We claim that there is a nontrivial correspondence between equivariant K-theories of these types of quiver with Manin (ADHM) quivers which arise in the study of moduli spaces of sheaves on surfaces in mathematics (we shall focus solely on the A-type quivers), and Atiyah-Drinfeld-Hitchin–ADE-type quivers and their algebras of representations of such quivers modulo the automorphisms of vertices. Currently in the study of quantum K-theory of the quiver varieties in question in terms of the tRS system $K\Psi^+_1$ related. This lead us to a clear understanding of quantum K-theory of the quiver varieties of integrals of motion of the trigonometric Ruijsenaars-Schneider (tRS) model was formulated...
Higher Symmetries

In this talk we shall discuss some higher symmetries which arise in physics and geometry.

\[ \mathcal{E} \sim U_{q_1,q_2} \left( \hat{gl}_1 \right) \sim \mathcal{E}_{q_1,q_2} \sim gl_{\infty}DAHA_{q_1,q_2}^S \sim DIM_{q_1,q_2} \sim D(\mathcal{A}_{\text{shuffle}}) \]

This algebra often appears in the BPS/CFT correspondence:

Connects BPS observables of \( \mathcal{N}=2 \) supersymmetric gauge theories with CFT correlators (Mathematically: relates structures arising on moduli spaces of sheaves (instantons) with vertex operator algebras or qVOAs)
BPS/CFT and Geometry

Mathematicians have now several *proofs* of BPS/CFT (AGT) in limiting cases (no fundamental matter), those proofs do not use the original class-S construction

Physics *proof* by Kimura and Pestun uses direct localization computations

One of our goals is to understand BPS/CFT **geometrically**

Namely we want describe instanton counting and vertex operator algebras in terms of **quantum geometry** (quantum cohomology or quantum K-theory) of some family of spaces

In other words we want (q)VOAs to **emerge** from quantum geometry
**Recent Developments**

**Vertex Algebras at the Corner** [Gaiotto Rapcak]

VOAs at junctions of supersymmetric intersections in N=4 SYM

**COHA and VOAs** [Rapcak Soibelman Yang Zhao]

Action of COHA on the moduli space of *spiked* instantons

**Quiver W-algebras** [Kimura Pestun]

4,5,6d quiver gauge theories on $R^4 \times S$ in Omega background

**The Magnificent Four** [Nekrasov]

D8 brane probed by D0 branes in B field

$U(1)^4 \subset \text{Spin}(8)$ + additional *nongeometric* $U(1)$ symmetry $q_1, q_2, q_3, q_4$
Nakajima Quiver Varieties

\[ \text{Rep}(v, w) — \text{linear space of quiver reps} \]

\[ \mu : T^* \text{Rep}(v, w) \rightarrow \text{Lie}(G)^* \quad \text{moment map} \]

Nakajima quiver variety

\[ X = \mu^{-1}(0) \text{ // } G \quad G = \prod GL(V_i) \]

Automorphism group

\[ \text{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times \]

Maximal torus

\[ T = \mathbb{T}(\text{Aut}(X)) \]

Tensorial polynomials of tautological bundles \( V_i, W_i \) and their duals generate \textit{classical} \( T \)-equivariant \( K \)-theory ring of \( X \)
Quasimaps

\[ q: \mathcal{C} \to X \]

is described by collection of vector bundles \( \mathcal{V}_i \) on \( \mathcal{C} \) of ranks \( v_i \) with section \( f \in H^0(\mathcal{C}, \mathcal{M} \oplus \mathcal{M}^* \otimes h) \) satisfying \( \mu = 0 \)

where 
\[
\mathcal{M} = \sum_{i \in I} \text{Hom}(\mathcal{V}_i, \mathcal{V}_i) \oplus \sum_{i, j \in I} Q_{ij} \otimes \text{Hom}(\mathcal{V}_i, \mathcal{V}_j)
\]

Degree \((v_1, \ldots, v_{n-1})\)

Evaluation map
\[
ev_p(f) = f(p) \in [\mu^{-1}(0)/G] \supset X
\]

Stable if \( f(p) \in X \)

for all but finitely many singular points

Resolve to make proper ev map
Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q$ base $\mathbb{P}^1$ keeping two fixed points (0, infinity). Define **vertex function** with quantum (Novikov) parameters $\z^\mathbf{d}=\prod_{i\in I} z_i^{d_i}$

$$V^{(\tau)}(z) = \sum_{\mathbf{d}=\mathbf{0}}^{\infty} z^{d_{ev_{p_2,\tau}}} \left(QM^d_{nonsing_{p_2}, \hat{\Theta}_{vir}(V_i|_{1})}\right) \in K_{T_q}(X)_{loc}[\hat{z}]$$

Define **quantum K-theory** as a ring with multiplication

$$A \otimes B = A \otimes B + \sum_{d=1}^{\infty} A \otimes_d B \z^d$$

$$\mathcal{F} \otimes = \sum_{\mathbf{d}=\mathbf{0}}^{\infty} z^{d_{ev_{p_1,p_3}}} \left(QM^d_{p_1,p_2,p_3}, ev_{p_2}^* (G^{-1}\mathcal{F}) \hat{\Theta}_{vir}\right) G^{-1} \xrightarrow{G^{-1}\mathcal{F}} G^{-1}$$

**gluing**

$$\mathcal{C}_0 = \mathcal{C}_{0,1} \cup_p \mathcal{C}_{0,2}$$
Vertex Functions

After classifying fixed points of space of nonsingular quasimaps we can compute the vertex

\[ V_p^{(\tau)}(z) = \sum_{d_{i,j} \in C} z^d q^{N(d)/2} \text{EHG} \quad \tau(x_{i,j} q^{-d_{i,j}}). \]

\[ E = \prod_{i=1}^{n-1} \prod_{j,k=1}^{v_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j} - d_{i,k}} \quad x_{i,j} \in \{a_1, \ldots, a_{w_n}\} \]

Vertex (trivial class)

\[ V = 2\phi_1 \left( \hbar, \frac{a_1}{a_2}, q \frac{a_1}{a_2}; q, z \right) \quad v_1 = 1, w_1 = 2 \]

Vortex

\[ N = 2^* \text{ quiver gauge theory on } X_3 = \mathbb{C}_{\epsilon_1} \times S^1_\gamma \]

Lagrangian depends on twisted masses \( a_1, a_2 \)

Fl parameter \( z \) and U(1) R-symmetry \( \log \hbar \)
The K-theory vertex function satisfies equation of motion of the trigonometric Ruijsenaars-Schneider model

\[ \hat{H}_d V = e_d(z_1, \ldots, z_{n-1})V \]

\[ \hat{H}_d = \sum_{I \subset \{1, \ldots, n\}, |I|=d} \left( \prod_{i \in I, j \notin I} \frac{a_i^\frac{1}{2} - a_j^{-\frac{1}{2}}}{a_i - a_j} \right) \prod_{i \in I} T_i^q \]

3d Mirror version (a.k.a. bispectral dual)

\[ \hat{H}_d^1 V = e_d(a_1, \ldots, a_{n-1})V \]

\[ \hat{H}_d^1(a_i, \hbar, T_a^q) = \hat{H}_d(z_i/z_{i+1}, \hbar^{-1}, T_z^q) \]
Spherical DAHA

Ruijsenaars-Schneider Hamiltonians form a maximal commuting subalgebra inside spherical double affine Hecke algebra for gl(n)

\[ \{ \hat{H}_1, \ldots, \hat{H}_n \} \subset \text{DAHA}_{q, \hbar}^S(gl_n) =: A_n \]

\( \hat{H}_d \) are also known as Macdonald operators

[Spherical gl(n) DAHA is a geometric quantization of the moduli space of flat GL(n;C) connections on a torus with one simple puncture]

If time will be tight then say in words that A and B are holonomies of electric and magnetic operators

Should be around 50% of time here!!!!

\[ \mathcal{M}_n = \{ A, B, C \}/GL(n; \mathbb{C}) \]

\[ ABA^{-1}B^{-1} = C \]

\[ C = \text{diag}(\hbar, \ldots, \hbar, \hbar^{1-n}) \]

\[ A_n = \mathbb{C}_J[\mathcal{M}_n] \]
Line Operators and Branes

$M_n$ is the moduli space of vacua in $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^3 \times S^1$ with gauge group U(n) and is described by VEVs of line operators wrapping the circle.

$A$ and $B$ are holonomies of electric and magnetic line operators

Omega background along real 2-plane $\mathbb{R}^2 \times \mathbb{R} \times S^1$

Line operators are forced to stay at the tip of the cigar and slide along the remaining line, hence non-commutativity

$H_{\mathcal{O} } = \text{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$

$\mathcal{H} = \text{Hom}(\mathcal{B}_{cc}, \mathcal{B})$

[Guokov-Witten]
[Nekrasov-Witten]
**Hitchin Moduli Space (n=2)**

- **SU(2) theory** \( x = Tr A \) **electric**
- **sl(2) flat connections** \( y = Tr B \) **magnetic** \( z = Tr AB \) **dyonic**

Nonabelian Hodge correspondence:

\[
\mathcal{M}_{\text{flat}}(SL(2; \mathbb{C}), T^2 \setminus \{\text{pt}\}) \simeq \mathcal{M}_H(SU(2), T^2 \setminus \{\text{pt}\})
\]

\[
\mathcal{M}_H : \quad x^2 + y^2 + z^2 + xyz = \hbar + \hbar^{-1} + 2 \quad \text{for } \hbar = 1
\]

Elliptic fibration with one singular fiber of Kodaira type I_0*

- **SU(2)** theory \( \xrightarrow{\text{SU(2)}} \) sl(2) flat connections

---

[Gukov]
[PK Gukov Nawata Saberi]
DAHA Modules

Algebra acts naturally by attaching open strings to closed strings

\[ B' \rightarrow \text{Hom}(B_{cc}, B') \quad \text{gives a functor} \quad \text{Hom}(B_{cc}, \cdot) \]

\[ \mathcal{B}_{cc} : \mathcal{L} \rightarrow \mathcal{M}_H \]

Algebra-deformation quantization of functions on \( \mathcal{M}_H \)

\[
\begin{align*}
F + B &= \frac{i}{\log q} \Omega_J \\
\Omega_J &= \frac{dx \wedge dy}{2z - xy}
\end{align*}
\]

Hilbert space comes from \((B_{cc}, B')\) strings

\[ \mathcal{A} = \text{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \]

\[ \mathcal{Q} \]

\[ \mathcal{H} = \text{Hom}(\mathcal{B}_{cc}, \mathcal{B}') \]

\[ \text{Fuk}(\mathcal{M}, \Omega) \cong \text{Rep}(\mathcal{A}) \]

Lagrangian A-brane

Module of DAHA

Dimension of a module

\[ \dim V = \int_{\mathcal{M}} \text{ch}(B') \wedge \text{ch}(B_{cc}) \wedge \text{Td}(\mathcal{M}) \]

compact branes

Finite dim reps
DAHA Reps

Start with a vertex function for $T^*F_n$

Specify equivariant parameters $a_k = q^{\lambda_k} \hbar^{n-k}$

q-hypergeometric series $\Rightarrow$ Macdonald polynomials with $\hbar = t^{-1}$

E.g. $k=2, n=2$

$V(z; \hbar q, q) = P_{(1,1)}(z|q, \hbar)$

$V(z; \hbar q^2, q) = P_{(2,0)}(z|q, \hbar)$

Raising and lowering operators of $sl(2)$ DAHA

$R_a = x + a_k^{-1} z$

$L_a = x + a_k z$

$R_a Z_a = r_a Z_{a+1}$

$L_a Z_a = l_a Z_{a-1}$

$0 \rightarrow \nu(\mathcal{D}_k^+ \oplus \mathcal{D}_k^-) \rightarrow \mathcal{U}_N \rightarrow \mathcal{W}_{n+1} \rightarrow 0$
Fock Space

Power-symmetric variables  \[ p_m = \sum_{l=1}^{n} z_{l}^m \]

Macdonald polynomials depend only on k and the partition

\[
\begin{align*}
P_{\square} &= \frac{1}{2} (p_1^2 - p_2), \\
P_{\square} &= \frac{1}{2} (p_1^2 - p_2) + \frac{1 - qt}{(1 + q)(1 - t)} p_2
\end{align*}
\]

Starting with Fock vacuum  \[ |0\rangle \]

Construct Hilbert space  \[ a_{-\lambda} |0\rangle \leftrightarrow p_\lambda \]

for each partition  \[ a_{-\lambda} |0\rangle = a_{-\lambda_1} \cdots a_{-\lambda_l} |0\rangle \]

Commutators  \[ [a_m, a_n] = m \frac{1 - q^{|m|}}{1 - \hbar |m|} \delta_{m,-n} \]
Vertex functions or quantum classes for $X$ are elements of quantum K-theory of $X$. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from $\mathbb{P}^1$ to $X$

$$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$$

with maximal torus $T = \mathbb{T}(U(n) \times U(1)_{\hbar} \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} \hbar^{n-k}$ restricts us to the Fock space representation of $(q,\hbar)$-Heisenberg algebra which is a DAHA module.

In other words, we can define the following action

$$K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)|_{a_k = q^{\lambda_k} \hbar^{n-k}}$$

$\lambda$ not more than $n$ columns

$$\mathcal{A}_n \xleftarrow{\text{Schiffmann Vaserot}} \mathcal{E}$$

$$K_{q,\hbar}(\bigoplus_k \text{Hilb}^k[\mathbb{C}^2])$$

$\mathbb{C}^\times_q \times \mathbb{C}^\times_\hbar$ fixed points are Macdonald polynomials
Large-n transition

Recall that $\text{Hilb}^k[\mathbb{C}^2] = \mathcal{M}_{1,k}^{\text{inst}}$

How did $\text{U}(1)$ 5d SYM appear?

Starting with M-theory on $n$ M5 branes wrapping $S^1 \times \mathbb{C}_q \times \mathbb{C}_{\hat{h}} \times T^* S^3$

Upon compactification on three sphere will get 3d quiver gauge theory on $T^* \text{Fl}_n$

When $n$ becomes large the background undergoes through the conifold transition and the resolved conifold becomes a deformed conifold $Y$: $S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times Y$

Reduction on $Y$ leads us to a 5d $\text{U}(1)$ theory with 8 supercharges
Flags vs ADHM

<table>
<thead>
<tr>
<th>$K_T(\text{QM}(\mathbb{P}^1, X))$</th>
<th>$K_{q,\hbar}(\text{Hilb}(\mathbb{C}^2))$</th>
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<tbody>
<tr>
<td>Kähler/quantum parameters of $X$</td>
<td>$z_1, z_2 \ldots$</td>
</tr>
<tr>
<td>Vertex function $V_q$</td>
<td>$x_1, x_2, \ldots$</td>
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<tr>
<td>$\mathbb{C}^\times_q$ acting on base curve</td>
<td>Classes of $(\mathbb{C}^\times)^2$ fixed points $[\mathcal{J}]$</td>
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<td>$\mathbb{C}_h^\times$ acting on cotangent fibers of $X$</td>
<td>$\mathbb{C}_h^\times$ acting on another $\mathbb{C} \subset \mathbb{C}^2$</td>
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<tr>
<td>Eigenvalues $e_r$ of tRS operators $T_r$</td>
<td>Chern polynomials $\mathcal{E}_r$ of $\Lambda^r \mathcal{U}$</td>
</tr>
</tbody>
</table>

**quantum deformation:**

Eigenvalues of **elliptic**

RS model at large $n$

$$E_r(\tilde{\zeta}) = \sum_{j \in \{1, \ldots, n\}} \prod_{i \in j} \frac{\theta_1(\hbar \zeta_i/\zeta_j | \mathfrak{p})}{\theta_1(\hbar \zeta_i/\zeta_j | \mathfrak{p})} \prod_{k} p_k$$

Eigenvalues of **quantum**

multiplication by

$$\mathcal{U} = \mathcal{W} + (1 - q)(1 - \hbar) \mathcal{V}|_{\mathcal{J}^\times}$$

Chern roots obey

$$\prod_{l=1}^{N} \frac{s_a - a_l}{s_a - q^{-1} \hbar^{-1} a_l} \cdot \prod_{b=1}^{k} \frac{s_a - q s_b}{s_a - q^{-1} s_b} = \mathfrak{z}$$
**ADHM & 1/2 ADHM**

\[ K_h(T^* \mathbb{F}l_n) \quad \text{ADHM (instanton moduli space)} \]

\[ \lim_{n \to \infty} \left[ \hbar^{n-1}(1 - \hbar) \left< W_{\square}^{U(n)} \right> \right]_{\lambda} = a - (1 - q)(1 - \hbar)e_1(s_1, \ldots, s_k)|_{\lambda} \]

**Claim:** \( \hbar \to \infty \) retracting the fibers, dimensional transmutation

\[ K(\mathbb{F}l_n) \quad 1/2 \text{ ADHM (vortex moduli space)} \]

Eigenvalues of **affine** qToda lattice at large \( n \)

\[ H_{1}^{\text{aff}} = p_1 \left( 1 - p^{\Lambda} \frac{3n}{31} \right) + \sum_{i=2}^{n} p_i \left( 1 - \frac{3i-1}{3i} \right) \]

Eigenvalues of **quantum** multiplication by

\[ E_1^{\Lambda}(\lambda) = a - (1 - q)e_1(s_1, \ldots, s_k) \]

Chern roots obey

\[ \prod_{l=1}^{N} (s_a - a_l) \cdot \prod_{b=1}^{k} \frac{q s_a - s_b}{s_a - q s_b} = \bar{p}^{\Lambda} \]