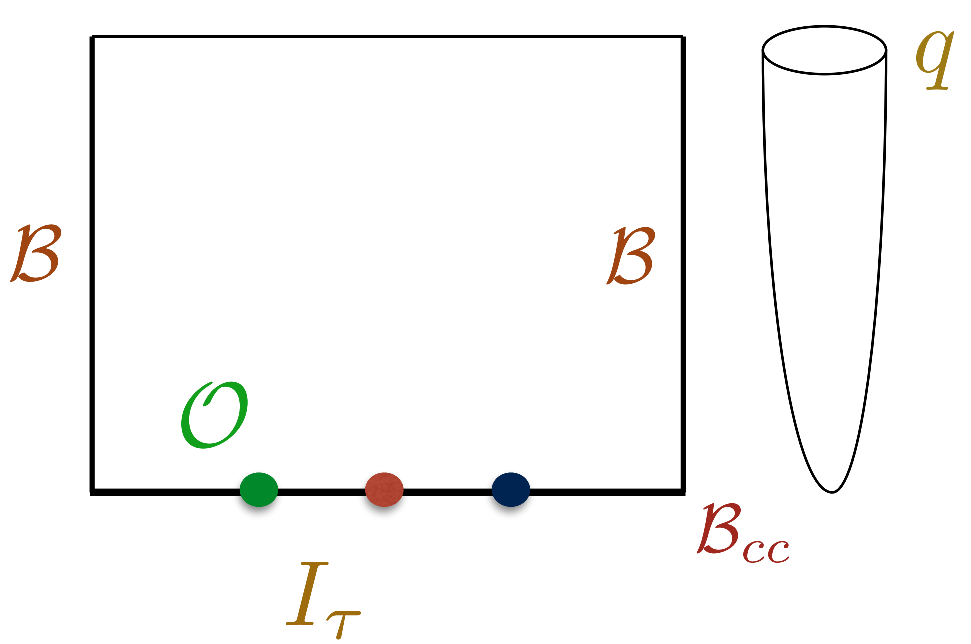
$\mathcal{M}_n$ is the moduli space of vacua in $\mathcal{N}$=2* gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n) and is described by VEVs of line operators wrapping the circle.

A and *B* are holonomies of *electric* and *magnetic* line operators

Omega background along real 2-plane $\mathbb{R}^2_q \times \mathbb{R} \times S^1$

Line operators are forced to stay at the tip of the cigar and slide along the remaining line, hence **non-commutativity**

[Gukov-Witten]
[Nekrasov-Witten]



algebra — open strings

$$\mathcal{A}_n = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$$

representations

(Hilbert space of SUSY QM)

$$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B})$$

Hitchin Moduli Space (n=2)

SU(2) theory → sl(2) flat connections

[Gukov]
[PK Gukov Nawata Saberi]

$x = TrA$ $y = TrB$ $z = TrAB$

electric   *magnetic*   *dyonic*

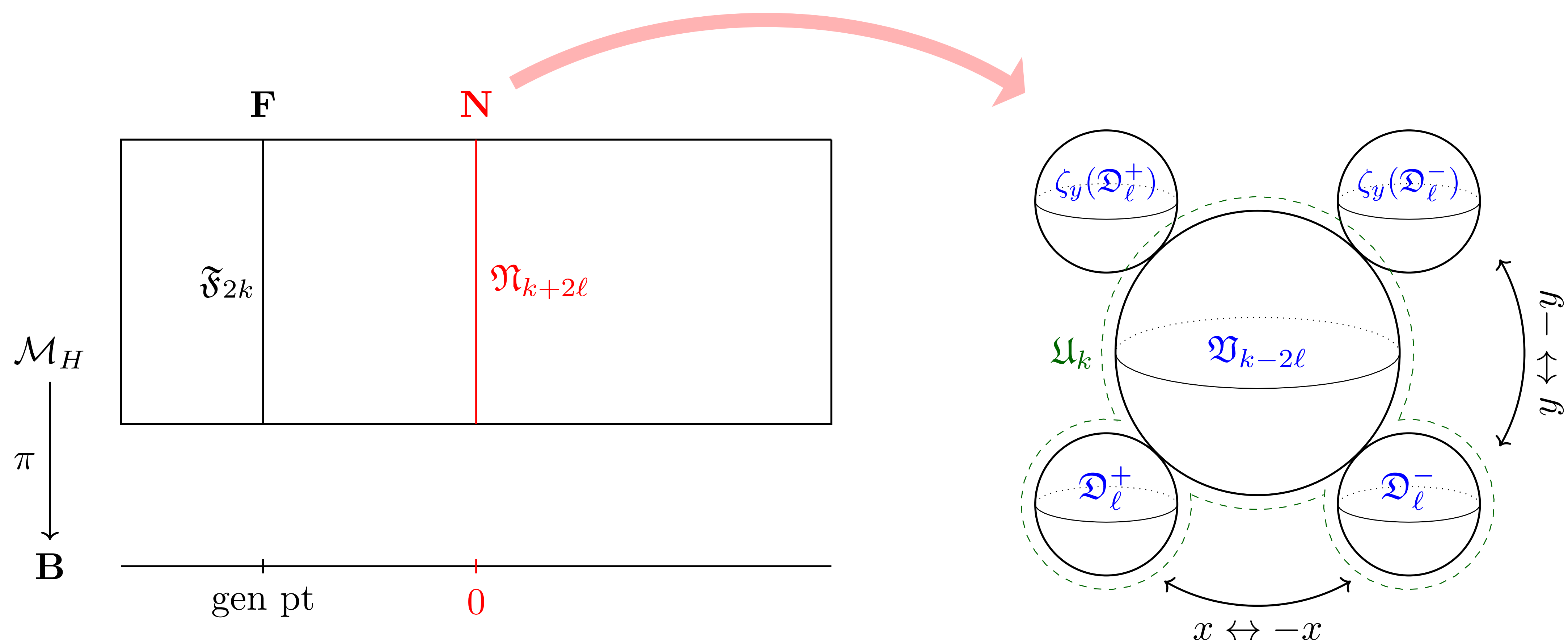
Nonabelian Hodge correspondence:

$$\mathcal{M}_{\text{flat}}(SL(2;\mathbb{C}),T^2\backslash\{\text{pt}\}) \simeq \mathcal{M}_H(SU(2),T^2\backslash\{\text{pt}\})$$

$$\mathcal{M}_H: \quad x^2+y^2+z^2+xyz=\hbar+\hbar^{-1}+2$$

for $\hbar=1$ $\mathcal{M}_n \simeq \dfrac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2}$

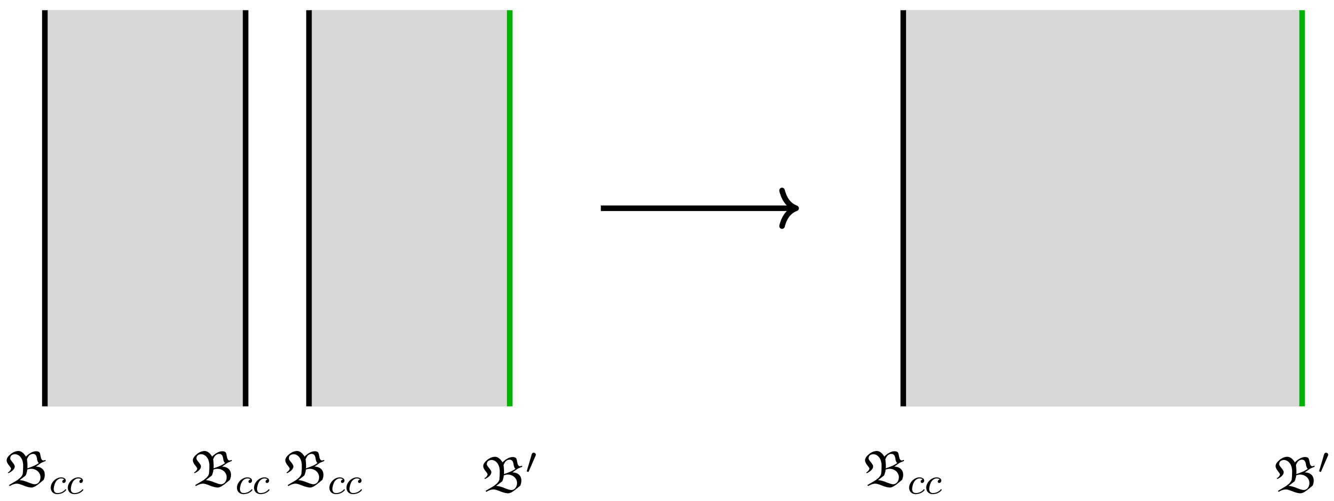
Elliptic fibration with one singular fiber of Kodaira type I_0^*



DAHA Modules

[Kapustin Orlov]
[Kapustin Witten]

Algebra acts naturally by attaching open strings to closed strings



Hilbert space comes from (Bcc, B') strings

$$\begin{array}{ccl} \mathcal{A} & = & \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{cc}) \\ \circlearrowright & & \qquad\circlearrowright \\ \mathcal{H} & = & \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}') \end{array}$$

$\mathcal{B}' \to \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ gives a functor $\mathrm{Hom}(\mathcal{B}_{cc}, \cdot)$

$$\boxed{\mathrm{Fuk}(\mathcal{M}, \Omega) \simeq \mathrm{Rep}(\mathcal{A})}$$

$$\mathfrak{B}_{cc} : \mathcal{L} \to \mathcal{M}_H$$

$$F + B = \frac{i}{\log q}\Omega_J$$

$$\Omega_J = \frac{dx \wedge dy}{2z - xy}$$

Algebra-deformation quantization of functions on M_H

$$[x, y]_q = (q - q^{-1})z$$
$$[z, x]_q = (q - q^{-1})y$$
$$[y, z]_q = (q - q^{-1})x$$

Lagrangian A-brane $\longrightarrow$ Module of DAHA

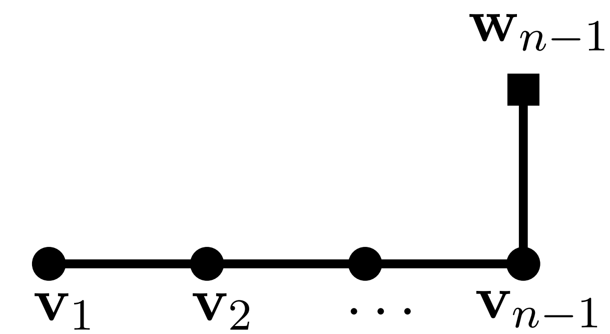
Dimension of a module

$$\mathrm{dim}V = \int_{\mathcal{M}} \mathrm{ch}(\mathcal{B}') \wedge \mathrm{ch}(\mathcal{B}_{cc}) \wedge \mathrm{Td}(\mathcal{M})$$

compact branes $\longleftrightarrow$ Finite dim reps

DAHA Reps

Start with a vertex function for T*Fn



Specify equivariant parameters $a_k = q^{\lambda_k} \hbar^{n-k}$

q-hypergeometric series → Macdonald polynomials with $\hbar = t^{-1}$

E.g. k=2, n=2

$\mathbf{v}_1 = 1,\ \mathbf{w}_1 = 2$

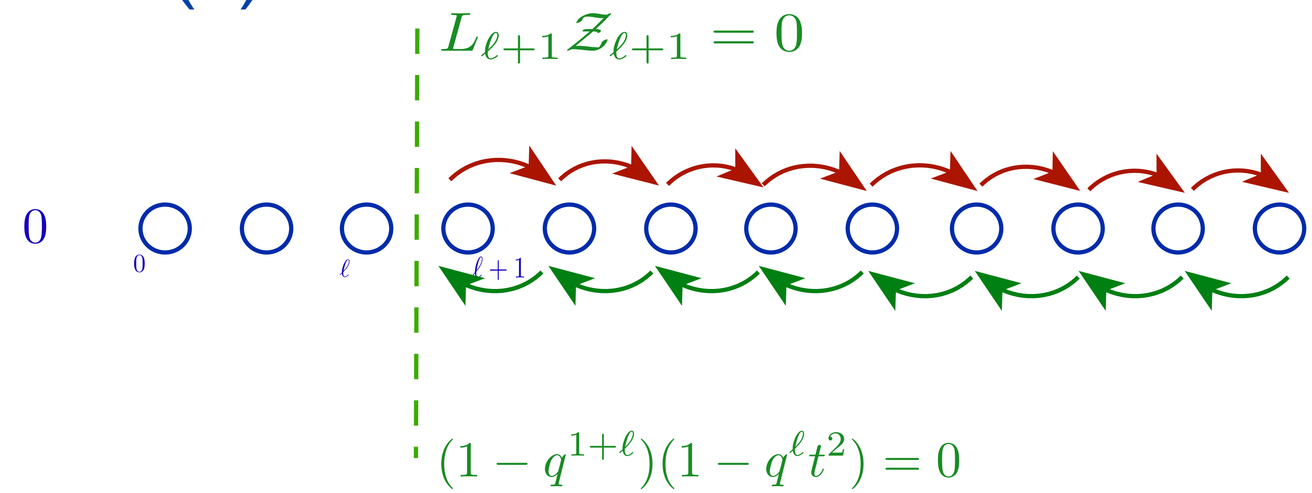
$$V(z;\hbar q, q) = P_{(1,1)}(z|q,\hbar)$$

$$V(z;\hbar q^2, q) = P_{(2,0)}(z|q,\hbar)$$

Raising and lowering operators of sl(2) DAHA

$$R_a = x + a_k^{-1} z$$
$$L_a = x + a_k z$$

$$R_a \mathcal{Z}_a = r_a \mathcal{Z}_{a+1}$$
$$L_a \mathcal{Z}_a = l_a \mathcal{Z}_{a-1}$$

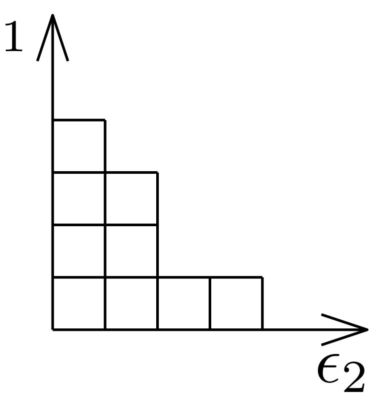


$$0 \longrightarrow \iota(\mathfrak{D}_k^+ \oplus \mathfrak{D}_k^-) \longrightarrow \mathfrak{U}_N \longrightarrow \mathfrak{V}_{n+1} \longrightarrow 0$$

Fock Space

$$q = e^{\epsilon_1}$$

$$\hbar = e^{\epsilon_2}$$



Power-symmetric variables $\qquad p_m = \sum_{l=1}^{n} z_l^m$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

Starting with Fock vacuum $\qquad |0\rangle$

Construct Hilbert space $\qquad a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $\qquad a_{-\lambda}|0\rangle = a_{-\lambda_1}\cdots a_{-\lambda_l}|0\rangle$

Commutators $\qquad [a_m, a_n] = m\frac{1-q^{|m|}}{1-\hbar^{|m|}}\delta_{m,-n}$

DAHA Action

[PK 1805.00986]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from P1 to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} \hbar^{n-k}$ restricts us to the Fock space representation of (q,h)-Heisenberg algebra which is a DAHA module

In other words, we can define the following action

$$
\begin{tikzcd}
\mathcal{A}_n \arrow[d, bend right] & & \mathfrak{E} \arrow[ll, "\text{[Schiffmann Vaserot]}"'] \arrow[d, bend right] \\
K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)\Big|_{a_k = q^{\lambda_k}\hbar^{n-k}} \arrow[rr] & & K_{q,\hbar}(\oplus_k \mathrm{Hilb}^k[\mathbb{C}^2])
\end{tikzcd}
$$

λ not more than n columns

$\mathbb{C}^\times_q \times \mathbb{C}^\times_\hbar$ fixed points are Macdonald polynomials

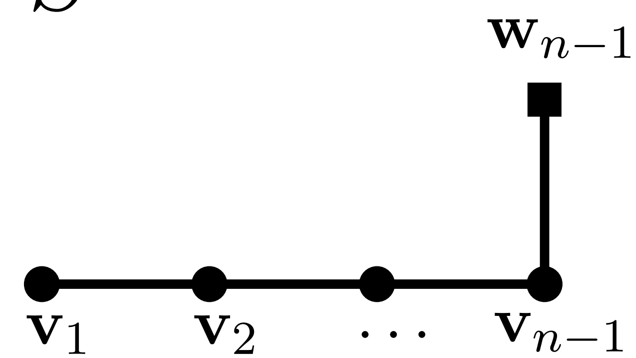
M-theory Description

Recall that $\mathrm{Hilb}^k[\mathbb{C}^2] = \mathcal{M}^{\mathrm{inst}}_{1,k}$ **How did U(1) 5d SYM appear?**

Starting with M-theory on $S^1 \times \mathbb{C}_q \times \mathbb{C}_\hbar \times T^* S^3$

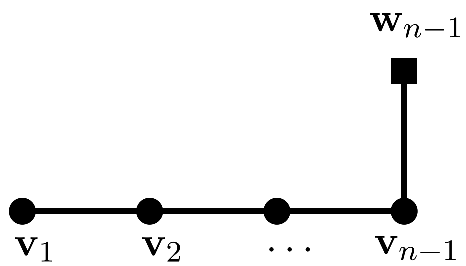
n M5 branes wrapping $S^1 \times \mathbb{C}_q \times S^3$

Upon compactification on three sphere will get *3d quiver gauge theory* on T*Fl$_n$



When n becomes large the background undergoes through the **conifold transition** and the *resolved* conifold becomes a *deformed* conifold Y: $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times Y$

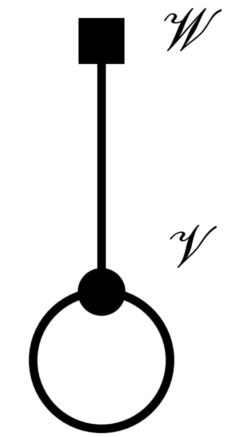
Reduction on Y leads us to a 5d U(1) theory with 8 supercharges



Flags vs ADHM

[PK Sciarappa]
[PK]

$K_T(\mathbf{QM}(\mathbb{P}^1, X))$	$K_{q,\hbar}(\mathrm{Hilb}(\mathbb{C}^2))$
Kähler/quantum parameters of X $z_1, z_2 \ldots$	Ring generators $x_1, x_2, \ldots$
Vertex function $\mathsf{V}_{\mathbf{q}}$	Classes of $(\mathbb{C}^\times)^2$ fixed points $[\mathcal{J}]$
$\mathbb{C}_q^\times$ acting on base curve	$\mathbb{C}_q^\times$ acting on $\mathbb{C} \subset \mathbb{C}^2$
$\mathbb{C}_\hbar^\times$ acting on cotangent fibers of X	$\mathbb{C}_\hbar^\times$ acting on another $\mathbb{C} \subset \mathbb{C}^2$
Eigenvalues e_r of tRS operators T_r	Chern polynomials $\mathcal{E}_r$ of $\Lambda^r \mathcal{U}$



quantum deformation:

Eigenvalues of **elliptic** RS model at large n

$$E_r(\vec{\zeta}) = \sum_{\substack{\mathcal{I} \subset \{1,\ldots,n\} \\ |\mathcal{I}| = r}} \prod_{\substack{i \in \mathcal{I} \\ j \notin \mathcal{I}}} \frac{\theta_1(\hbar \zeta_i / \zeta_j | \mathfrak{p})}{\theta_1(\hbar \zeta_i / \zeta_j | \mathfrak{p})} \prod_{i \in \mathcal{I}} p_k$$

Eigenvalues of **quantum** multiplication by

$$\mathscr{U} = \mathscr{W} + (1-q)(1-\hbar)\mathscr{V}|_{\mathcal{J}_{\vec{\lambda}}}$$

Chern roots obey

$$\prod_{l=1}^{N} \frac{s_a - \mathsf{a}_l}{s_a - q^{-1}\hbar^{-1}\mathsf{a}_l} \cdot \prod_{\substack{b=1 \\ b \neq a}}^{k} \frac{s_a - q s_b}{s_a - q^{-1} s_b} \frac{s_a - \hbar s_b}{s_a - \hbar^{-1} s_b} \frac{s_a - q^{-1}\hbar^{-1} s_b}{s_a - q\hbar s_b} = \mathfrak{z}$$

Quiver W-algebras

[Kimura Pestun]

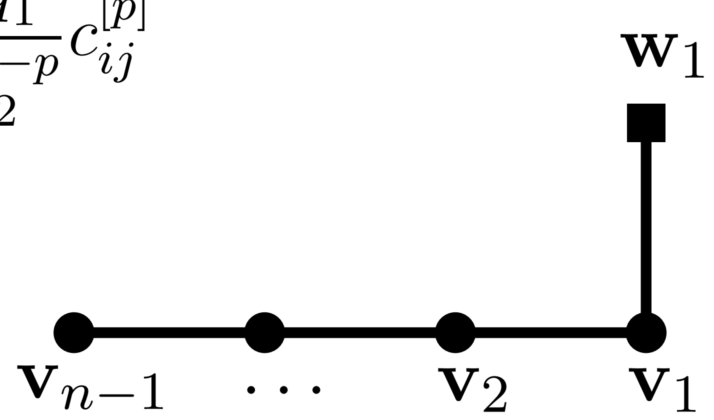
BPS/CFT correspondence from localization and operator formalism

qW algebra

Construction of qW algebra from free-boson representation of extended Nekrasov partition function [Kimura Pestun]

$$\mathcal{Z}_{\text{Nek}} = \widehat{\mathcal{Z}}_{\text{Nek}}|0\rangle \qquad [s_{i,p}, s_{j,p'}] = -\delta_{p+p',0}\frac{1}{p}\frac{1-q_1^p}{1-q_2^{-p}}c_{ij}^{[p]}$$

Start with quiver gauge theory on $\mathbb{C}_{q_1} \times \mathbb{C}_{q_2} \times S^1$



Moduli space of vacua is the space of A_{n-1} periodic monopoles with $\mathbf{w}_1$ Dirac singularities whose charges are given by the number of colors [Nekrasov Pestun Shatashvili]

Quantization of this moduli space in carefully chosen complex structure gives qW(q1,q2) algebra modulo Virasoro constraints!

$$\widehat{\mathbb{C}}[\mathcal{M}_{\text{mon}}] = \frac{qW_{q_1,q_2}}{\text{Vir}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1})} \qquad T_{i,-k}|\psi\rangle = 0, \quad k > \mathbf{v}_i$$

'Double' quantization

There is an integrable systems associated with the moduli space of periodic monopoles. Its classical description is given by the Seiberg-Witten curve of the theory. [Nekrasov, Pestun]

[Nekrasov, Pestun, Shatashvili]

In order to **quantize** the integrable system one turns on one of the Omega background parameters — $\mathbf{q_1}$ $[\mathcal{H}_i, \mathcal{H}_j] = 0$

Finally, **qW-algebra/Vir** for the quiver is the $\mathbf{q_2}$-deformation of the ring of commuting Hamiltonians of the quantum integrable system [cf with Aganagic Frenkel Okounkov]

Virasoro constrains can be removed by taking $\mathbf{v}_i \to \infty$

We shall provide physical and geometric interpretation of both Virasoro constraints and the limit

Partition Function

Y-observables and conjugate higher times

$$Z_{\mathsf{T}}(t) = \mathrm{ch}_{\mathsf{T}}\, \mathfrak{q}^{\gamma} \pi_{!} \prod_{i \in \Gamma_0} [\det \hat{\mathbf{Y}}_i]^{\kappa_i} \exp\Big(\sum_{p=1}^{\infty} t_{i,p} \mathbf{Y}_i^{[p]}\Big)$$

Equivariant localization yields the sum over fixed points of the torus action $\mathcal{X}_i = \{x_{i,\alpha,s_1}\}_{\alpha \in [1 \ldots \mathsf{n}_i], s_1 \in [1 \ldots \infty]}$, $\quad \mathcal{X} = \sqcup_{i \in \Gamma_0} \mathcal{X}_i \quad$ $x_{i,\alpha,s_1} = \nu_{i,\alpha} q_1^{s_1 - 1} q_2^{\lambda_{s_1}}$

State can be thought of as ordered product of screening charges

$$|Z_{\mathsf{T}}\rangle = \sum_{\mathcal{X} \in \mathfrak{M}^{\mathsf{T}}} \prod_{x \in \mathcal{X}}^{\succ} S_{\mathsf{i}(x),x} |1\rangle \qquad S_{i,x} =: \exp\Big(\sum_{p>0} s_{i,-p} x^p + s_{i,0} \log x + \tilde{s}_{i,0} + \sum_{p>0} s_{i,p} x^{-p}\Big):$$

with familiar commutation relations $\quad [s_{i,p}, s_{j,p'}] = -\delta_{p+p',0} \dfrac{1}{p} \dfrac{1 - q_1^p}{1 - q_2^{-p}} c_{ij}^{[p]}$

qW-algebra is recovered from $\quad [S(x), T(x)] = 0$

Gauge Origami

Why do these two completely different theories lead to exactly the same algebra with some strange identification of parameters?

$$q \leftrightarrow q_1 \qquad t \leftrightarrow q_2$$

Extended partition function satisfied qKZ equation which is related to tRS model we have talked about earlier

Large-n limits were needed in the previous two stories, however they were applied to different observables

Gauge Origami [Nekrasov]

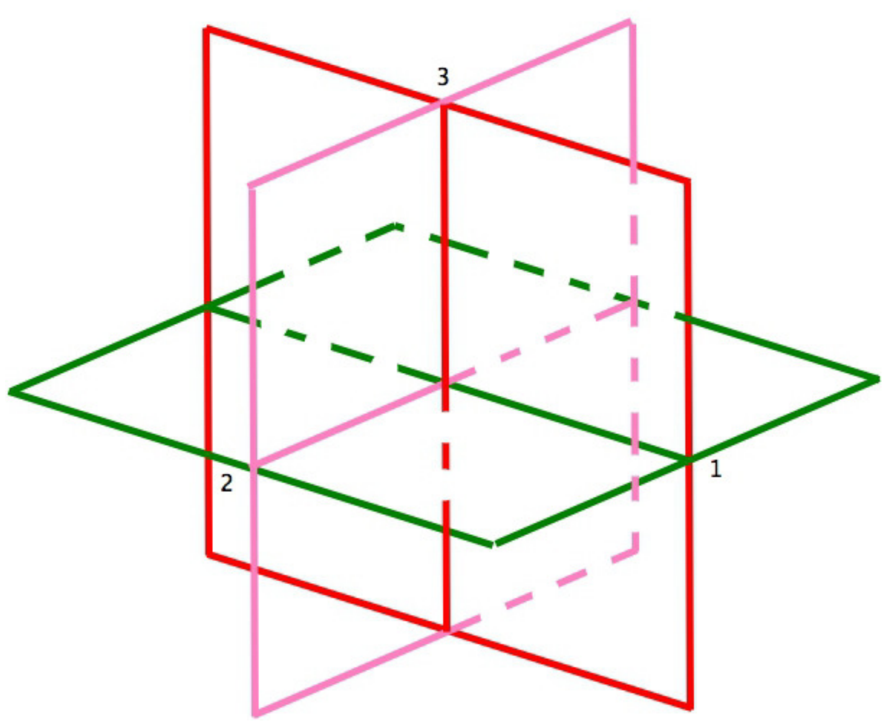
Type IIB on Calabi-Yau 4 $\mathcal{X}_4 \times \Sigma$

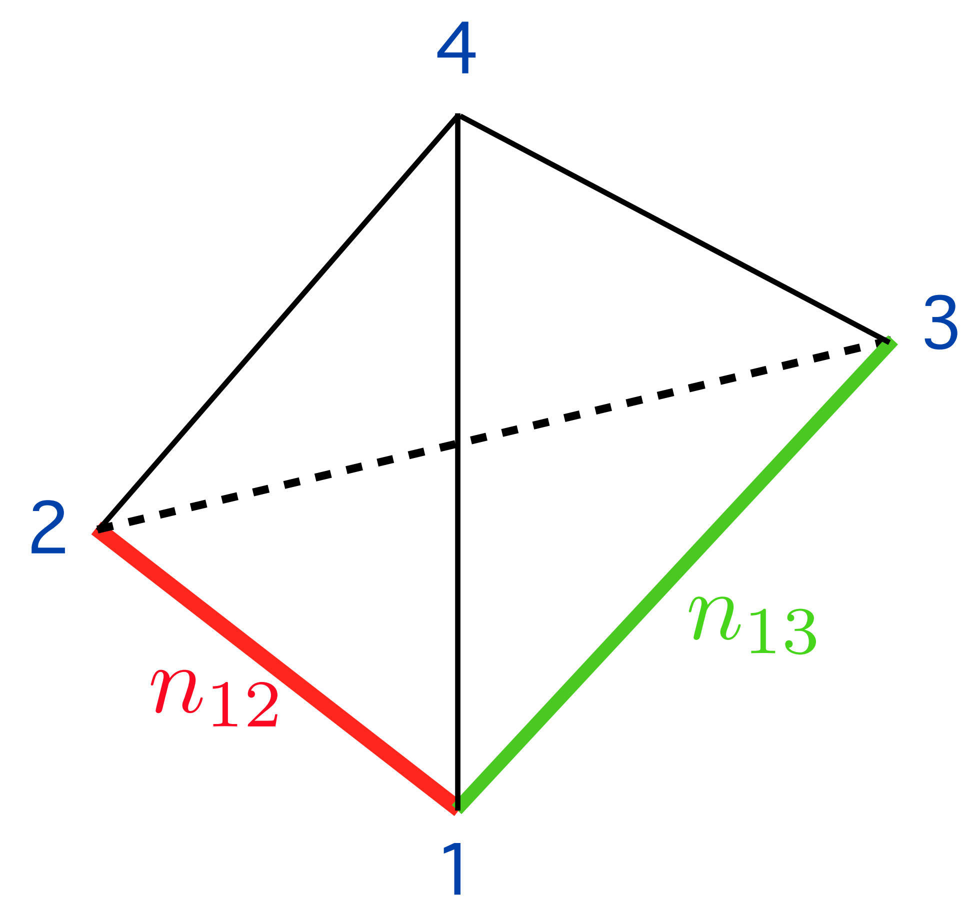
singular hypersurface $Z_2 \subset \mathcal{X}_4$

Wrap D3 branes on 2-planes in Z_2 pointlike on Σ

Local model: $\cup_{a<b} \mathbb{C}^2_{ab} \subset \mathbb{C}^4$

For example, when $1 \le a, b \le 3$





$$\mathcal{X}_4 = \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3} \times \mathbb{C}_{\epsilon_4} \qquad \sum_a \epsilon_a = 0$$

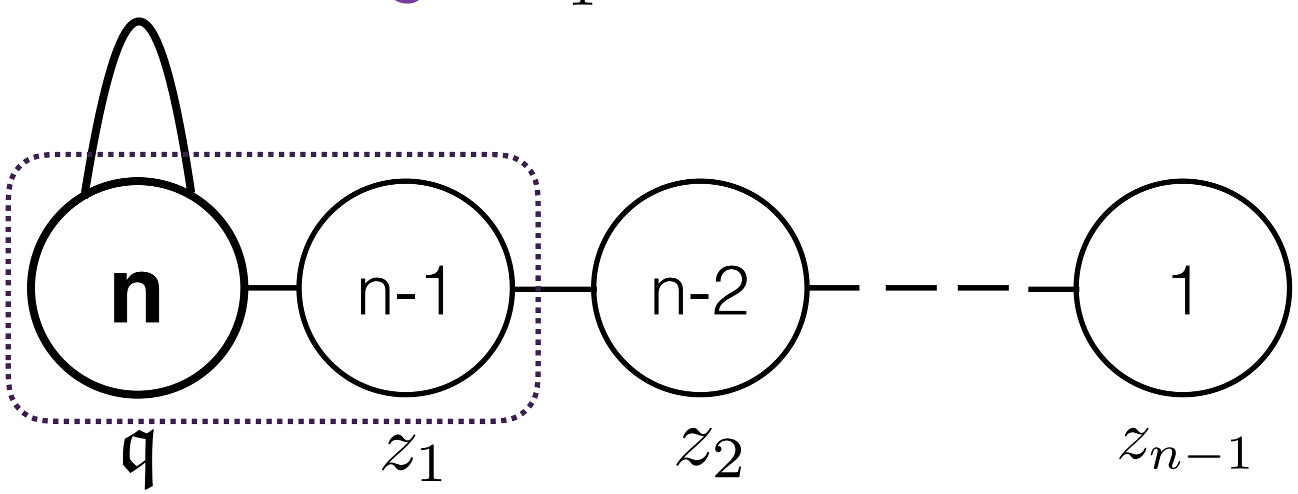
Folded Instantons

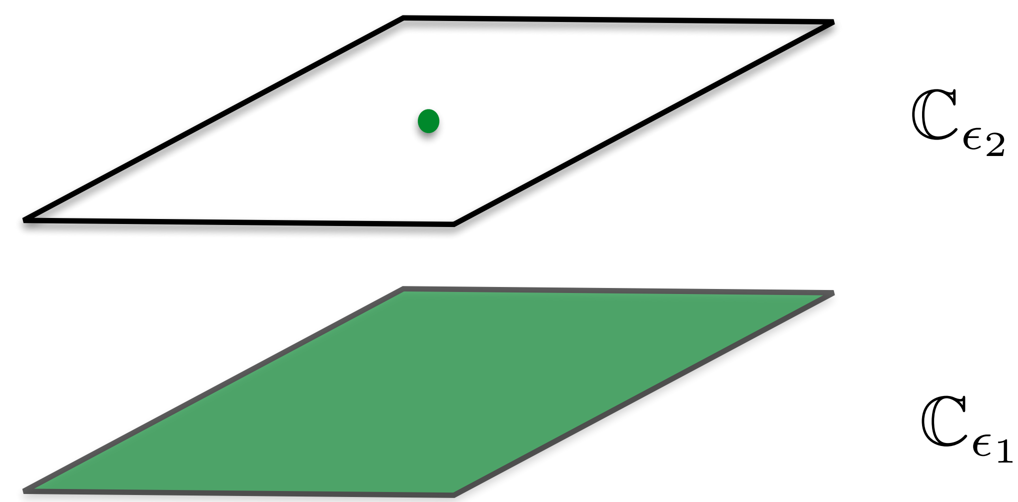
Take n_{12}=n, n_{13} = 2

In the presence of Abelian orbifold

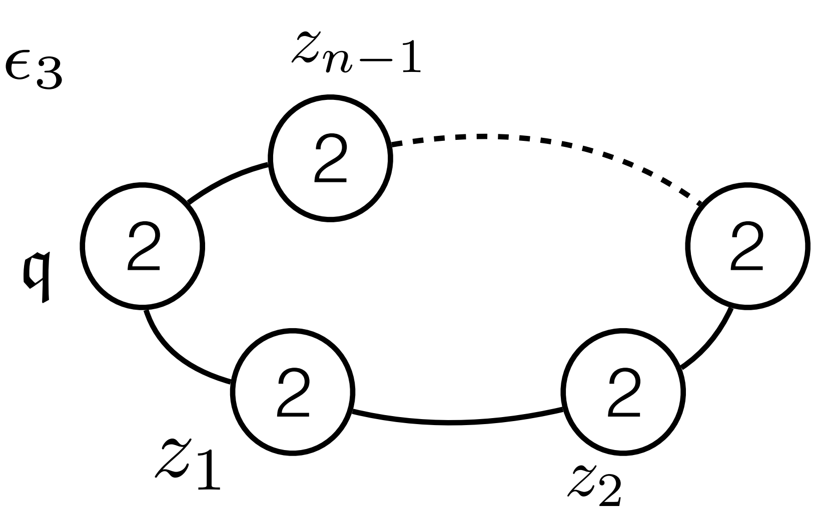
$$\Gamma = \mathrm{diag}(1\ \omega\ 1\ \omega^{-1}) \qquad \omega^n = 1$$
$$\epsilon_1\,\epsilon_2\,\epsilon_3\,\epsilon_4$$

Produces U(n) $\mathcal{N}$=1* theory on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$ with maximal monodromy defect along $\mathbb{C}_{\epsilon_1}$ and adjoint mass ϵ_3





Together with necklace quiver with n U(2) gauge groups on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_3}$



Gauge coupling constants

W-algebras from Origami

Origami partition function combines instanton and perturbative data of both theories

$$\mathcal{Z}^{\Gamma} = \mathcal{Z}^{\text{pert}} \cdot \sum_{\lambda} \left[\prod_{\omega \in \Gamma^{\vee}} \mathfrak{q}_{\omega}^{k_{\omega}} \right] \varepsilon \left[-\tilde{T}_{\lambda}^{\Gamma} \right]$$

Taking limits $\mathfrak{q} \to 0\,, \qquad \epsilon_2 \to 0$

we get 3d quiver defect gauge theory T*Fln on $\mathbb{C}_{\epsilon_1} \times S^1$

and finite linear 5d quiver on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_3} \times S^1$

Locus $a_k = q_1^{\lambda_k} q_3^{n-k}$ truncates vortex functions to polynomials and simultaneously Higgses the 5d theory (truncates instanton series)

$$[a_i, a_j] = \frac{1}{j}\delta_{i+j,0}\frac{1-q_1^{|j|}}{1-q_2^{|j|}}$$

ADHM & 1/2 ADHM

[PK Koroteeva Gorsky Vainshtein]

$$K_\hbar(T^*\mathbb{F}l_n) \longleftrightarrow$$ ADHM (**instanton** moduli space)

$$\lim_{n\to\infty}\left[\hbar^{n-1}(1-\hbar)\left\langle W_{\square}^{U(n)}\right\rangle\right]\Big|_\lambda = a-(1-q)(1-\hbar)e_1(s_1,\dots,s_k)|_\lambda$$

Claim: $\hbar \to \infty$ retracting the fibers, dimensional transmutation

[Hanany Tong]

$$K(\mathbb{F}l_n) \longleftrightarrow$$ 1/2 ADHM (**vortex** moduli space)

Eigenvalues of **affine** qToda lattice at large n $\longleftrightarrow$ Eigenvalues of **quantum** multiplication by

$$\mathcal{E}_1^\Lambda(\lambda) = a-(1-q)e_1(s_1,\dots,s_k)$$

$$H_1^{\text{aff}} = \mathfrak{p}_1\left(1-\mathfrak{p}^\Lambda\frac{\mathfrak{z}_n}{\mathfrak{z}_1}\right)+\sum_{i=2}^n \mathfrak{p}_i\left(1-\frac{\mathfrak{z}_{i-1}}{\mathfrak{z}_i}\right)$$

Chern roots obey

$$\prod_{l=1}^N (s_a-\mathrm{a}_l)\cdot\prod_{\substack{b=1\\ b\neq a}}^k \frac{qs_a-s_b}{s_a-qs_b} = \widetilde{\mathfrak{p}}^\Lambda$$

Subscheme $\mathscr{Z}_k \subset \mathrm{Hilb}^k[\mathbb{C}^2]$

q-Heisenberg algebra preserving $\oplus_k K_q(\mathscr{Z}_k)$