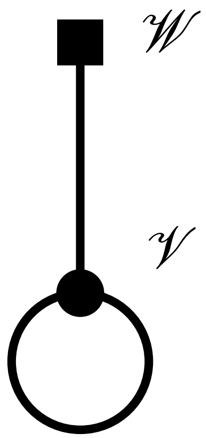
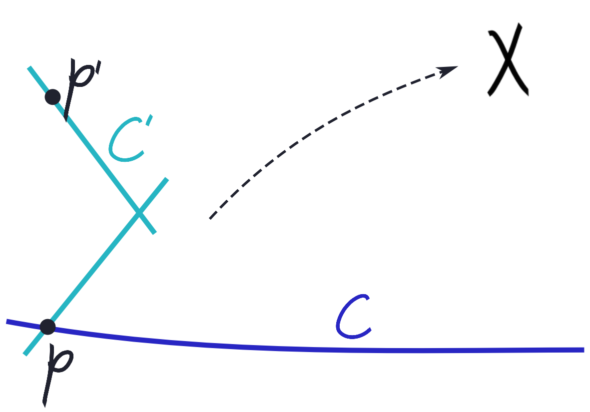
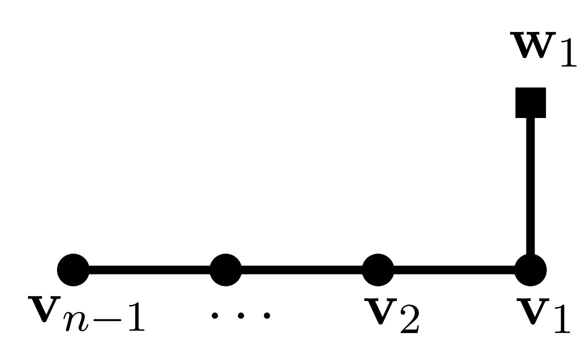
DAHA, Elliptic Hall Algebra ADHM & 1/2 ADHM



Peter Koroteev





Talk at workshop `Higher Symmetries'
Aspen, CO March 19th 2019

Higher Symmetries

In this talk we shall discuss some **higher symmetries** which arise in *physics* and *geometry*.

$$\mathfrak{E} \simeq U_{q_1,q_2}\left(\widehat{\widehat{\mathfrak{gl}_1}}\right) \simeq \mathscr{E}_{q_1,q_2} \simeq \mathfrak{gl}_\infty \mathrm{DAHA}^S_{q_1,q_2} \simeq \mathrm{DIM}_{q_1,q_2} \simeq D(\mathscr{A}_{\text{shuffle}})$$

Quantum Toroidal $\mathfrak{gl}_1$	Elliptic Hall algebra	spherical $\mathfrak{gl}_\infty$ DAHA	Ding-Iohara-Miki algebra	Drinfeld double of shuffle algebra

This algebra often appears in the **BPS/CFT** correspondence:

Connects BPS observables of $\mathcal{N}=2$ supersymmetric gauge theories with CFT correlators (*Mathematically*: relates structures arising on moduli spaces of sheaves (instantons) with vertex operator algebras or qVOAs)

BPS/CFT and Geometry

Mathematicians have now several ***proofs*** of BPS/CFT (AGT) in limiting cases (no fundamental matter), those proofs do not use the original class-S construction [Schiffmann Vaserot] [Negut]

Physics **proof*** by Kimura and Pestun uses direct localization computations

One of our goals is to understand BPS/CFT **geometrically**

Namely we want describe instanton counting and vertex operator algebras in terms of **quantum geometry** (quantum cohomology or quantum K-theory) of some family of spaces

In other words we want (q)VOAs to **emerge** from quantum geometry

Recent Developments

Vertex Algebras at the Corner [Gaiotto Rapcak]

VOAs at junctions of supersymmetric intersections in N=4 SYM

COHA and VOAs [Rapcak Soibelman Yang Zhao]

Action of COHA on the moduli space of *spiked* instantons

Quiver W-algebras [Kimura Pestun]

4,5,6d quiver gauge theories on R^4 x S in Omega background

The Magnificent Four [Nekrasov]

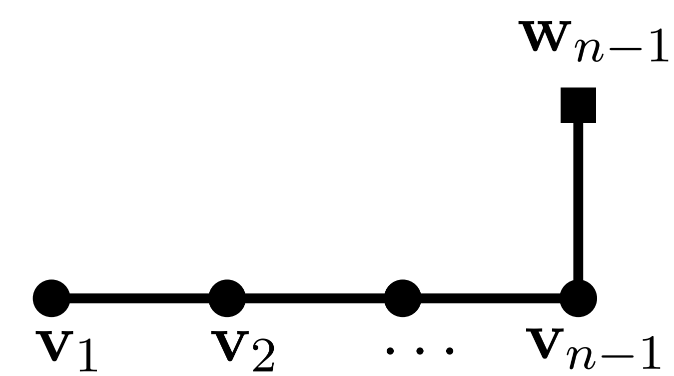
D8 brane probed by D0 branes in B field

$U(1)^4 \subset \mathrm{Spin}(8)$ + additional *nongeometric* U(1) symmetry

q_1, q_2, q_3, q_4

Nakajima Quiver Varieties

Rep(**v**,**w**) — linear space of quiver reps



$\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map

Nakajima quiver variety $X = \mu^{-1}(0)/\!\!/G$ $\quad G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

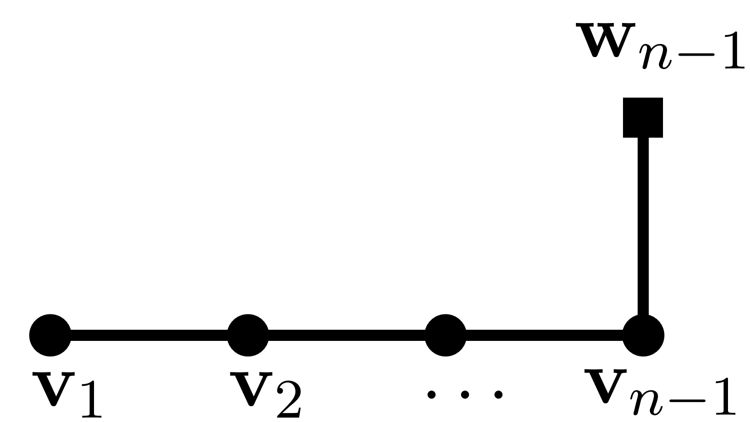
Tensorial polynomials of tautological bundles $\mathbf{V}_i$, $\mathbf{W}_i$ and their duals generate *classical T-equivariant K-theory* ring of X value of a quasimap defines a map to a quotient stack which contains stable locus as an open subset

Quasimaps

$- - \to X$ is described by collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ with section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$

where $\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$

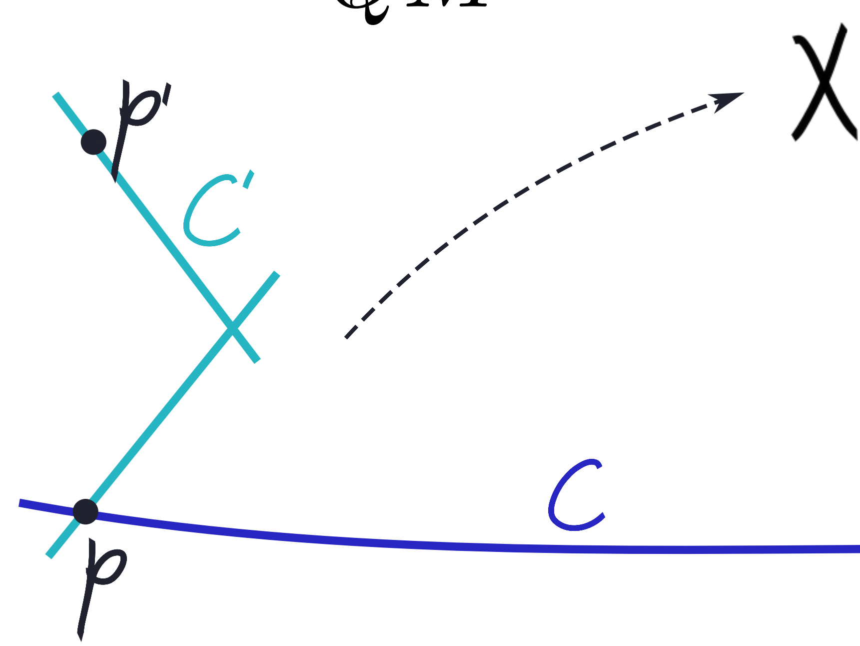
Degree $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$



Evaluation map

$$\mathrm{ev}_p(f) = f(p) \in [\mu^{-1}(0)/G] \supset X$$

$QM^{\mathbf{d}}$



Stable if $f(p) \in X$

for all but finitely many singular points

Resolve to make *proper* ev map

Vertex Function (g

Say this in words: equivariant pushforward, etc.

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q$
base $\mathbb{P}^1$ keeping two fixed points (0, infinity)

Define **vertex function** with quantum (Novikov) parameters $z^{\mathbf{d}} = \prod_{i \in I} z_i^{d_i}$

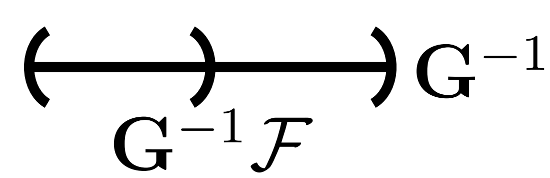
$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_2,*}\left(QM^{\mathbf{d}}_{\mathrm{nonsing}\, p_2}, \widehat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathscr{V}_i|_{p_1}) \right) \in K_{\mathsf{T}_q}(X)_{loc}[[z]]$$

[Okounkov]
[PK Pushkar Smirnov Zeitlin]

Define **quantum K-theory** as a ring with multiplication

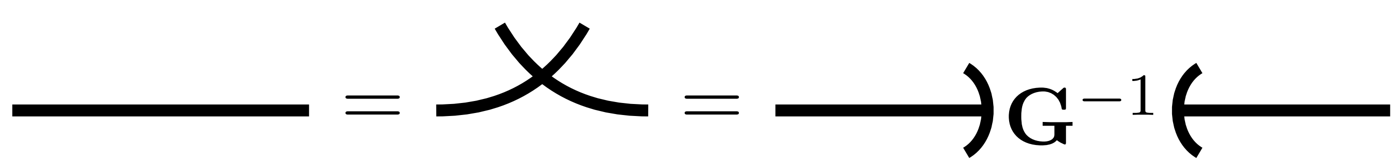
$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathcal{F} \circledast = \sum_{\mathbf{d}=\vec{0}}^{\infty} z^{\mathbf{d}} \mathrm{ev}_{p_1,p_3*}\left(\mathrm{QM}^{\mathbf{d}}_{p_1,p_2,p_3}, \mathrm{ev}^*_{p_2}(\mathbf{G}^{-1}\mathcal{F}) \widehat{\mathcal{O}}_{\mathrm{vir}} \right) \mathbf{G}^{-1}$$



gluing

$$\mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}$$

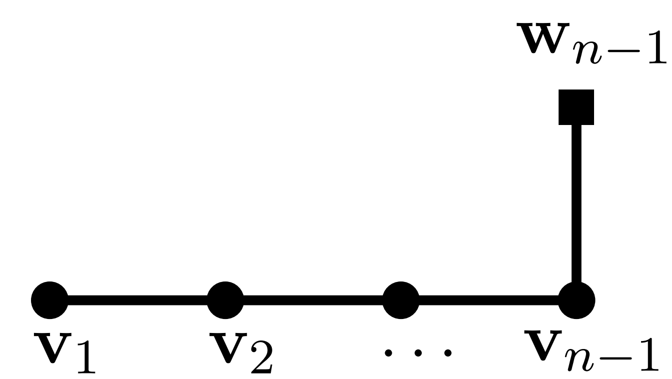


Vertex Functions

After classifying fixed points of space of nonsingular quasimaps we can compute the vertex

$$V_{\boldsymbol{p}}^{(\tau)}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j}q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1}\prod_{j,k=1}^{\mathbf{v}_i}\{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}} \qquad x_{i,j}\in\{a_1,\dots a_{\mathbf{w}_n}\}$$

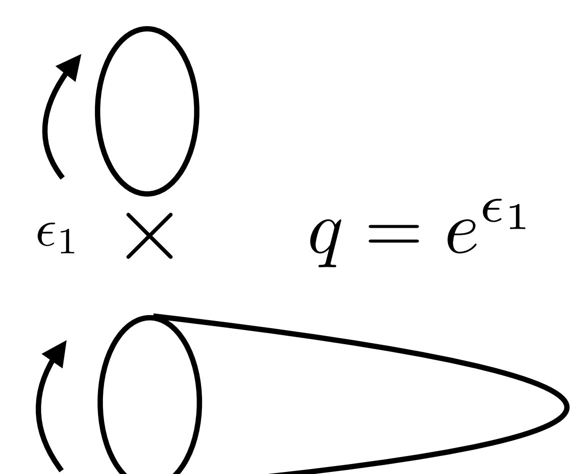


Vertex (trivial class)

$$V = {}_2\phi_1\left(\hbar, \hbar\frac{a_1}{a_2}, q\frac{a_1}{a_2}; q; z\right) \qquad \mathbf{v}_1 = 1,\ \mathbf{w}_1 = 2$$

Vortex

$\mathcal{N} = 2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1}\times S^1_\gamma$



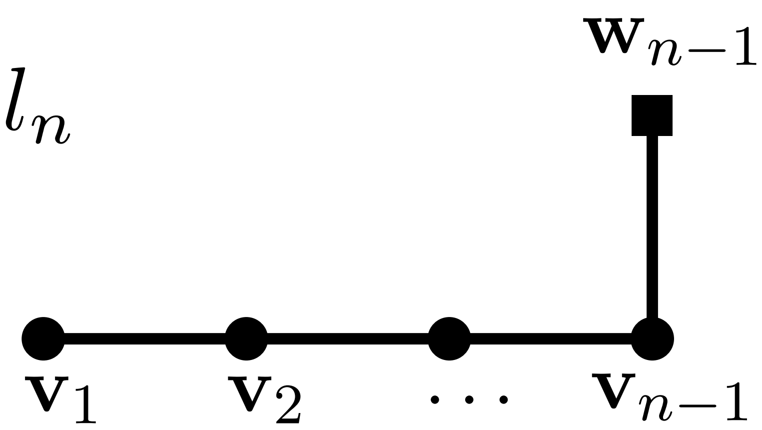
$q = e^{\epsilon_1}$

Lagrangian depends on twisted masses a_1, a_2
FI parameter z and U(1) R-symmetry $\log\hbar$

Difference Equations

[PK Pushkar Smirnov Zeitlin]
[PK, PK Zeitlin]

$X = T^*\mathbb{F}l_n$



Quantum K-theory Ring $q \to 0$

$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[z_i^{\pm 1}, a_i^{\pm 1}, T_q^{\pm 1}\hbar, q]}{\mathcal{I}_{\mathrm{tRS}}}$$

The K-theory vertex function satisfies equation of motion of *trigonometric Ruijsenaars-Schneider* model

$$\hat{H}_d V = e_d(z_1, \ldots, z_{n-1}) V$$

$$\hat{H}_d = \sum_{I\subset\{1,\ldots,n\}, |I|=d} \Big(\prod_{i\in I, j\notin I} \frac{a_i \hbar^{\frac{1}{2}} - a_j \hbar^{-\frac{1}{2}}}{a_i - a_j} \Big) \prod_{i\in I} T_i^q$$

3d Mirror version (a.k.a. bispectral dual)

$$\hat{H}^!_d V = e_d(a_1, \ldots, a_{n-1}) V$$

$$\hat{H}^!_d(a_i, \hbar, T_a^q) = \hat{H}_d(z_i/z_{i+1}, \hbar^{-1}, T_z^q)$$

If time will be tight then say in words that A and B are holonomies of electric and magnetic operators

Should be around 50% of time here!!!!

Spherical DAHA

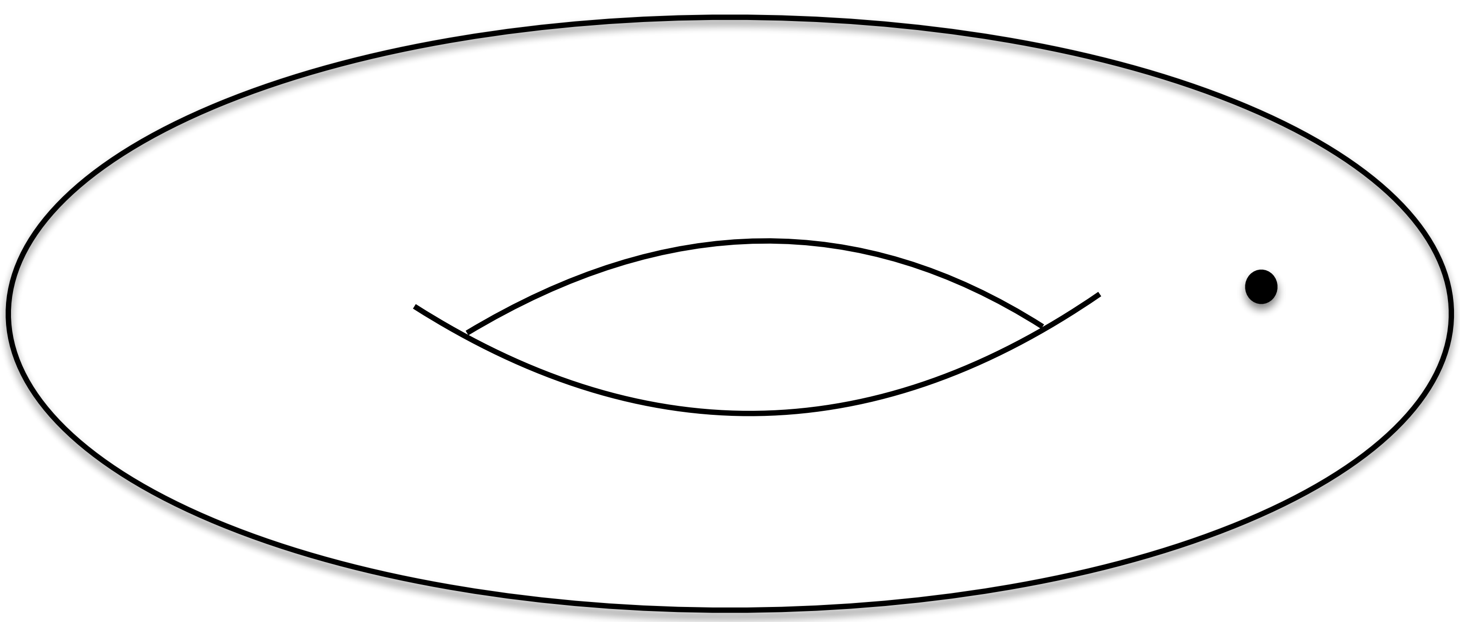
...ijsenaars-Schneider Hamiltonians form a maximal commuting subalgebra inside **spherical double affine Hecke algebra for gl(n)**

$$\{\hat{H}_1, \ldots, \hat{H}_n\} \subset \mathrm{DAHA}^{\mathfrak{S}_n}_{q,\hbar}(\mathfrak{gl}_n) =: \mathcal{A}_n$$

$\hat{H}_d$ are also known as Macdonald operators

[Oblomkov]

Spherical gl(n) DAHA is a **geometric quantization** of the moduli space of flat GL(n;C) connections on a torus with one simple puncture



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \mathrm{diag}(\hbar, \ldots, \hbar, \hbar^{1-n})$$

$$\mathcal{A}_n = \widehat{\mathbb{C}_J[\mathcal{M}_n]}$$

Line Operators and Branes

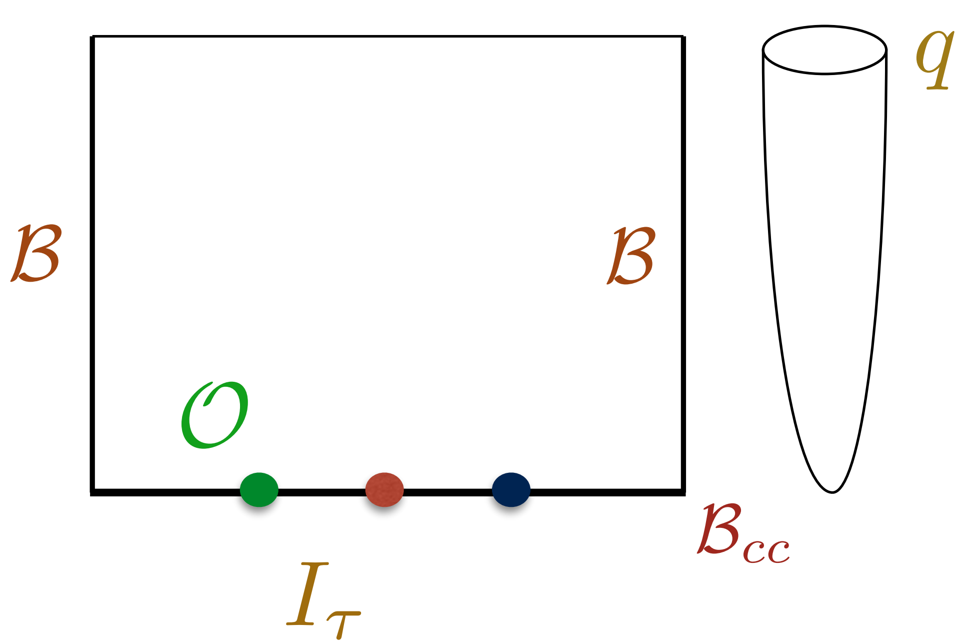
$\mathcal{M}_n$ is the moduli space of vacua in $\mathcal{N}$=2* gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n) and is described by VEVs of line operators wrapping the circle.

A and *B* are holonomies of *electric* and *magnetic* line operators

Omega background along real 2-plane $\mathbb{R}^2_q \times \mathbb{R} \times S^1$

Line operators are forced to stay at the tip of the cigar and slide along the remaining line, hence **non-commutativity**

[Gukov-Witten]
[Nekrasov-Witten]



algebra — open strings

$$\mathcal{A}_n = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$$

representations

(Hilbert space of SUSY QM)

$$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B})$$

Hitchin Moduli Space (n=2)

[Gukov]
[PK Gukov Nawata Saberi]

SU(2) theory → sl(2) flat connections

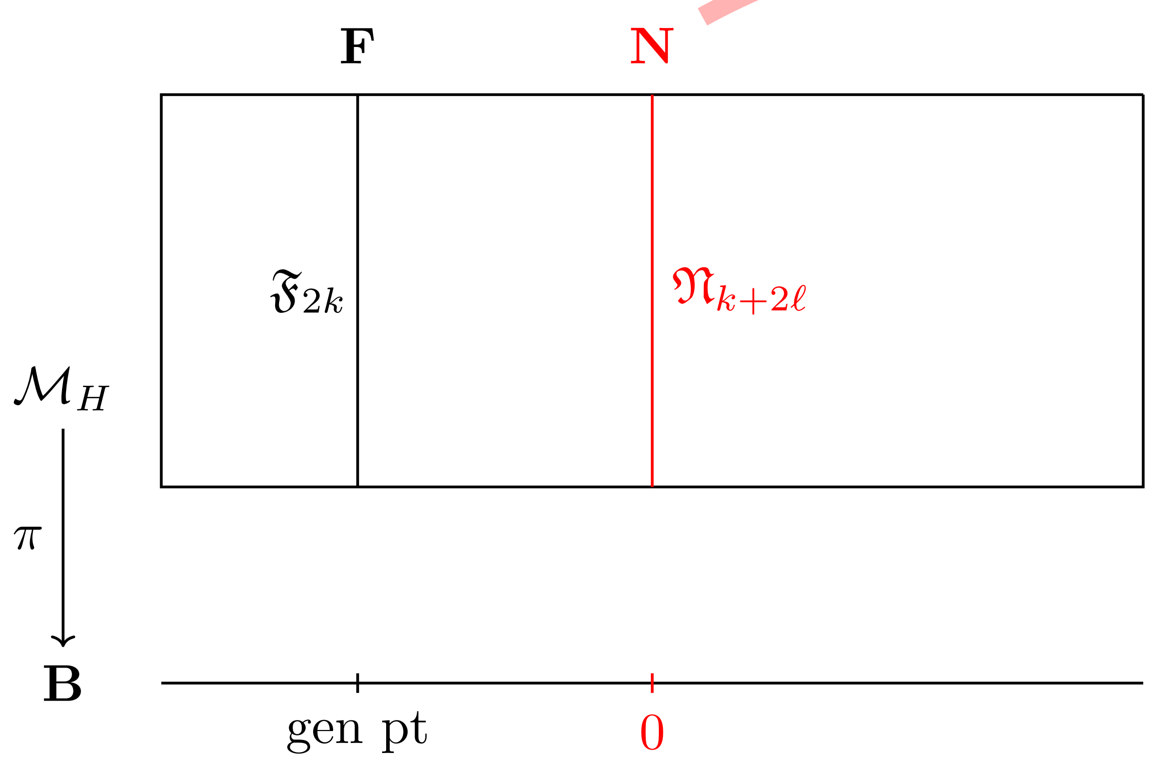
$x = TrA$ — electric
$y = TrB$ — magnetic
$z = TrAB$ — dyonic

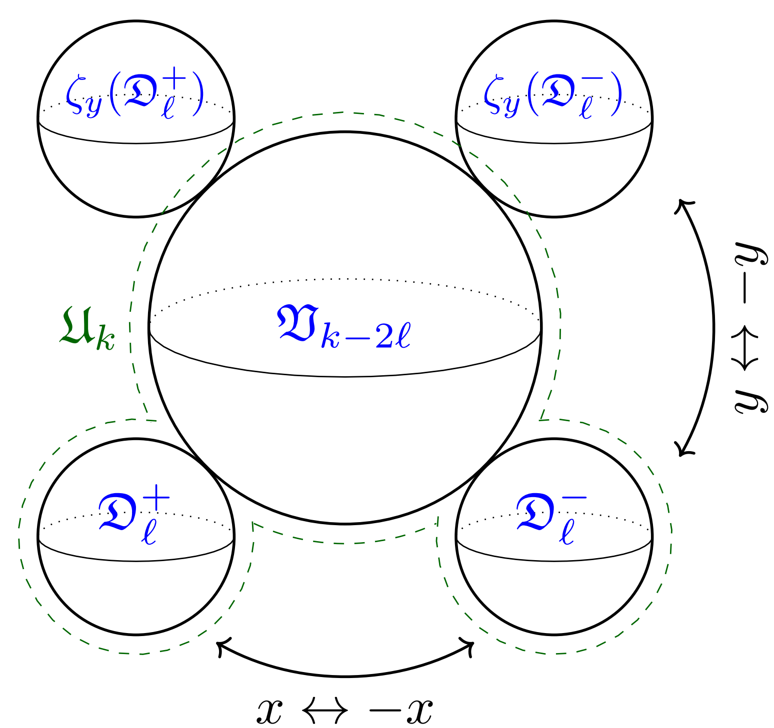
Nonabelian Hodge correspondence: $$\mathcal{M}_{\text{flat}}(SL(2;\mathbb{C}), T^2\backslash\{\text{pt}\}) \simeq \mathcal{M}_H(SU(2), T^2\backslash\{\text{pt}\})$$

$$\mathcal{M}_H: \quad x^2+y^2+z^2+xyz = \hbar + \hbar^{-1} + 2$$

for $\hbar$=1 $\quad \mathcal{M}_n \simeq \dfrac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2}$

Elliptic fibration with one singular fiber of Kodaira type I_0^*



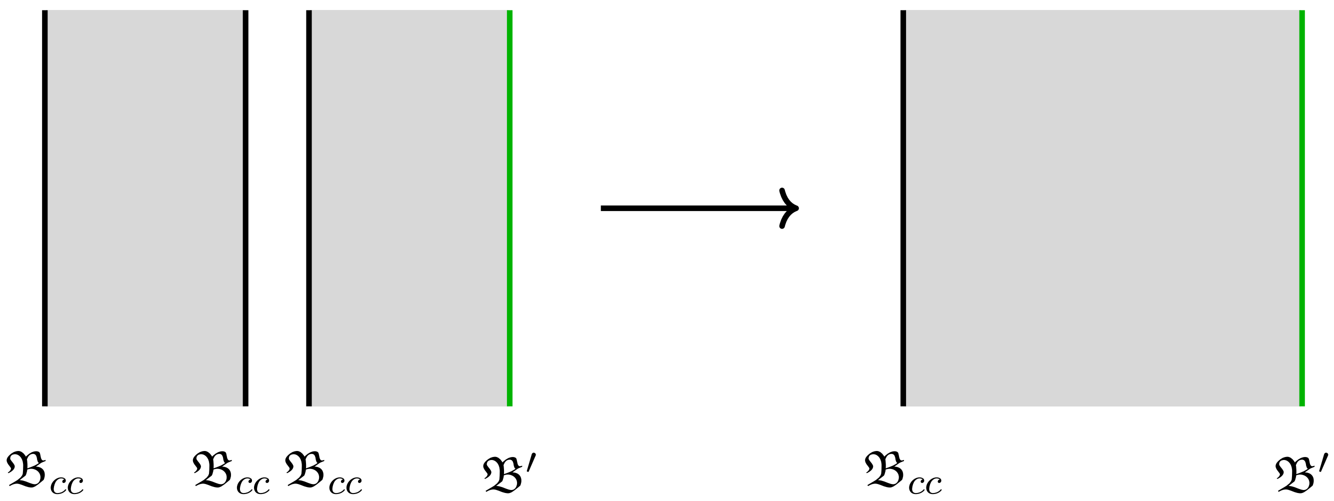


DAHA Modules

[Kapustin Orlov]
[Kapustin Witten]

Algebra acts naturally by attaching open strings to closed strings

Hilbert space comes from (Bcc, B′) strings



$$\begin{aligned} \mathcal{A} &= \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{cc}) \\ \circlearrowleft & \qquad\qquad \circlearrowleft \\ \mathcal{H} &= \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}') \end{aligned}$$

$\mathcal{B}' \to \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ gives a functor $\mathrm{Hom}(\mathcal{B}_{cc}, \cdot)$

$$\boxed{\mathrm{Fuk}(\mathcal{M}, \Omega) \simeq \mathrm{Rep}(\mathcal{A})}$$

$$\mathfrak{B}_{cc} : \mathcal{L} \to \mathcal{M}_H$$

$$F + B = \frac{i}{\log q}\Omega_J$$

$$\Omega_J = \frac{dx \wedge dy}{2z - xy}$$

Algebra-deformation quantization of functions on M_H

$$[x, y]_q = (q - q^{-1})z$$
$$[z, x]_q = (q - q^{-1})y$$
$$[y, z]_q = (q - q^{-1})x$$

Lagrangian A-brane $\longrightarrow$ Module of DAHA

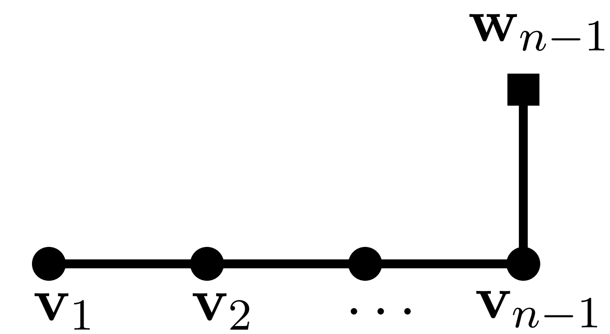
Dimension of a module

$$\mathrm{dim}V = \int_{\mathcal{M}} \mathrm{ch}(\mathcal{B}') \wedge \mathrm{ch}(\mathcal{B}_{cc}) \wedge \mathrm{Td}(\mathcal{M})$$

compact branes $\longleftrightarrow$ Finite dim reps

DAHA Reps

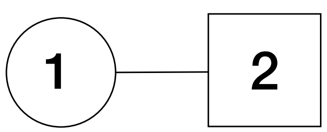
Start with a vertex function for T*Fn



Specify equivariant parameters $a_k = q^{\lambda_k} \hbar^{n-k}$

q-hypergeometric series ⟶ Macdonald polynomials with $\hbar = t^{-1}$

E.g. k=2, n=2



$$V(z; \hbar q, q) = P_{(1,1)}(z|q, \hbar)$$

$$V(z; \hbar q^2, q) = P_{(2,0)}(z|q, \hbar)$$

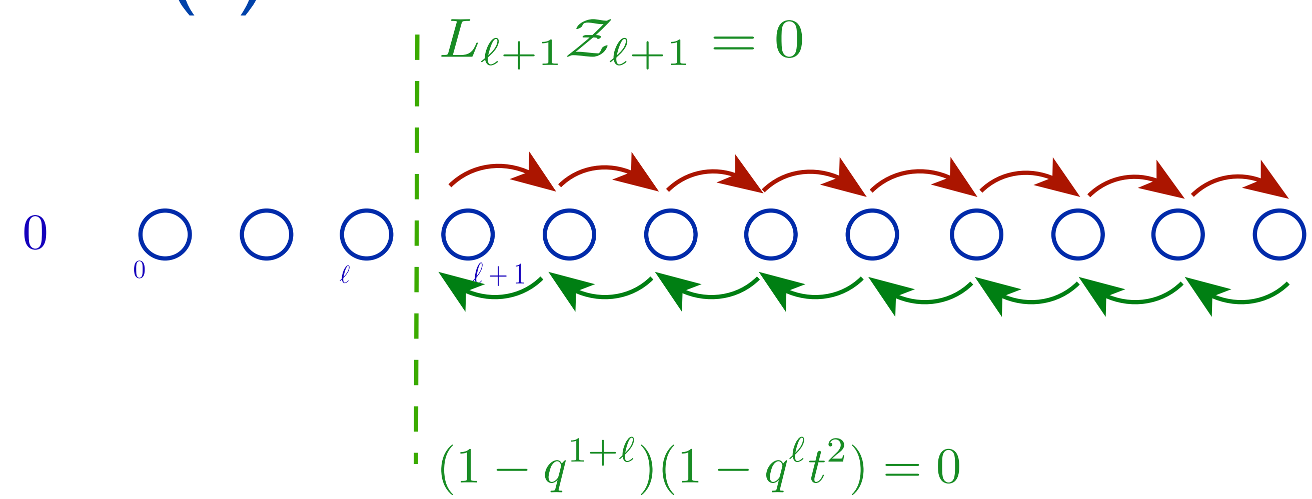
Raising and lowering operators of sl(2) DAHA

$$R_a = x + a_k^{-1} z$$

$$L_a = x + a_k z$$

$$R_a \mathcal{Z}_a = r_a \mathcal{Z}_{a+1}$$

$$L_a \mathcal{Z}_a = l_a \mathcal{Z}_{a-1}$$

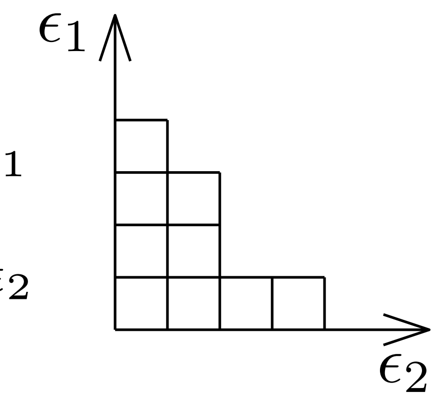


$$0 \longrightarrow \iota(\mathfrak{D}_k^+ \oplus \mathfrak{D}_k^-) \longrightarrow \mathfrak{U}_N \longrightarrow \mathfrak{V}_{n+1} \longrightarrow 0$$

Fock Space

$$q = e^{\epsilon_1}$$

$$\hbar = e^{\epsilon_2}$$



Power-symmetric variables $\quad p_m = \sum_{l=1}^{n} z_l^m$

Macdonald polynomials depend only on k and the partition

$$P_{\Box\Box} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\substack{\Box\\ \Box}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

Starting with Fock vacuum $\quad |0\rangle$

Construct Hilbert space $\quad a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $\quad a_{-\lambda}|0\rangle = a_{-\lambda_1}\cdots a_{-\lambda_l}|0\rangle$

Commutators $\quad [a_m, a_n] = m\frac{1-q^{|m|}}{1-\hbar^{|m|}}\delta_{m,-n}$

DAHA Action

[PK 1805.00986]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from P1 to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} \hbar^{n-k}$ restricts us to the Fock space representation of (q,h)-Heisenberg algebra which is a DAHA module

In other words, we can define the following action

$$
\begin{tikzcd}
\mathcal{A}_n \arrow[d, bend right] & & \mathfrak{E} \arrow[ll, "\text{[Schiffmann Vaserot]}"'] \arrow[d, bend right] \\
K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)\Big|_{a_k = q^{\lambda_k}\hbar^{n-k}} \arrow[rr] & & K_{q,\hbar}(\oplus_k \mathrm{Hilb}^k[\mathbb{C}^2])
\end{tikzcd}
$$

λ not more than n columns

$\mathbb{C}^\times_q \times \mathbb{C}^\times_\hbar$ fixed points are Macdonald polynomials

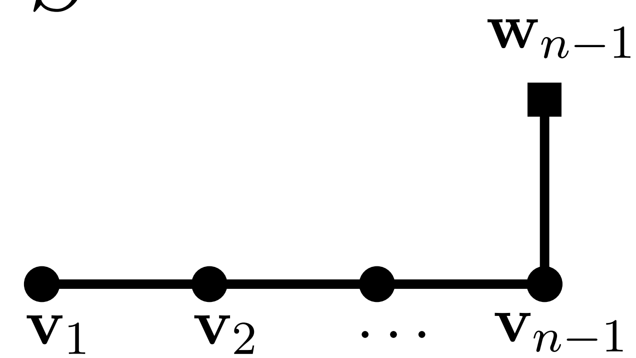
Large-n transition

Recall that $\mathrm{Hilb}^k[\mathbb{C}^2] = \mathcal{M}^{\mathrm{inst}}_{1,k}$ How did U(1) 5d SYM appear?

Starting with M-theory on $S^1 \times \mathbb{C}_q \times \mathbb{C}_\hbar \times T^* S^3$

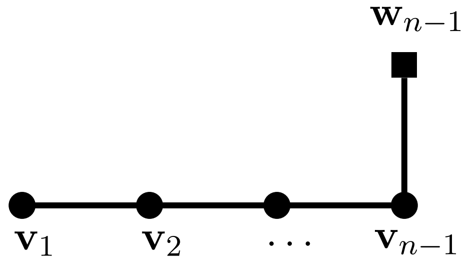
n M5 branes wrapping $S^1 \times \mathbb{C}_q \times S^3$

Upon compactification on three sphere will get *3d quiver gauge theory* on T^*Fl_n



When n becomes large the background undergoes through the **conifold transition** and the *resolved* conifold becomes a *deformed* conifold Y: $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times Y$

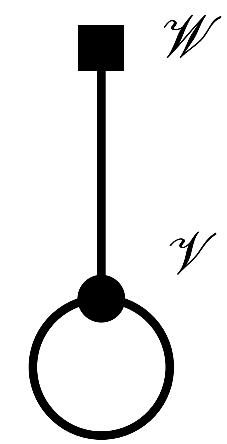
Reduction on Y leads us to a 5d U(1) theory with 8 supercharges



Flags vs ADHM

[PK Sciarappa]

$K_T(\mathbf{QM}(\mathbb{P}^1, X))$	$K_{q,\hbar}(\mathrm{Hilb}(\mathbb{C}^2))$
Kähler/quantum parameters of X $z_1, z_2 \dots$	Ring generators $x_1, x_2, \dots$
Vertex function $\mathsf{V}_\mathbf{q}$	Classes of $(\mathbb{C}^\times)^2$ fixed points $[\mathcal{J}]$
$\mathbb{C}^\times_q$ acting on base curve	$\mathbb{C}^\times_q$ acting on $\mathbb{C} \subset \mathbb{C}^2$
$\mathbb{C}^\times_\hbar$ acting on cotangent fibers of X	$\mathbb{C}^\times_\hbar$ acting on another $\mathbb{C} \subset \mathbb{C}^2$
Eigenvalues e_r of tRS operators T_r	Chern polynomials $\mathcal{E}_r$ of $\Lambda^r \mathcal{U}$



quantum deformation:

Eigenvalues of **elliptic** RS model at large n

$$E_r(\vec{\zeta}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})}{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})} \prod_{i\in\mathcal{I}} p_k$$

Eigenvalues of **quantum** multiplication by

$$\mathscr{U} = \mathscr{W} + (1-q)(1-\hbar)\mathscr{V}|_{\mathcal{J}_{\vec{\lambda}}}$$

Chern roots obey

$$\prod_{l=1}^{N} \frac{s_a - \mathrm{a}_l}{s_a - q^{-1}\hbar^{-1}\mathrm{a}_l} \cdot \prod_{\substack{b=1\\ b\neq a}}^{k} \frac{s_a - q s_b}{s_a - q^{-1} s_b} \frac{s_a - \hbar s_b}{s_a - \hbar^{-1} s_b} \frac{s_a - q^{-1}\hbar^{-1} s_b}{s_a - q\hbar s_b} = \mathfrak{z}$$

ADHM & 1/2 ADHM

[PK Koroteeva Gorsky Vainshtein]

$$K_\hbar(T^*\mathbb{F}l_n) \longleftrightarrow$$ ADHM (instanton moduli space)

$$\lim_{n\to\infty}\left[\hbar^{n-1}(1-\hbar)\left\langle W_{\square}^{U(n)}\right\rangle\right]\Big|_\lambda = a-(1-q)(1-\hbar)e_1(s_1,\dots,s_k)|_\lambda$$

Claim: $\hbar \to \infty$ retracting the fibers, dimensional transmutation

[Hanany Tong]

$$K(\mathbb{F}l_n) \longleftrightarrow$$ 1/2 ADHM (vortex moduli space)

Eigenvalues of **affine** qToda lattice at large n ⟷ Eigenvalues of **quantum** multiplication by

$$H_1^{\text{aff}} = \mathfrak{p}_1\left(1-\mathfrak{p}^\Lambda\frac{\mathfrak{z}_n}{\mathfrak{z}_1}\right)+\sum_{i=2}^{n}\mathfrak{p}_i\left(1-\frac{\mathfrak{z}_{i-1}}{\mathfrak{z}_i}\right)$$

$$\mathcal{E}_1^\Lambda(\lambda) = a-(1-q)e_1(s_1,\dots,s_k)$$

Chern roots obey

$$\prod_{l=1}^{N}(s_a-\mathrm{a}_l)\cdot\prod_{\substack{b=1\\ b\neq a}}^{k}\frac{qs_a-s_b}{s_a-qs_b}=\widetilde{\mathfrak{p}}^\Lambda$$