BPS/CFT and Large-n Limit

Peter Koroteev

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

THE UNIVERSITY OF CALIFORNIA DAVIS

Talk at Great Lakes String Meeting
University of Cincinnati May 6th 2017

Large-N Approach

Gauge theories are known to have effective descriptions when the rank of the gauge group becomes large $U(N)$ $N \to \infty$

For supersymmetric gauge theories we expect to do exact computations in the effective large-N theory

There are similar ideas which work in mathematics (stable limits)

String theory is a powerful tool study Large-N dualities

AdS/CFT, Gopakumar-Vafa BPS/CFT (i.e. AGT)

N=2 Theories at Large-n

We want to apply the idea of large-N asymptotics to theories or with 8 supercharges, such as N=2 in 4d or N=1 in 5d, in particular to those which have Seiberg-Witten description

Along the way we want to understand the origins of the **BPS/CFT** correspondence, which relates BPS counting in gauge theories with CFT correlators.

Starting from a gauge theory whose gauge group has finite rank we expect to see how the CFT ***emerges*** when rank becomes large

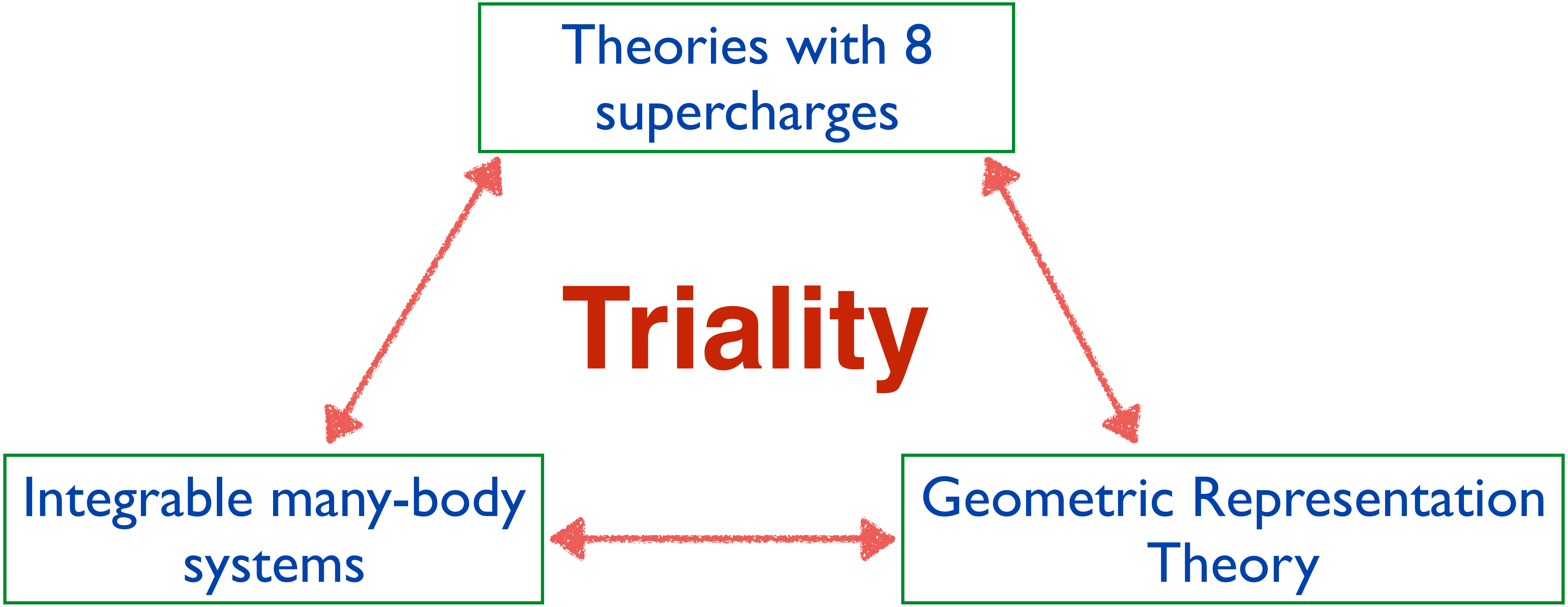
N=2 Theories at Large-n

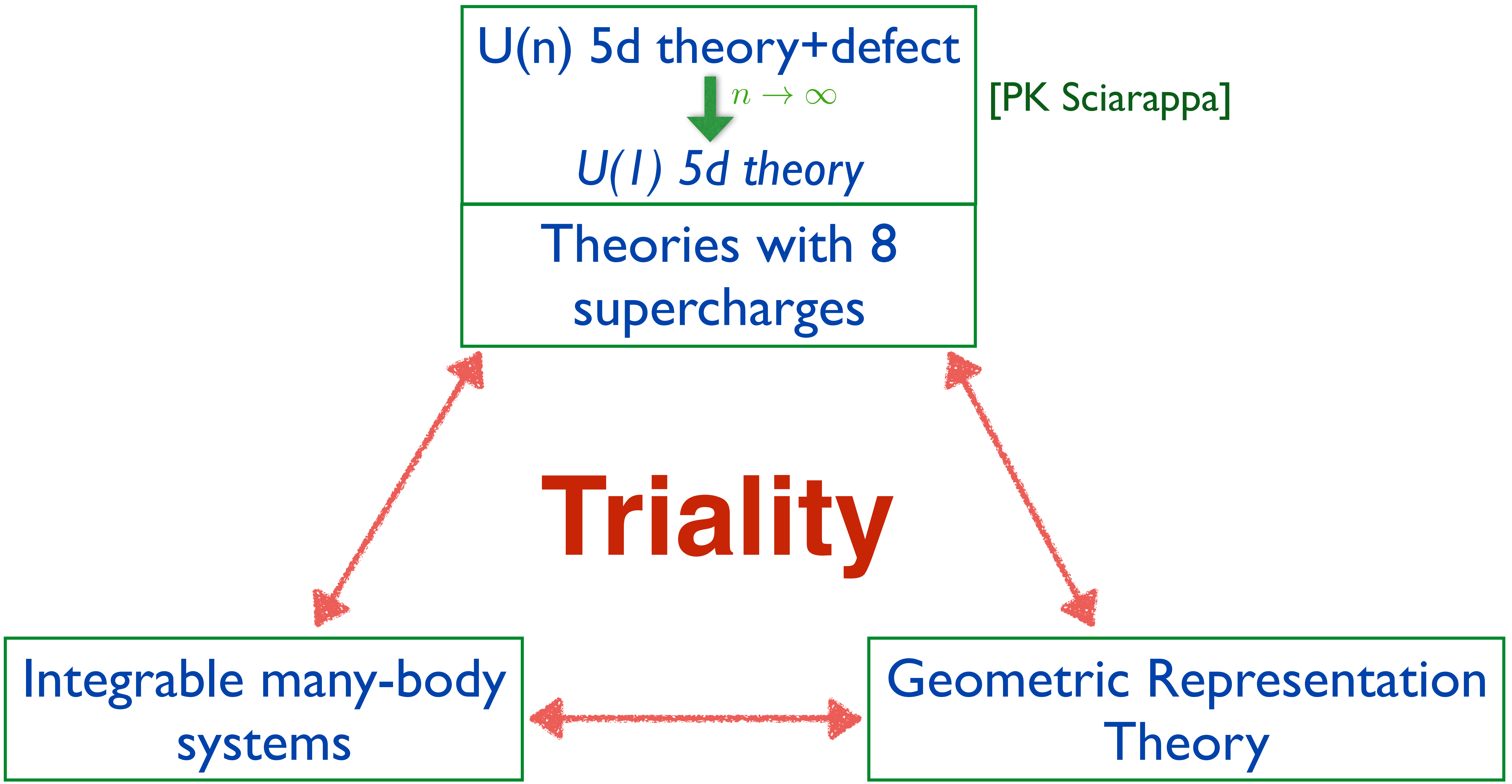
We want to apply the idea of large-N asymptotics to theories or with 8 supercharges, such as N=2 in 4d or N=1 in 5d, in particular to those which have Seiberg-Witten description

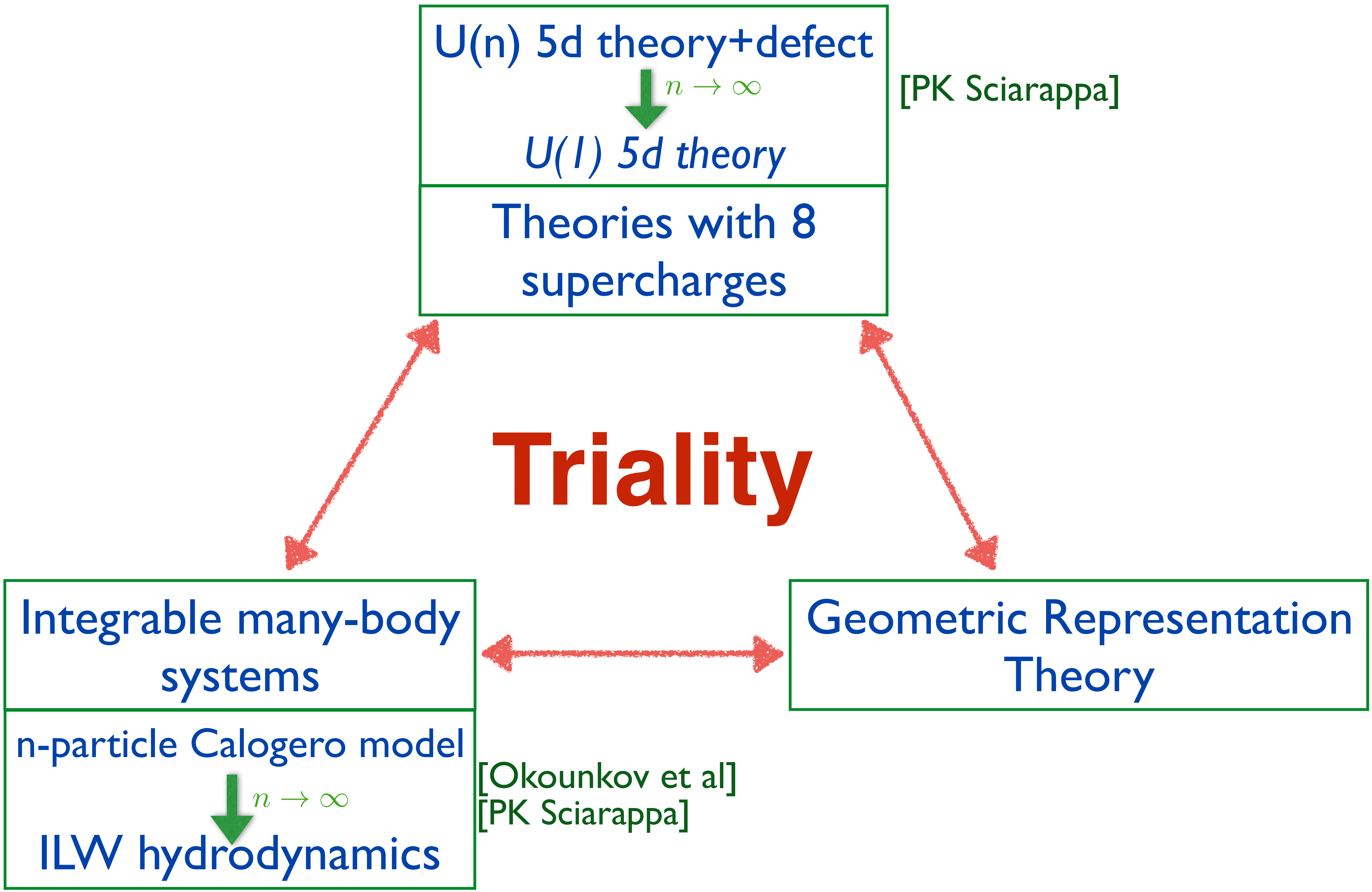
Along the way we want to understand the origins of the **BPS/CFT** correspondence, which relates BPS counting in gauge theories with CFT correlators.

Starting from a gauge theory whose gauge group has finite rank we expect to see how the CFT ***emerges*** when rank becomes large

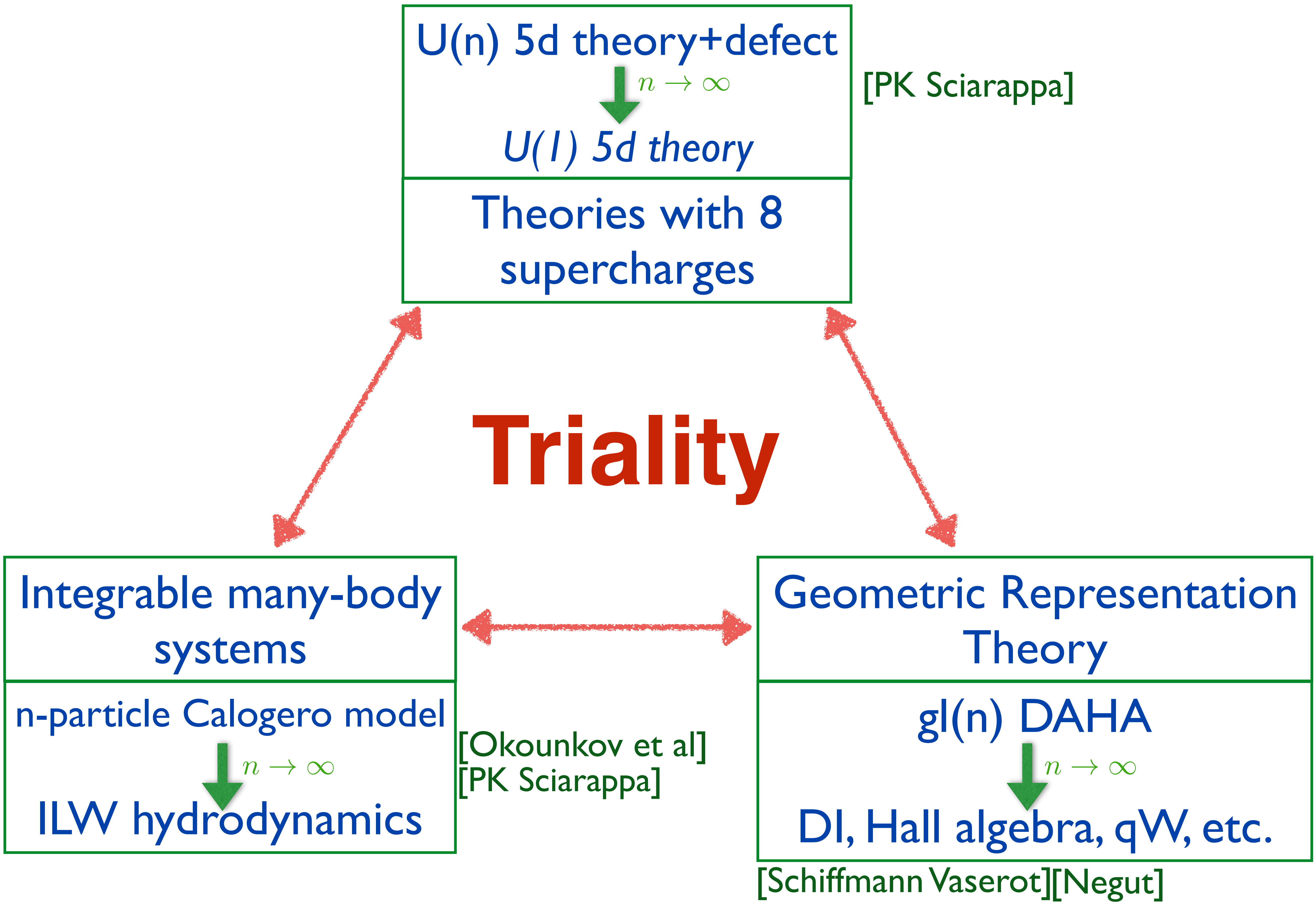
It is a part of a bigger story...

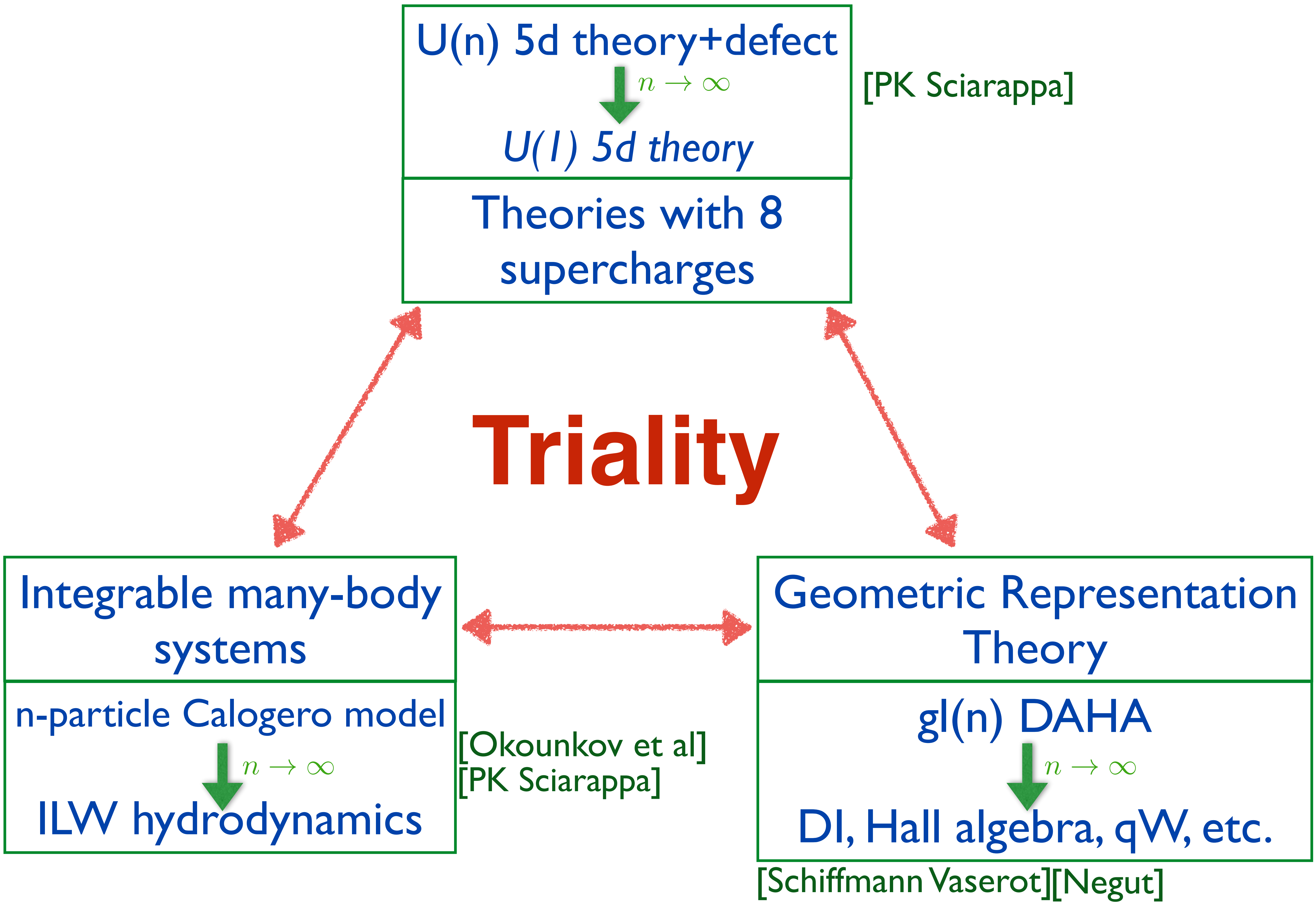

Theories with 8 supercharges
Triality
Integrable many-body systems
Geometric Representation Theory


U(n) 5d theory+defect
$n \to \infty$
U(1) 5d theory
[PK Sciarappa]
Theories with 8 supercharges
Triality
Integrable many-body systems
Geometric Representation Theory



U(n) 5d theory+defect
$n \to \infty$
U(1) 5d theory
Theories with 8 supercharges
[PK Sciarappa]
Triality
Integrable many-body systems
n-particle Calogero model
$n \to \infty$
ILW hydrodynamics
[Okounkov et al]
[PK Sciarappa]
Geometric Representation Theory


U(n) 5d theory+defect
$n \to \infty$
U(1) 5d theory
Theories with 8 supercharges
[PK Sciarappa]
Triality
Integrable many-body systems
n-particle Calogero model
$n \to \infty$
ILW hydrodynamics
[Okounkov et al]
[PK Sciarappa]
Geometric Representation Theory
gl(n) DAHA
$n \to \infty$
DI, Hall algebra, qW, etc.
[Schiffmann Vaserot][Negut]

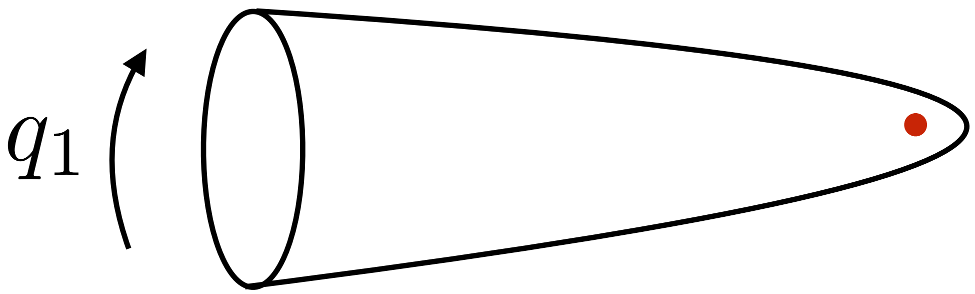


Large-n limits are manifest in each description!

BPS/CFT (AGT)

Gauge theory in Omega background

$$\mathbb{R}^2_{q_1} \times \mathbb{R}^2_{q_2} \times S^1$$

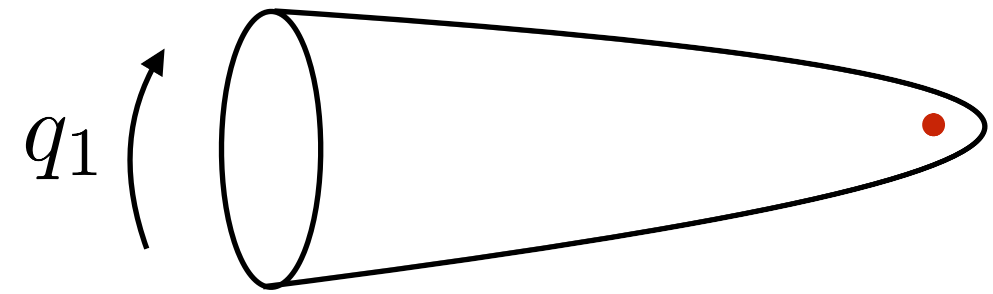


Nekrasov used it to count instantons which localize on the tip

BPS/CFT (AGT)

Gauge theory in Omega background

$$\mathbb{R}^2_{q_1} \times \mathbb{R}^2_{q_2} \times S^1$$



Nekrasov used it to count instantons which localize on the tip

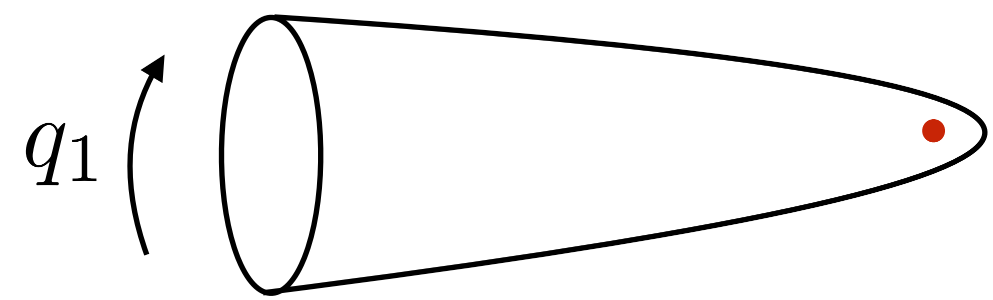
AGT states that $\mathcal{Z}_{\text{Nek}} = \mathcal{F}_{\text{CFT}}$

Omega background data is matched with the CFT central charge and (q)VOA data

BPS/CFT (AGT)

Gauge theory in Omega background

$$\mathbb{R}^2_{q_1} \times \mathbb{R}^2_{q_2} \times S^1$$



Nekrasov used it to count instantons which localize on the tip

AGT states that $\mathcal{Z}_{\text{Nek}} = \mathcal{F}_{\text{CFT}}$

Omega background data is matched with the CFT central charge and (q)VOA data

Mathematicians have now several proofs of AGT in limiting cases (no fundamental matter), but those proofs do not use the original class-S theory construction

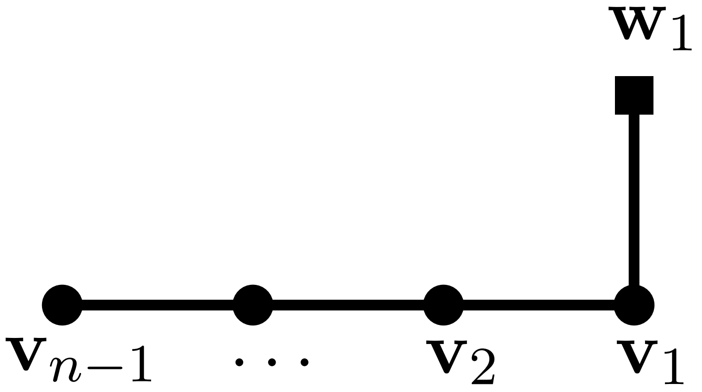
Recent proof by Kimura and Pestun uses direct localization computations

qW algebra

Construction of qW algebra from free-boson representation of Nekrasov partition function **with defects** [Kimura Pestun]

$$\mathcal{Z}_{\mathrm{Nek}} = \widehat{\mathcal{Z}}_{\mathrm{Nek}}|0\rangle \qquad [a_i, a_j] = \frac{1}{j}\delta_{i+j,0}\frac{1-q_1^{|j|}}{1-q_2^{|j|}}$$

Start with quiver gauge theory compactified on a circle

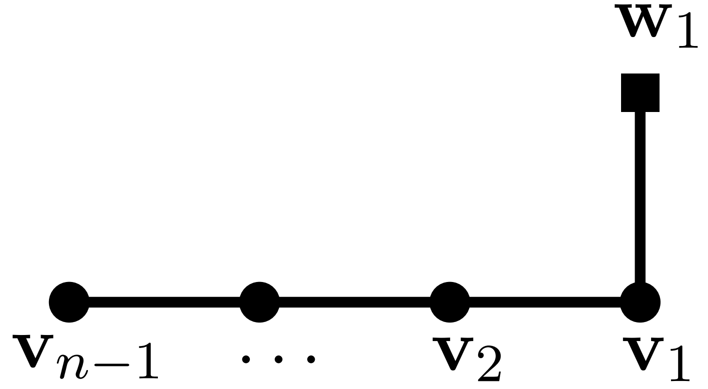


qW algebra

Construction of qW algebra from free-boson representation of Nekrasov partition function **with defects** [Kimura Pestun]

$$\mathcal{Z}_{\mathrm{Nek}} = \widehat{\mathcal{Z}}_{\mathrm{Nek}}|0\rangle \qquad [a_i, a_j] = \frac{1}{j}\delta_{i+j,0}\frac{1-q_1^{|j|}}{1-q_2^{|j|}}$$

Start with quiver gauge theory compactified on a circle



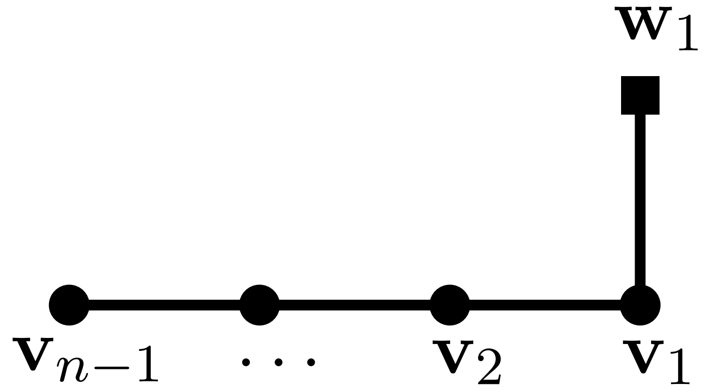
Moduli space of vacua is the space of A_{n-1} periodic monopoles with $\mathbf{w}_1$ Dirac singularities whose charges are given by the number of colors

qW algebra

Construction of qW algebra from free-boson representation of Nekrasov partition function **with defects** [Kimura Pestun]

$$\mathcal{Z}_{\mathrm{Nek}} = \widehat{\mathcal{Z}}_{\mathrm{Nek}}|0\rangle \qquad [a_i, a_j] = \frac{1}{j}\delta_{i+j,0}\frac{1-q_1^{|j|}}{1-q_2^{|j|}}$$

Start with quiver gauge theory compactified on a circle



Moduli space of vacua is the space of A_{n-1} periodic monopoles with $\mathbf{w}_1$ Dirac singularities whose charges are given by the number of colors

Quantization of this moduli space in carefully chosen complex structure gives qW(q1,q2) algebra modulo Virasoro constraints!

$$\widehat{\mathbb{C}}[\mathcal{M}_{\mathrm{mon}}] = \frac{qW_{q_1,q_2}}{\mathrm{Vir}(\mathbf{v}_1,\ldots,\mathbf{v}_{n-1})} \qquad T_{i,-k}|\psi\rangle = 0, \quad k > \mathbf{v}_i$$

‘Double’ quantization

There is an integrable systems associated with the moduli space of periodic monopoles. Its classical description is given by the Seiberg-Witten curve of the theory. [Nekrasov, Pestun]

[Nekrasov, Pestun, Shatashvili]

In order to **quantize** the integrable system one turns on one of the Omega background parameters — **q**ı

$$[\mathcal{H}_i, \mathcal{H}_j] = 0$$

'Double' quantization

There is an integrable systems associated with the moduli space of periodic monopoles. Its classical description is given by the Seiberg-Witten curve of the theory. [Nekrasov, Pestun]

[Nekrasov, Pestun, Shatashvili]

In order to **quantize** the integrable system one turns on one of the Omega background parameters — $\mathbf{q_1}$

$$[\mathcal{H}_i, \mathcal{H}_j] = 0$$

Finally, **qW-algebra/Vir** for the quiver is the $\mathbf{q_2}$-deformation of the ring of commuting Hamiltonians of the quantum integrable system [cf with Aganagic Frenkel Okounkov]

Virasoro constrains can be removed by taking $\mathbf{v}_i \to \infty$

‘Double’ quantization

There is an integrable systems associated with the moduli space of periodic monopoles. Its classical description is given by the Seiberg-Witten curve of the theory. [Nekrasov, Pestun]

[Nekrasov, Pestun, Shatashvili]

In order to **quantize** the integrable system one turns on one of the Omega background parameters — $\mathbf{q_1}$ $[\mathcal{H}_i, \mathcal{H}_j] = 0$

Finally, **qW-algebra/Vir** for the quiver is the $\mathbf{q_2}$-deformation of the ring of commuting Hamiltonians of the quantum integrable system [cf with Aganagic Frenkel Okounkov]

Virasoro constrains can be removed by taking $\mathbf{v}_i \to \infty$

We shall provide physical and geometric interpretation of both Virasoro constraints and the limit

Theories with defects

These Virasoro constraints can be interpreted as equations of motion for a different integrable system

[Alday Tachikawa]
[Nawata]
[Bullimore Kim PK]
[PK Sciarappa]
[Aganagic Shakirov Haouzi]

For balanced A-quivers (with adjusted masses) or quivers with adjoint hypers with codimension-2 defects the corresponding integrable system is **trigonometric Ruijsenaars-Schneider model**

$$D_n^{(1)}(\tau_i, p_\tau^i) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{t\tau_i - \tau_j}{\tau_i - \tau_j} p_\tau^i$$

Partition functions of defect theories are the eigenfunctions

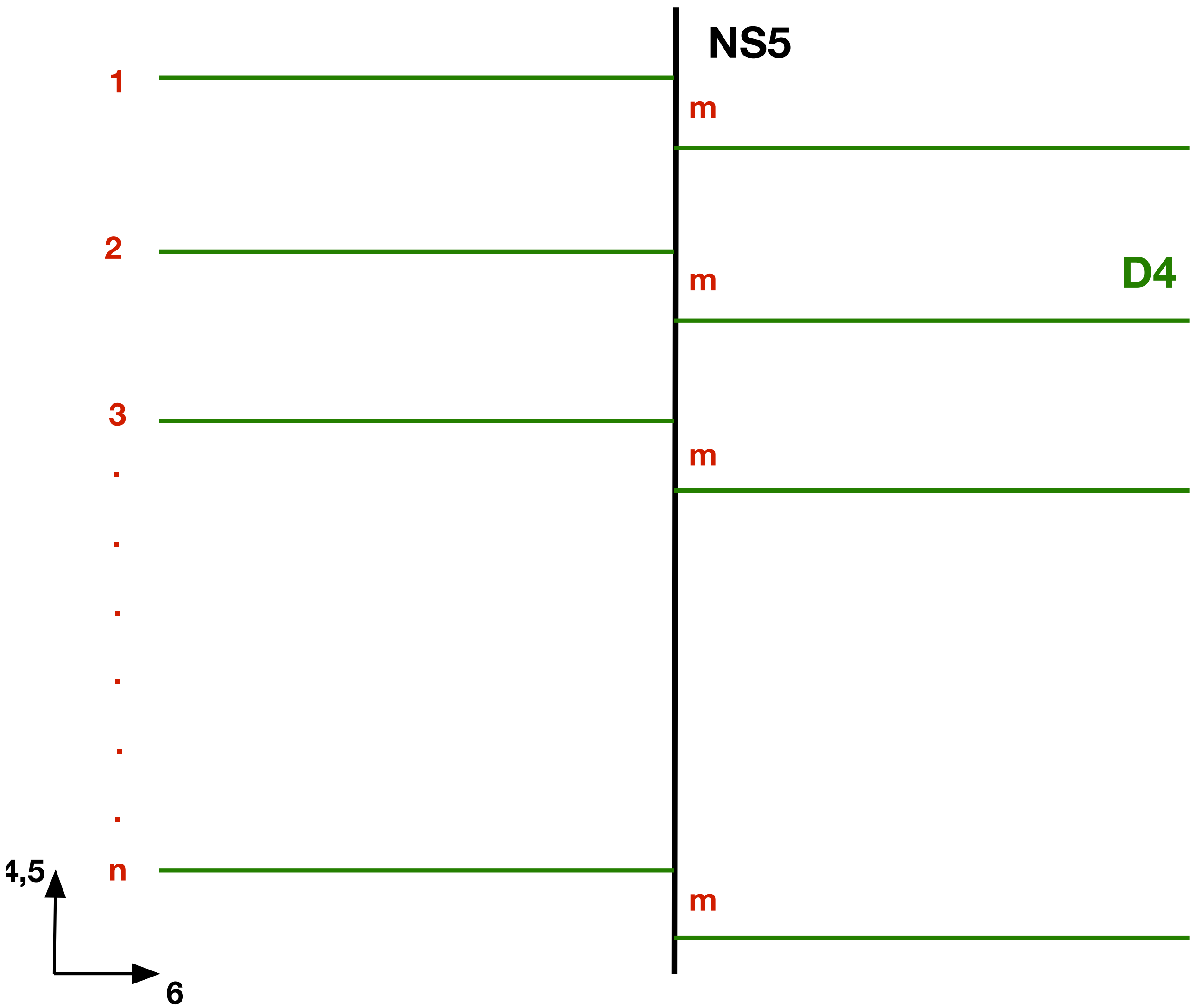
$$D_n^{(r)}(\tau, p_\tau)\mathcal{Z} = e_r(\mu)\mathcal{Z}$$

Defects from branes

N=1* U(n) 5d theory on $\mathbb{R}^4 \times S^1_\gamma$.

NS5 012345

D4 0123(4 6)



Complex scalar

$$\mu_a = e^{-i\gamma a_a}$$

Adjoint hyper

$$t = e^{-i\gamma m}$$

Omega backgr
in 23-plane

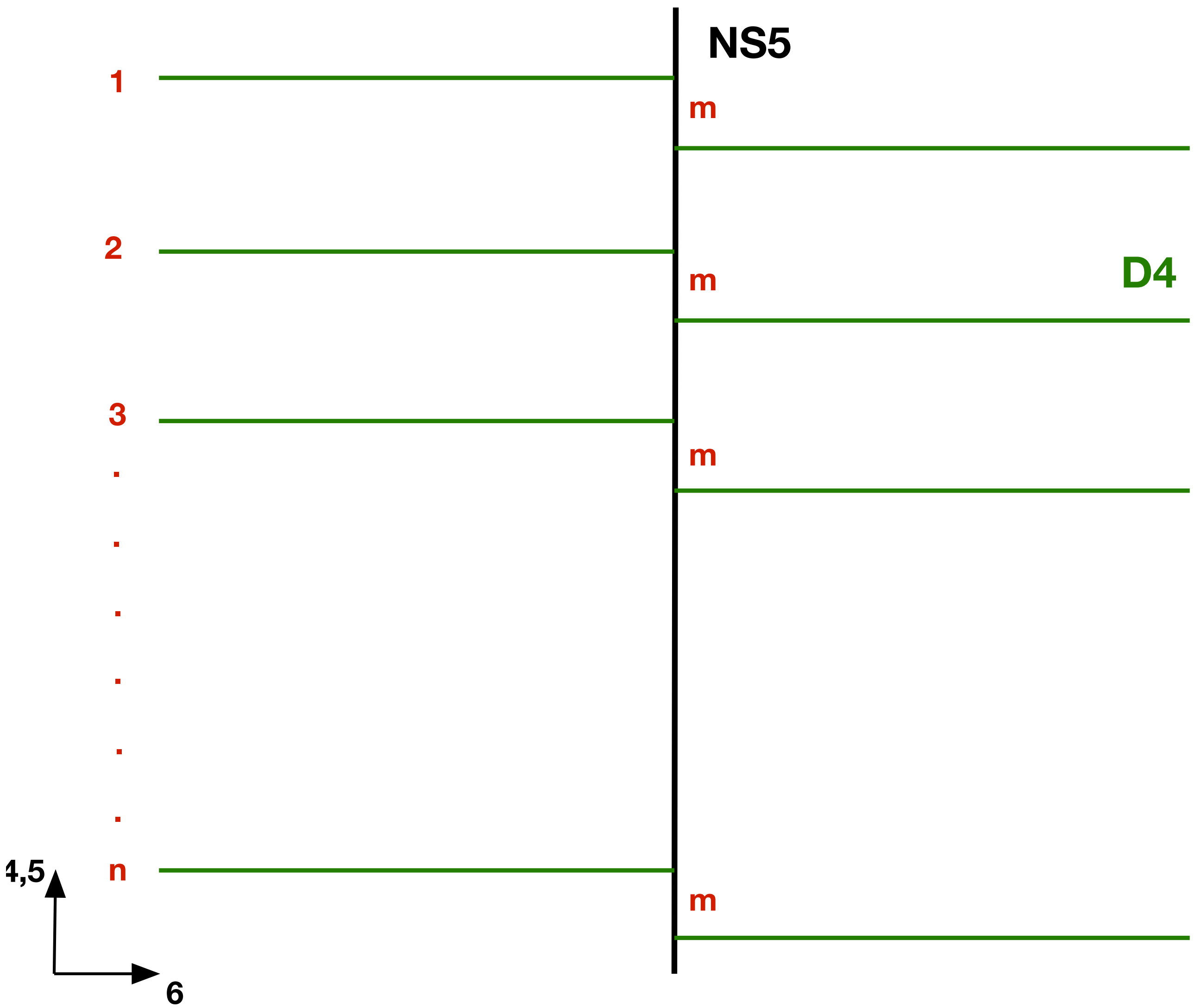
$$q = e^{i\gamma \epsilon_1}$$

Defects from branes

N=1* U(n) 5d theory on $\mathbb{R}^4 \times S^1_\gamma.$

NS5 012345

D4 0123(4 6)



Complex scalar

$$\mu_a = e^{-i\gamma a_a}$$

Adjoint hyper

$$t = e^{-i\gamma m}$$

Omega backgr
in 23-plane

$$q = e^{i\gamma \epsilon_1}$$

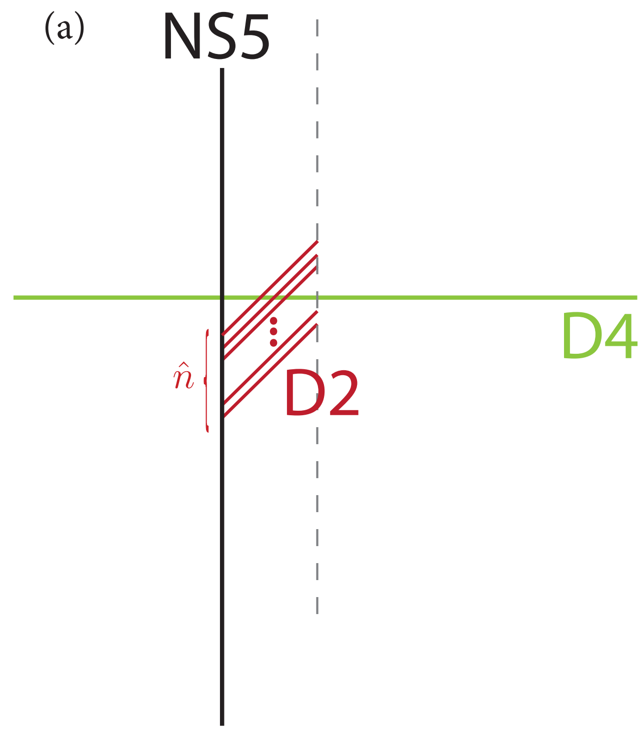
Higgs branch locus

$$\mu_a = q^{\lambda_a} t^{n-a}$$

System undergoes geometric transition

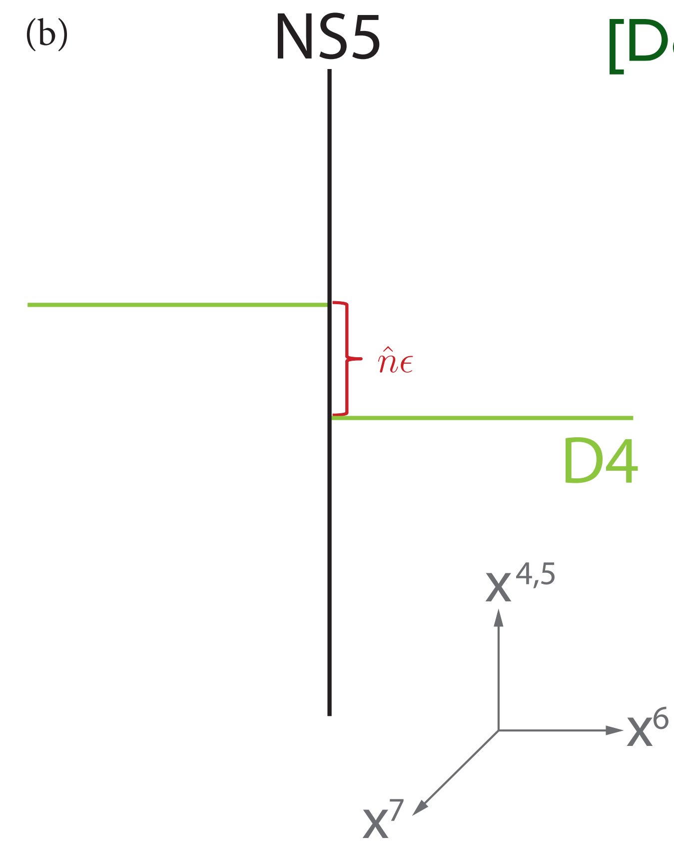
Confold Transition

Higgs



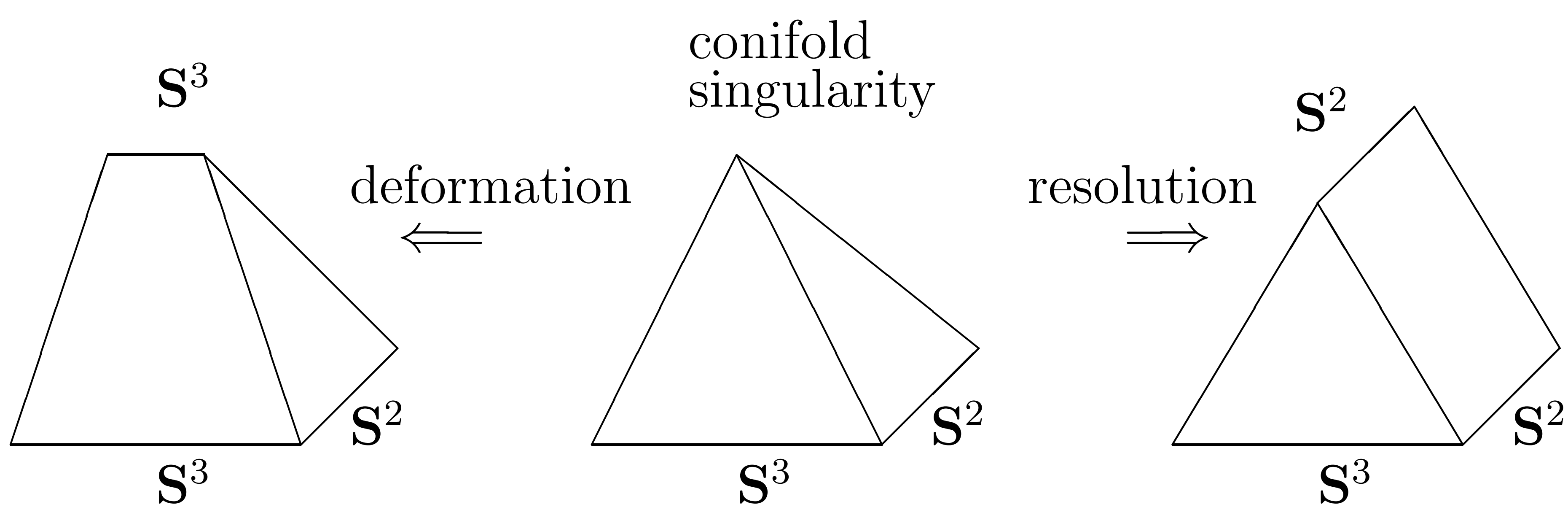
$$\mu_i = q^{\lambda_i} t^{n-i}$$

Coulomb



[Dorey Hollowood Lee]

[Gopakumar Vafa]



M-theory construction

Starting with M-theory on $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times T^* S^3$

With n M5 branes wrapping $S^1 \times \mathbb{C}_q \times S^3$ (with $S^3 \subset T^* S^3$)

This setup provides us with the U(n) theory on M5 branes with 8 supercharges

M-theory construction

Starting with M-theory on $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times T^* S^3$

With n M5 branes wrapping $S^1 \times \mathbb{C}_q \times S^3$ (with $S^3 \subset T^* S^3$)

This setup provides us with the U(n) theory on M5 branes with 8 supercharges

When n becomes large the background undergoes through the conifold transition and the resolved conifold becomes a deformed conifold

So we are left with M-theory on $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times Y$

Reduction on Y leads us to a 5d U(1) N=1 super Yang-Mills theory

U(1) Instantons

In other words, there is a direct connection, via the large-n limit, between instantons in N=1* U(n) theory and instantons of U(1) SYM!

Heisenberg algebra (***elliptic Hall algebra***) which we have seen earlier also appears in the study of moduli space of U(1) (non-commutative) instantons

$$[a_i, a_j] = \frac{1}{j}\delta_{i+j,0}\frac{1-q_1^{|j|}}{1-q_2^{|j|}}$$

[Nakjima]
[Schiffmann Vaserot]

U(1) Instantons

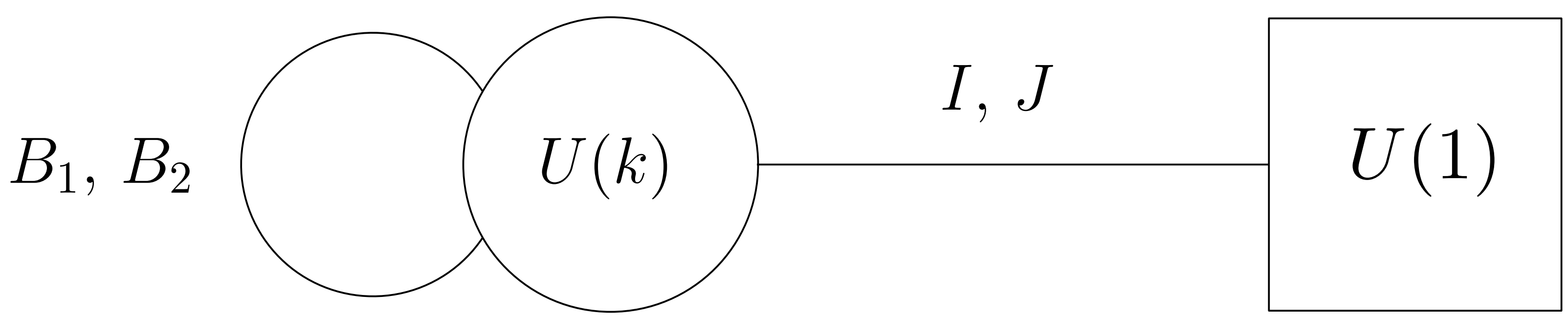
In other words, there is a direct connection, via the large-n limit, between instantons in N=1* U(n) theory and instantons of U(1) SYM!

Heisenberg algebra (***elliptic Hall algebra***) which we have seen earlier also appears in the study of moduli space of U(1) (non-commutative) instantons

$$[a_i, a_j] = \frac{1}{j}\delta_{i+j,0}\frac{1-q_1^{|j|}}{1-q_2^{|j|}}$$

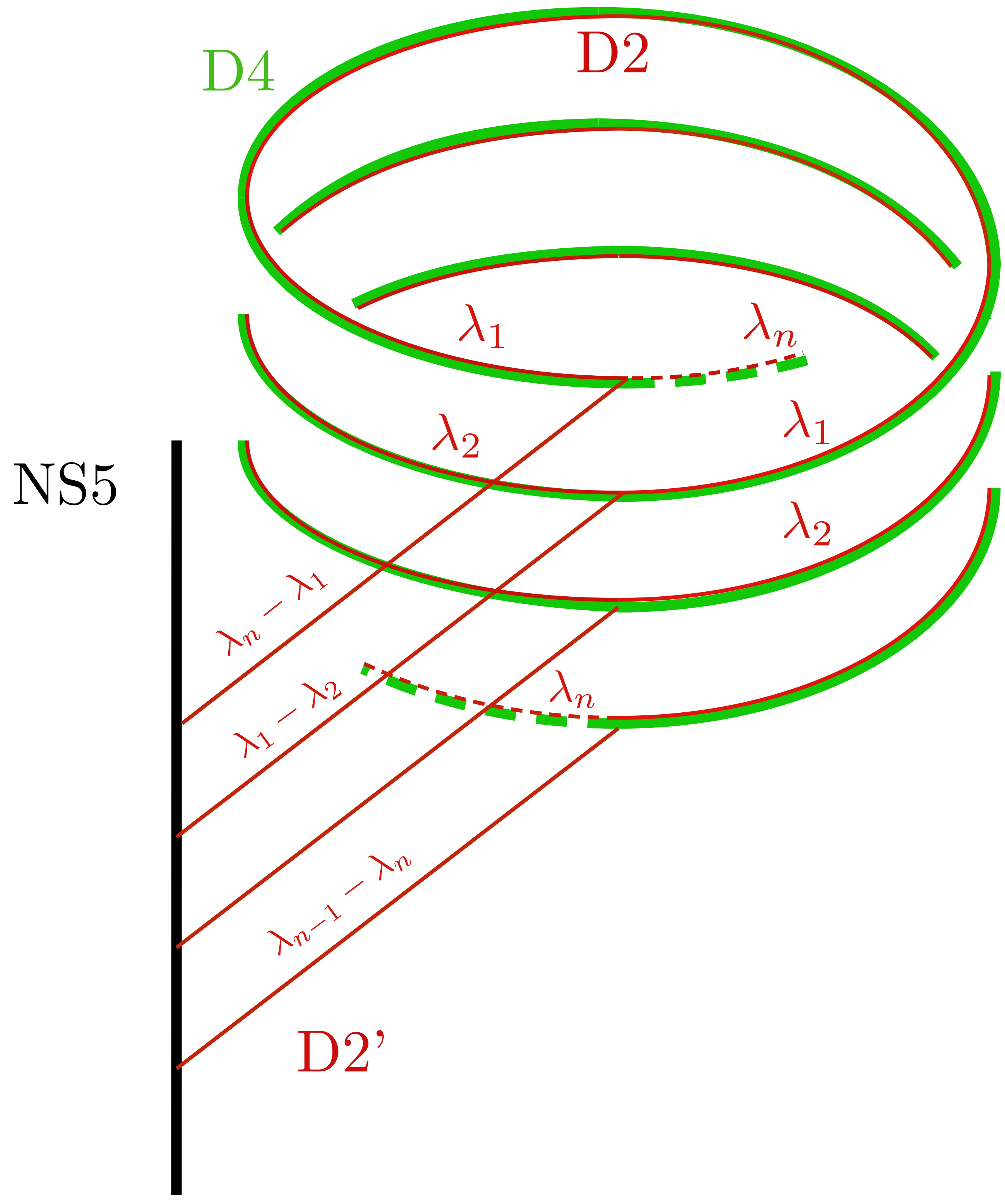
[Nakjima]
[Schiffmann Vaserot]

Moduli space $\mathcal{M}_{k,1}$ described by ADHM quiver



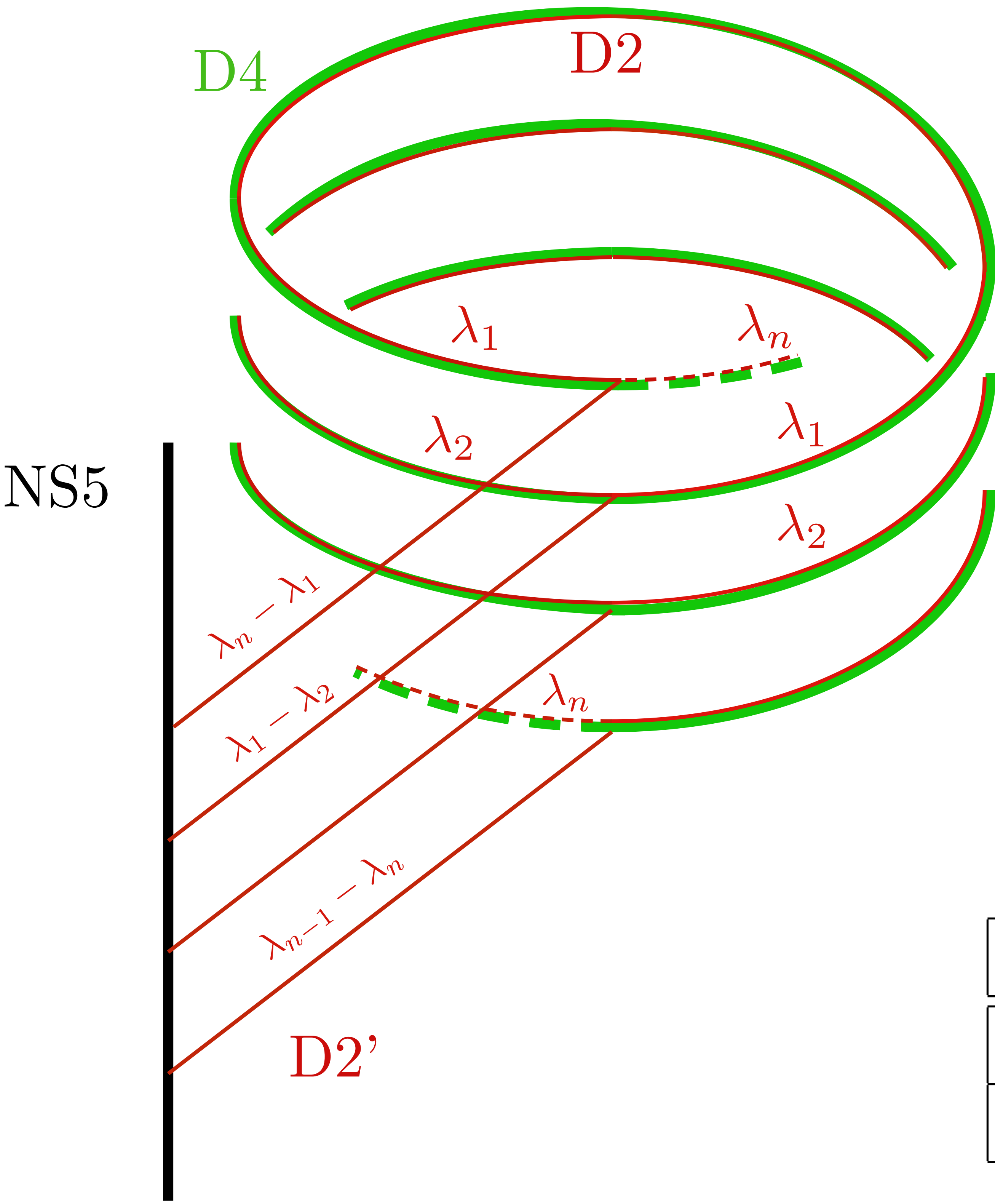
It is related to the Kimura-Pestun answer by a Nahm transform!

ADHM from branes



	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D4	x	x	x	x	$\cos\delta$		$\sin\delta$			
D2'	x	x						x		
D2	x	x			$-\sin\delta$		$\cos\delta$			

ADHM from branes



	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D4	x	x	x	x	$\cos\delta$		$\sin\delta$			
D2'	x	x						x		
D2	x	x			$-\sin\delta$		$\cos\delta$			

$$\delta \sim \tan^{-1} m/R$$

At large-n limit arrive at ADHM!!

	0	1	2	3	4	5	6	7	8	9
D6	x	x	x	x	x		x		x	
D2	x	x							x	

Stable limit

Trigonometric Ruijsenaars-Schneider (Macdonald) operators form a maximal commuting subalgebra inside **DAHA for gl(n)**

DAHA generators act on defect partition functions and may change vortex number. Its representation theory is quite complicated, but at least in low ranks it resembles that of gl(n) itself [Gukov, Nawata, PK, Sabieri]

Stable limit

Trigonometric Ruijsenaars-Schneider (Macdonald) operators form a maximal commuting subalgebra inside **DAHA for gl(n)**

DAHA generators act on defect partition functions and may change vortex number. Its representation theory is quite complicated, but at least in low ranks it resembles that of gl(n) itself [Gukov, Nawata, PK, Sabieri]

Schiffmann and Vaserot showed that large-n limit of **gl(n) DAHA** is given by **elliptic Hall algebra** (a.k.a Ding-Iohara algebra)

Another math result shows that qW algebra can be obtained as representation of Ding-Iohara algebra, which is a Hopf algebra itself. **Thus the symmetry of CFT emerges directly in the limit**

The Duality

[PK Sciarappa]

There exists a stable limit of the equivariant Chern character of the universal bundle over the U(n) instanton moduli space in terms of the same character only for U(1) instantons $\mathcal{M}_{k,1}$

$$\left\langle W_{\Box}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

The Duality

[PK Sciarappa]

There exists a stable limit of the equivariant Chern character of the universal bundle over the U(n) instanton moduli space in terms of the same character only for U(1) instantons $\mathcal{M}_{k,1}$

$$\left\langle W_{\square}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

With elliptic parameter turned on

elliptic RS	**3d ADHM theory**	**3d/5d coupled theory, $n \to \infty$**
coupling t	twisted mass $e^{-i\gamma\epsilon_2}$	5d $\mathcal{N} = 1^*$ mass deformation $e^{-i\gamma m}$
quantum shift q	twisted mass $e^{i\gamma\epsilon_1}$	Omega background $e^{i\gamma\widetilde{\epsilon}_1}$
elliptic parameter p	FI parameter	5d instanton parameter
eigenstates λ	ADHM Coulomb vacua	5d Coulomb branch parameters
eigenvalues	$\langle \mathrm{Tr}\,\sigma \rangle$	$\langle W_{\square}^{U(\infty)} \rangle$ in NS limit $\widetilde{\epsilon}_2 \to 0$

What's next?

Add more equivariant parameters:

from 4 to 5 to 6d

from cohomology to K-theory to elliptic cohomology

Math questions:

Quantization of integrable systems vs.

Enumerative geometry, etc.

....