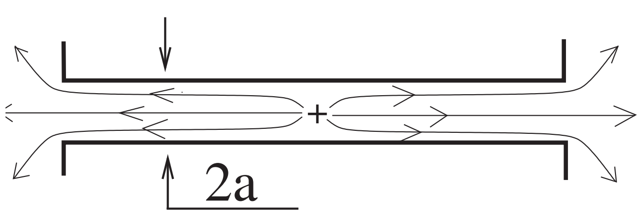
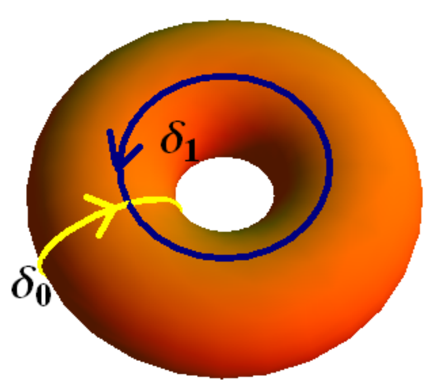
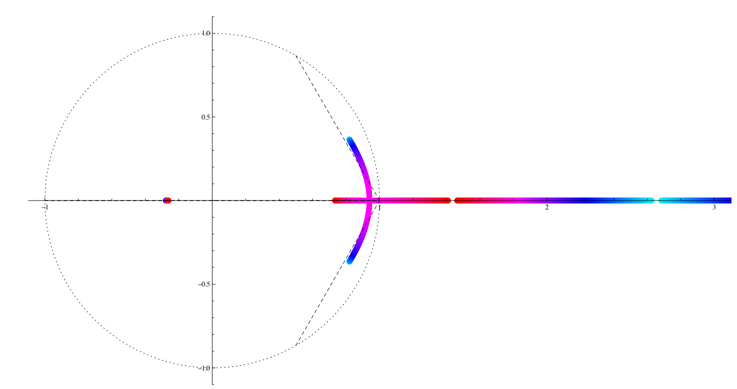
1D Statistical Mechanics
&
Resurgence

Peter Koroteev







(b) $\alpha = 1$

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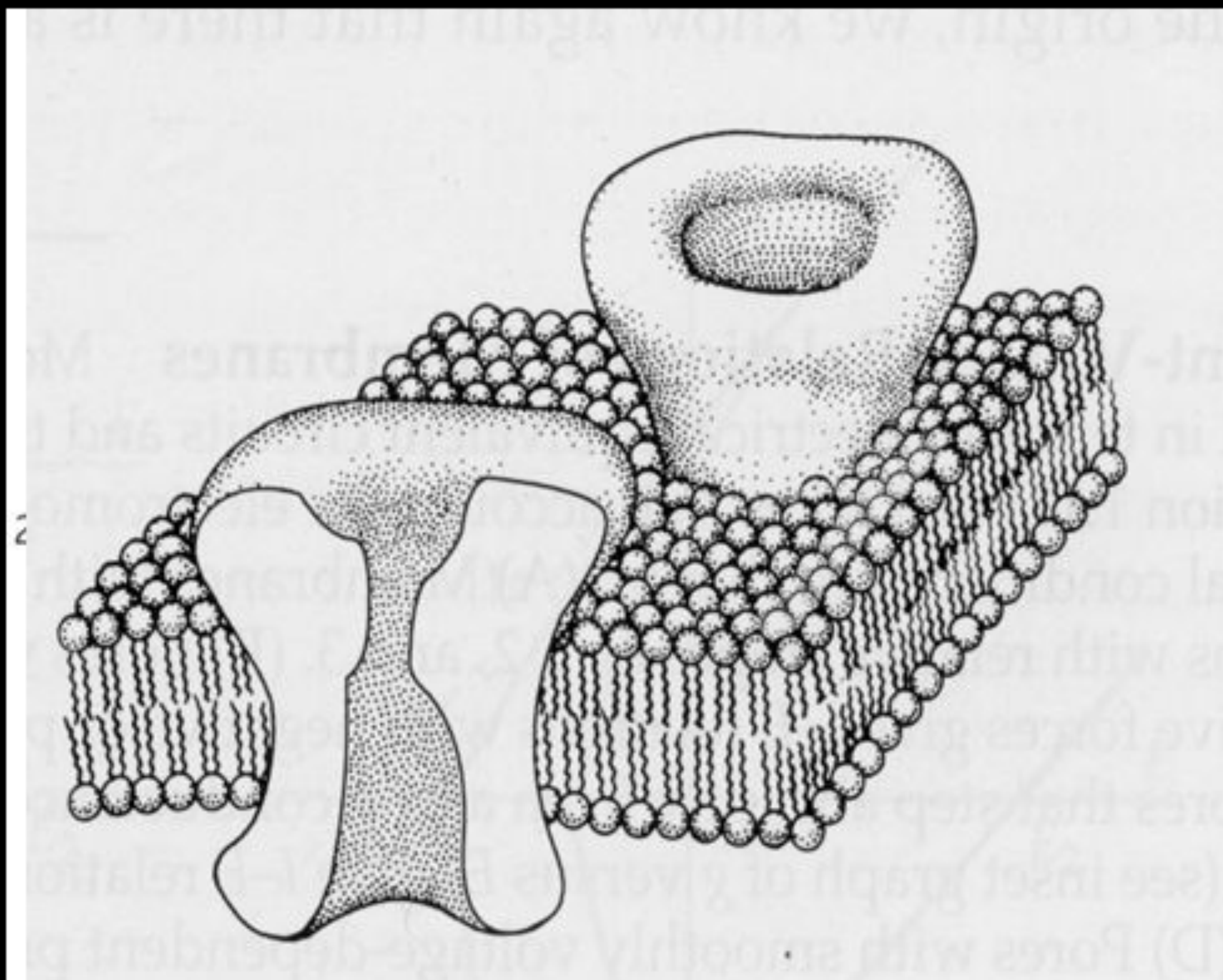
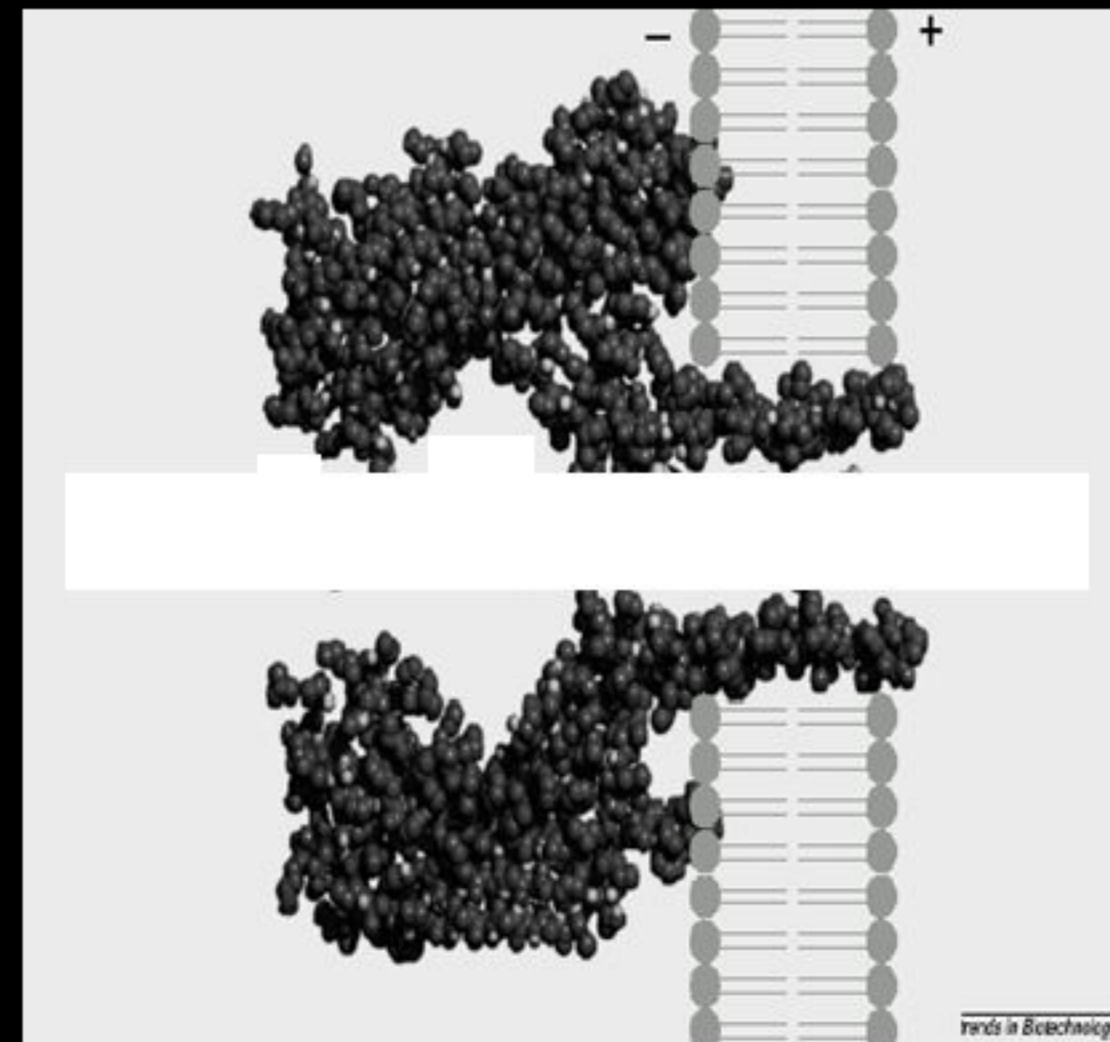
Talk at HEP seminar
Arizona State University, Tempe AZ, September 19th 2018

Ion channels of cell membranes



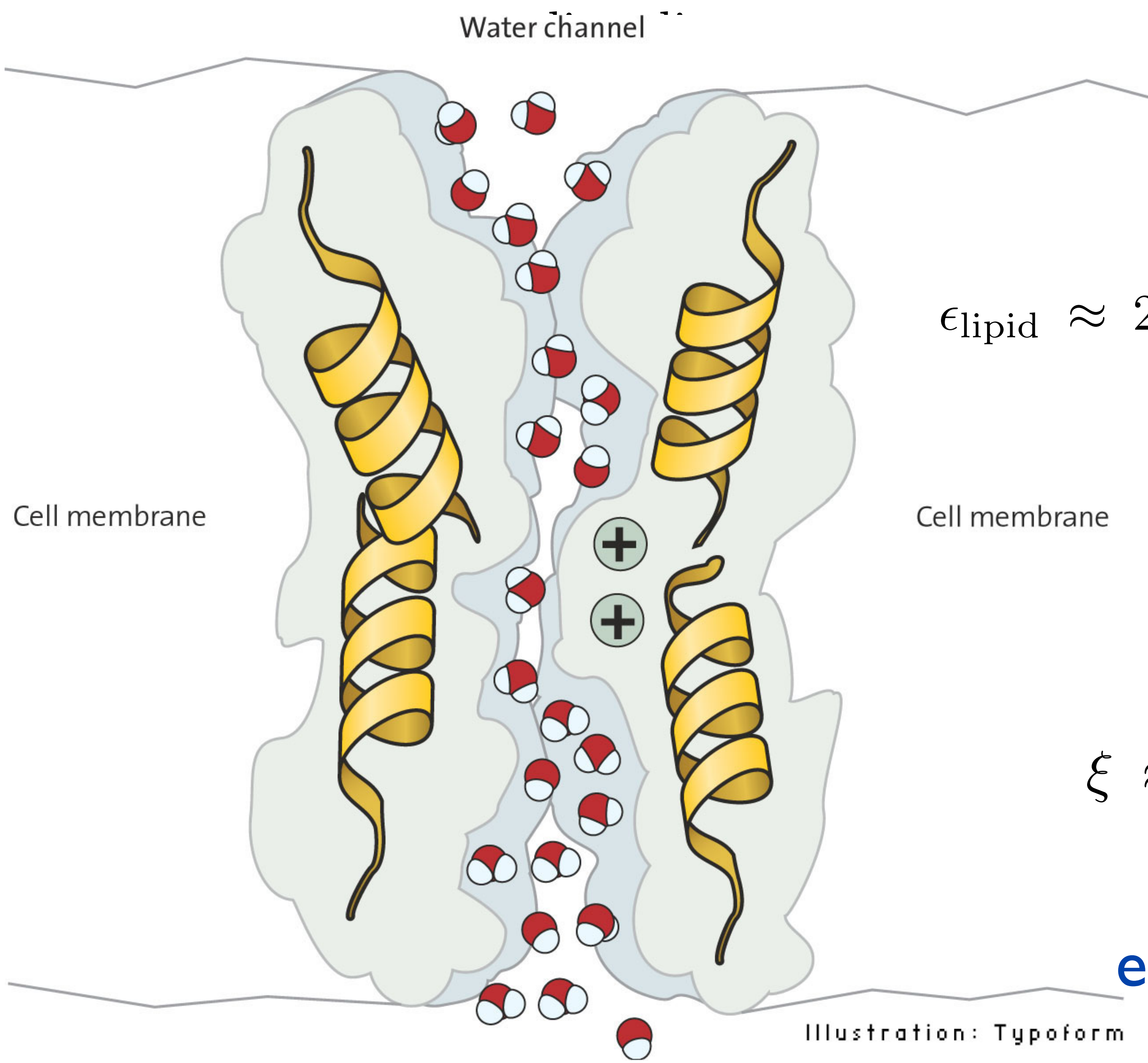


α-Hemolysin

Nobel Prize in Chemistry 2003

to Peter Agre "*for the discovery of water channels*" and Roderick MacKinnon "*for structural and mechanistic studies of ion channels*."

$$\epsilon_{\text{water}} \approx 80$$



$$\epsilon_{\text{lipid}} \approx 2$$

channels of radii

$$a \approx 6\mathring{A}$$

and length

$$L \approx 120\mathring{A}$$

at scale

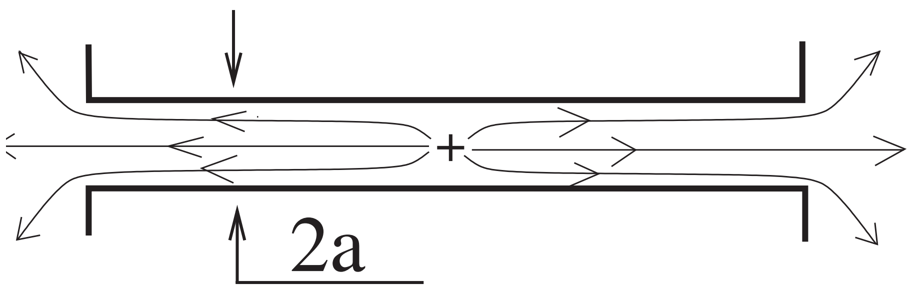
$$\xi \approx a\sqrt{\epsilon_{\text{water}}/\epsilon_{\text{lipid}}}\,\ln(\epsilon_{\text{water}}/\epsilon_{\text{lipid}})$$
$$\approx 140\mathring{A}$$

electric field stays inside channel

Cylindrical Cow

[Kamenev Zheng Larkin Shklovskii]

Ion channels in cell membranes



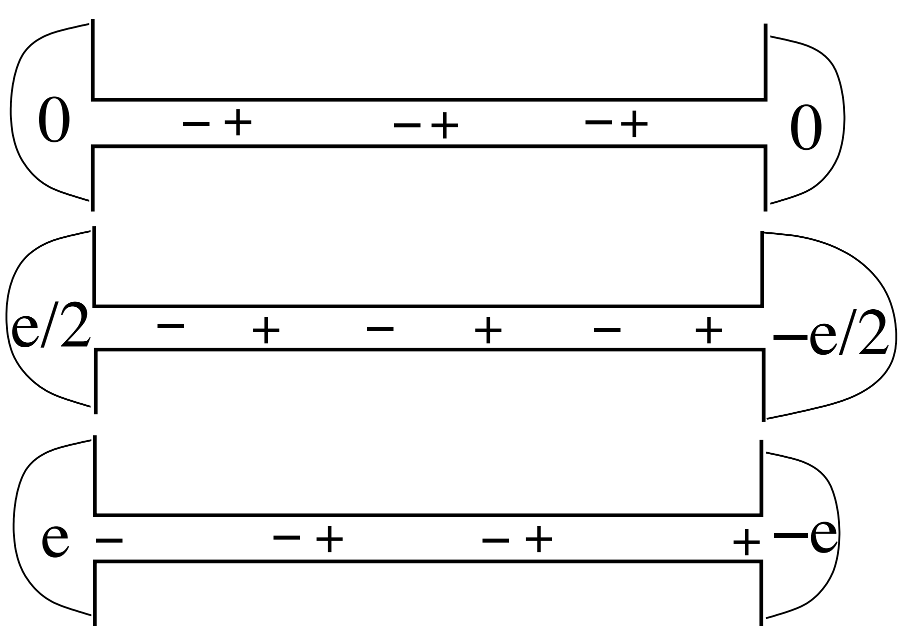
$$U(x_1 - x_2) \approx eE_0|x_1 - x_2|$$

discontinuity $\quad E_0 = 2e/a^2\epsilon_{\mathrm{water}}$

Energy to move an ion thru channel $\quad U(L/4) \approx 4k_B T_{\mathrm{room}}$

typically impedes ion transport

If screened by then barrier drops to $k_B T_{\mathrm{room}}$ which is enough to pass an ion



boundary charge (order parameter) $\quad q$

1D Coulomb Gas

Potential Energy $U = -\frac{eE_0}{2}\sum_{i,j}\sigma_i\sigma_j|x_i - x_j|$ ions $n_1 e$ $-n_2 e$

Grand canonical partition function

$$\mathcal{Z}_L = \sum_{N_1,N_2=0}^{\infty}\frac{f_1^{N_1}f_2^{N_2}}{N_1!N_2!}\prod_{i=1}^{N_1}\int_0^L dx_i \prod_{j=1}^{N_2}\int_0^L dx_j\; e^{-U/k_BT}$$

We want to rewrite the partition function as quantum mechanical amplitude between two *boundary states of charge q*

density $\delta[\rho(x) - \sum_j \sigma_j\delta(x - x_j)]$ elevate into exp $\int [d\theta]\exp(i\theta(x)[\ldots])$

Resum x integrals $\sum_N [f\int dx\, e^{i\sigma\theta(x)}]^N/N! = \exp\{f\int dx\, e^{i\sigma\theta(x)}\}$

Integrate over coordinates (Inverse 1D Laplace) $\exp\{(T/eE_0)\int dx\,\theta\partial_x^2\theta\}$

QM amplitude

$$\mathcal{Z}_L = \left\langle q\middle|\mathcal{X}e^{-\frac{eE_0}{k_BT}\int_0^L dx\,\hat{H}}\middle|q\right\rangle$$

1D Coulomb Gas as Quantum Mechanics

Amplitude $$\mathcal{Z}_L = \left\langle q \middle| \mathcal{X} e^{-\frac{eE_0}{k_B T}\int_0^L dx\, \hat{H}} \middle| q \right\rangle = \sum_m |\langle q|m\rangle|^2 e^{-\frac{eE_0 L}{k_B T}\epsilon_m(q)}$$

Hamiltonian $$\hat{H} = (i\partial_\theta - q)^2 - \left(\alpha_1 e^{in_1\theta} + \alpha_2 e^{-in_2\theta}\right)$$

concentration of ions $\alpha_{1,2} = f_{1,2} k_B T / eE_0$

H acts on Hilbert space of periodic functions $\psi_m(\theta) \;=\; \psi_m(\theta + 2\pi)$

scalar product $\langle q|m\rangle \;=\; \int_0^{2\pi} d\theta e^{-iq\theta}\psi_m(\theta)$

Thermodynamical Transfer Properties

Boundary charge **q** — **Bloch quasi-momentum**.

$$\langle q|m\rangle = \int_0^{2\pi} d\theta e^{-iq\theta}\psi_m(\theta)$$

Momentum is periodic with unit period because integral values of charge can be screened by boundary ions

Pressure proportional to first eigenvalue $P = k_B T \frac{\partial \ln \mathcal{Z}_L}{\partial L} \xrightarrow{L\to\infty} -eE_0\epsilon_0(q)$

Energy barrier for ion transport proportional to the width of first Bloch band $U_0 = eE_0 L\Delta_0$

Thus transfer properties are defined in term of $\epsilon_0, \Delta_0, n_1, n_2$

We shall find those using semiclassical methods

Symmetries

[T. Gulden, M. Janas, A. Kamenev, P. Koroteev]

$$\hat{H} = (i\partial_\theta - q)^2 - \left(\alpha_1 e^{in_1\theta} + \alpha_2 e^{-in_2\theta}\right)$$

- PT symmetry. P: $\theta \to -\theta$
  T: complex conjugation

- Isospectrality. $\theta \to \theta + \theta_0$

$$\alpha_1^{n_2}\alpha_2^{n_1} = \mathrm{const}$$

after scaling

$$\hat{H} = \alpha\left[\hat{p}^2 - \left(\frac{1}{n_1}\,e^{in_1\theta} + \frac{1}{n_2}\,e^{-in_2\theta}\right)\right]$$

quantization

$$\hat{p} = \alpha^{-1/2}(-i\partial_\theta + q)\,; \qquad [\theta, \hat{p}] = i\alpha^{-1/2}$$

concentration = Planck constant!

Numerics

In Fourier modes basis

$$\hat{H}_{m,m'} = (m-q)^2\delta_{m,m'} - \alpha\left(\frac{1}{n_1}\delta_{m+n_1,m'} + \frac{1}{n_2}\delta_{m-n_2,m'}\right)$$

quasimomentum

$$q \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

$$= \begin{pmatrix} (-2-q)^2 & 0 & -\frac{\alpha}{2} & 0 & 0 \\ -\alpha & (-1-q)^2 & 0 & -\frac{\alpha}{2} & 0 \\ 0 & -\alpha & (0-q)^2 & 0 & -\frac{\alpha}{2} \\ 0 & 0 & -\alpha & (1-q)^2 & 0 \\ 0 & 0 & 0 & -\alpha & (2-q)^2 \end{pmatrix}$$

Normalized energy

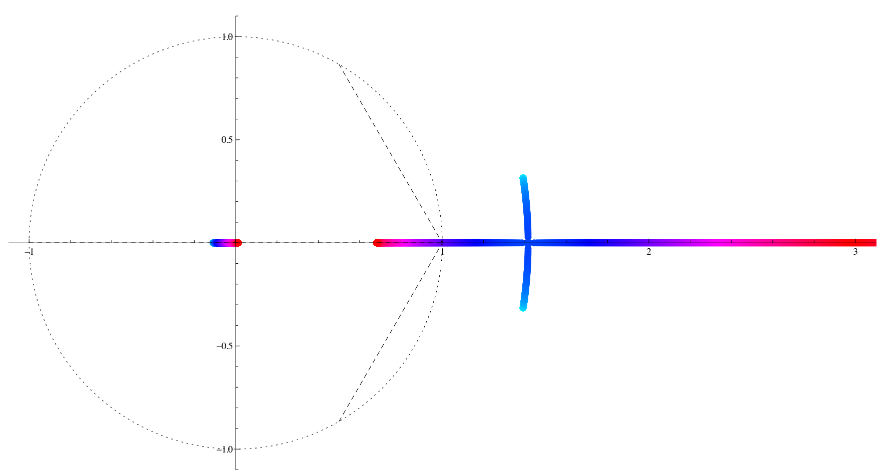
$$u = \frac{n_1 n_2}{n_1+n_2}\frac{\epsilon}{\alpha}$$

(2,1) gas

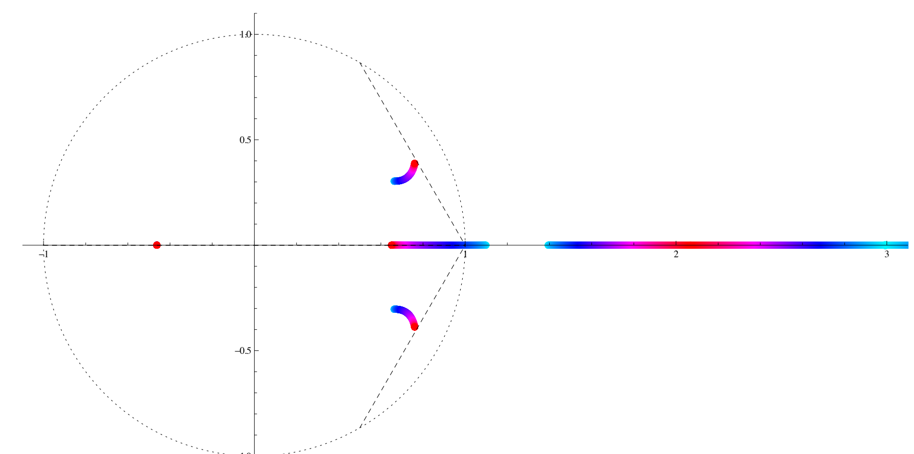
3 branches

#of bands inside |u|=1 $\sim\sqrt{\alpha}$

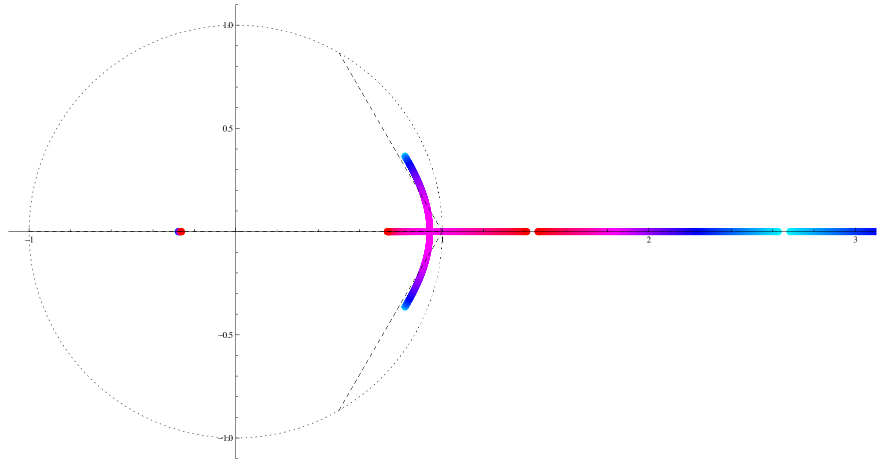
bands wide outside



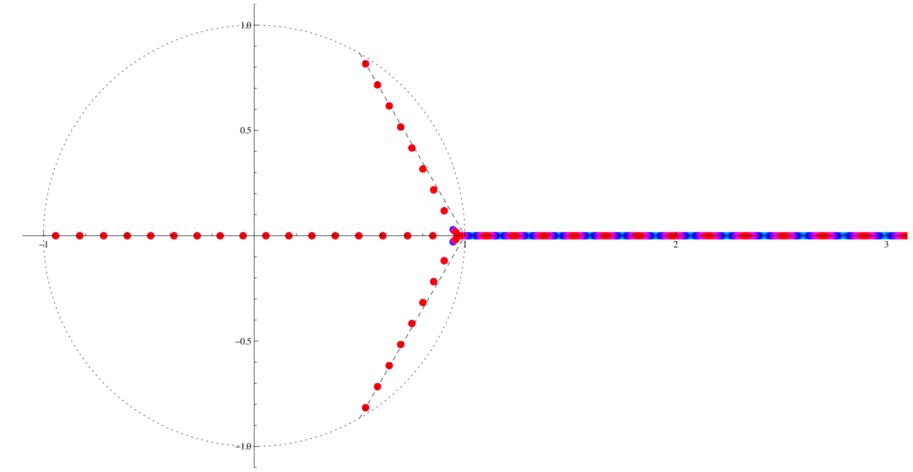
(a) $\alpha = 0.5$



(c) $\alpha = 2$



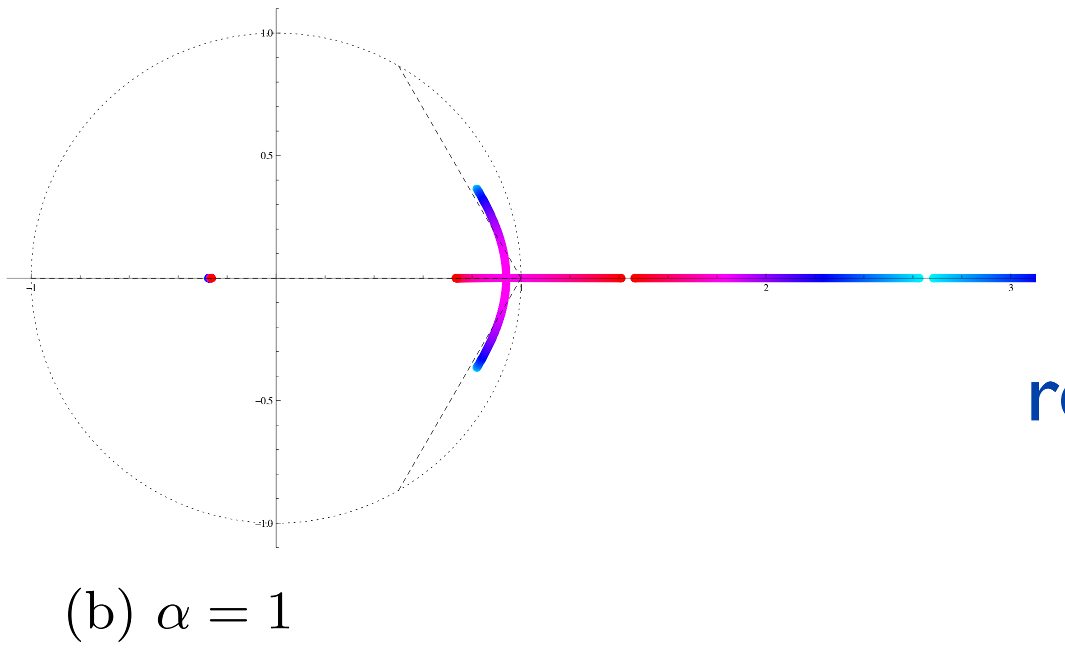
(b) $\alpha = 1$



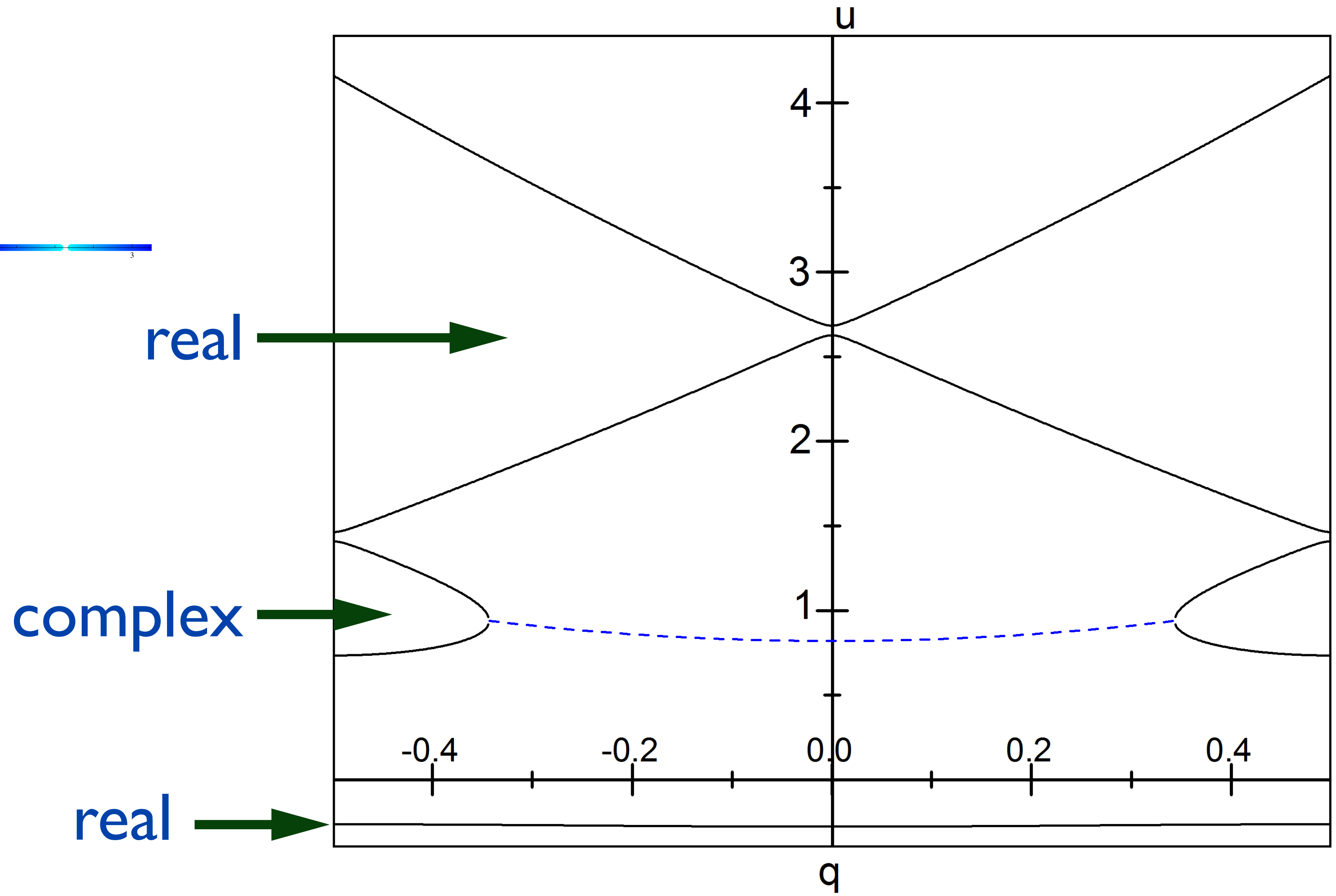
(d) $\alpha = 200$

Band Structure

u-plane

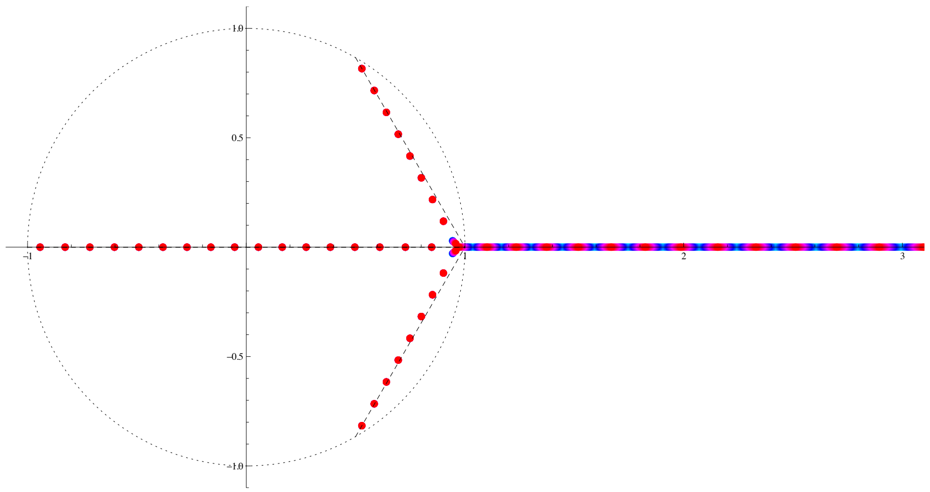


(b) $\alpha = 1$

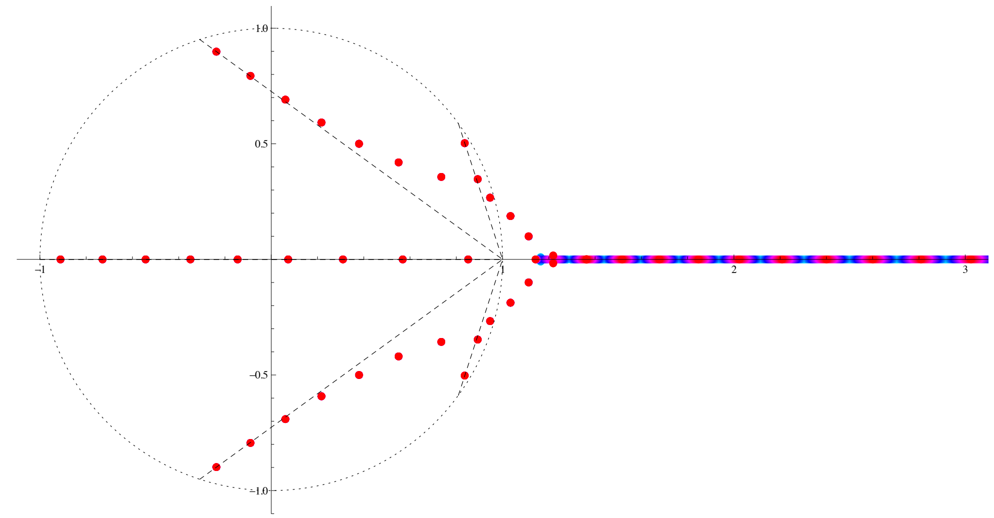


Strong coupling (a = 200)

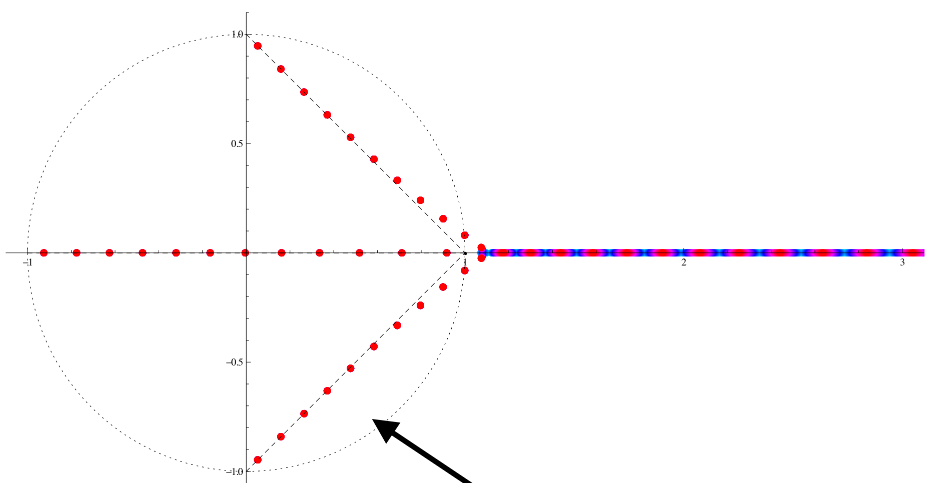
$$\alpha \sim \hbar$$



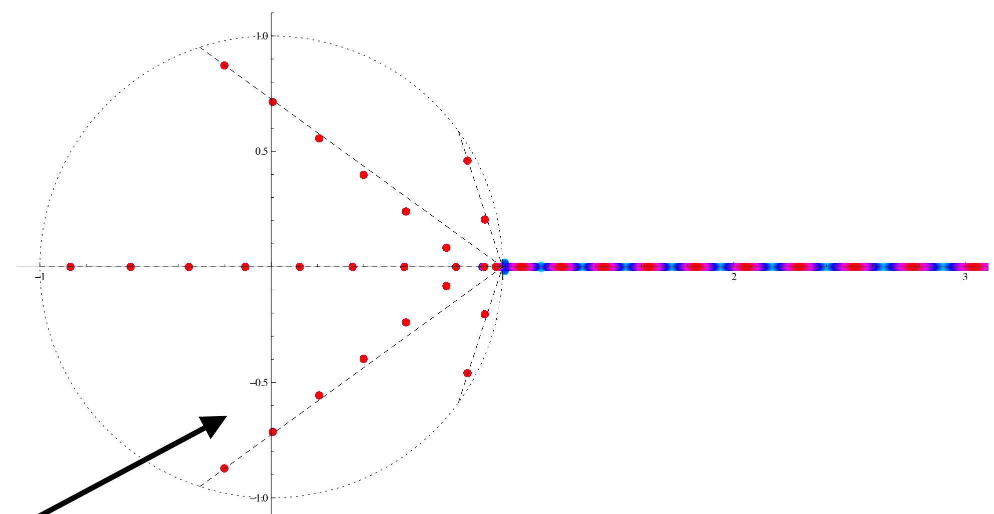
(a) $(2,1)$



(c) $(4,1)$



(b) $(3,1)$



(d) $(3,2)$

$$u = -(1)^{1/(n_1+n_2)}$$

Analytic Solution

First consider $n_1=n_2=1$

Energy equation $\quad 2u = p^2 - 2\cos\theta$

Action form $\quad \lambda = p\,d\theta$

Symplectic form $\quad d\lambda = dp \wedge d\theta$

or $z = e^{i\theta}$

$$2u = p^2 - \left(z + \frac{1}{z}\right)$$

$$\lambda = p\frac{dz}{z}$$

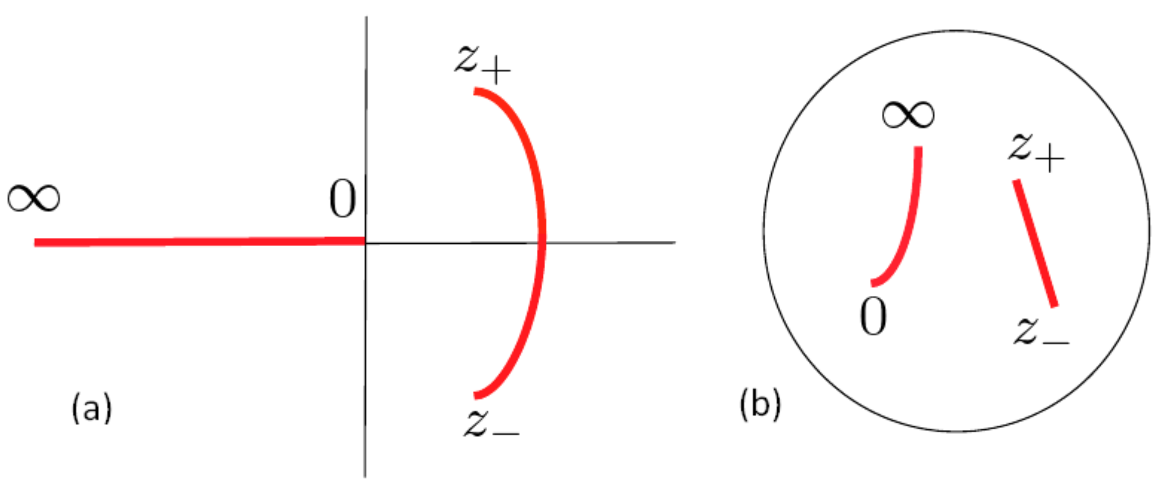
Double cover of complex plane

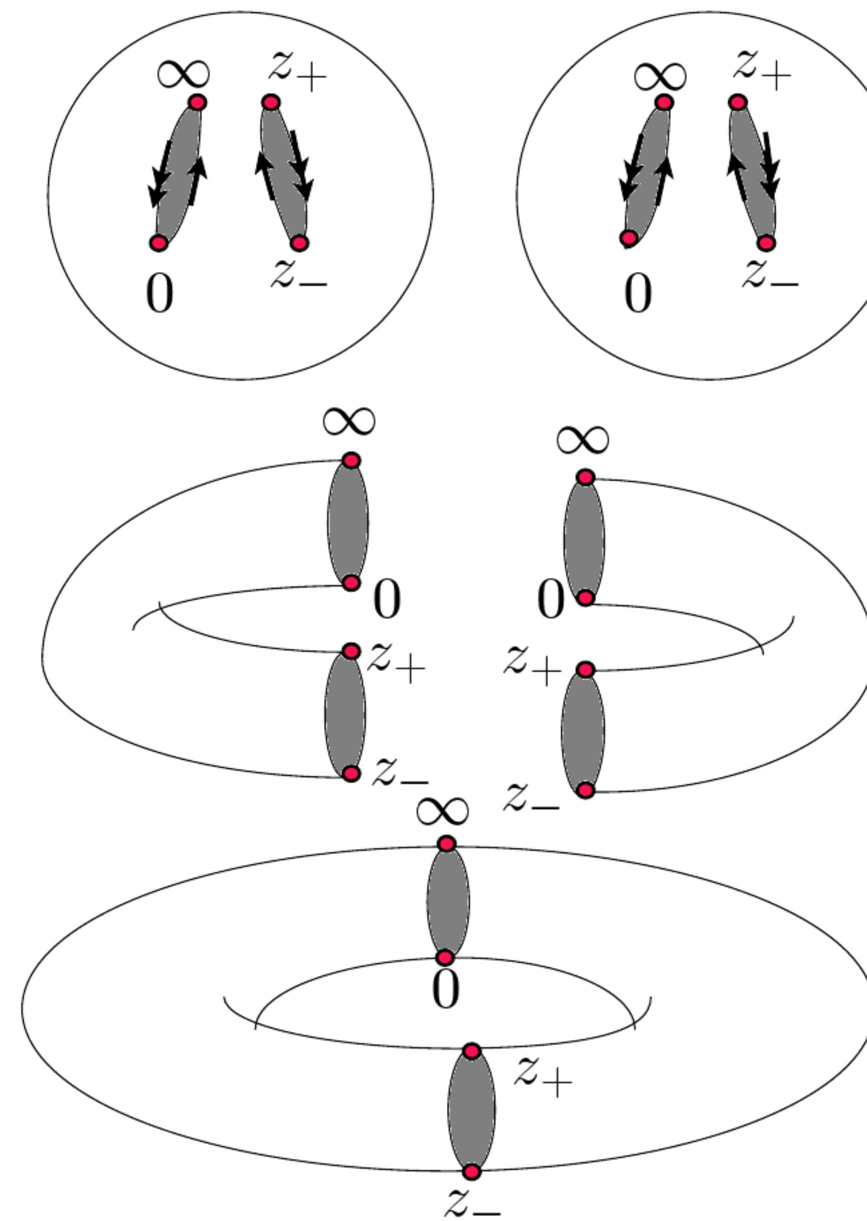
$p^2 = 0$ at $z_\pm = -u \pm i\sqrt{1-u^2}$

and

$$p \sim z^{1/2}, \qquad (z \sim \infty)$$

$$p \sim (z - z_\pm)^{1/2}, \qquad (z \sim z_\pm)$$

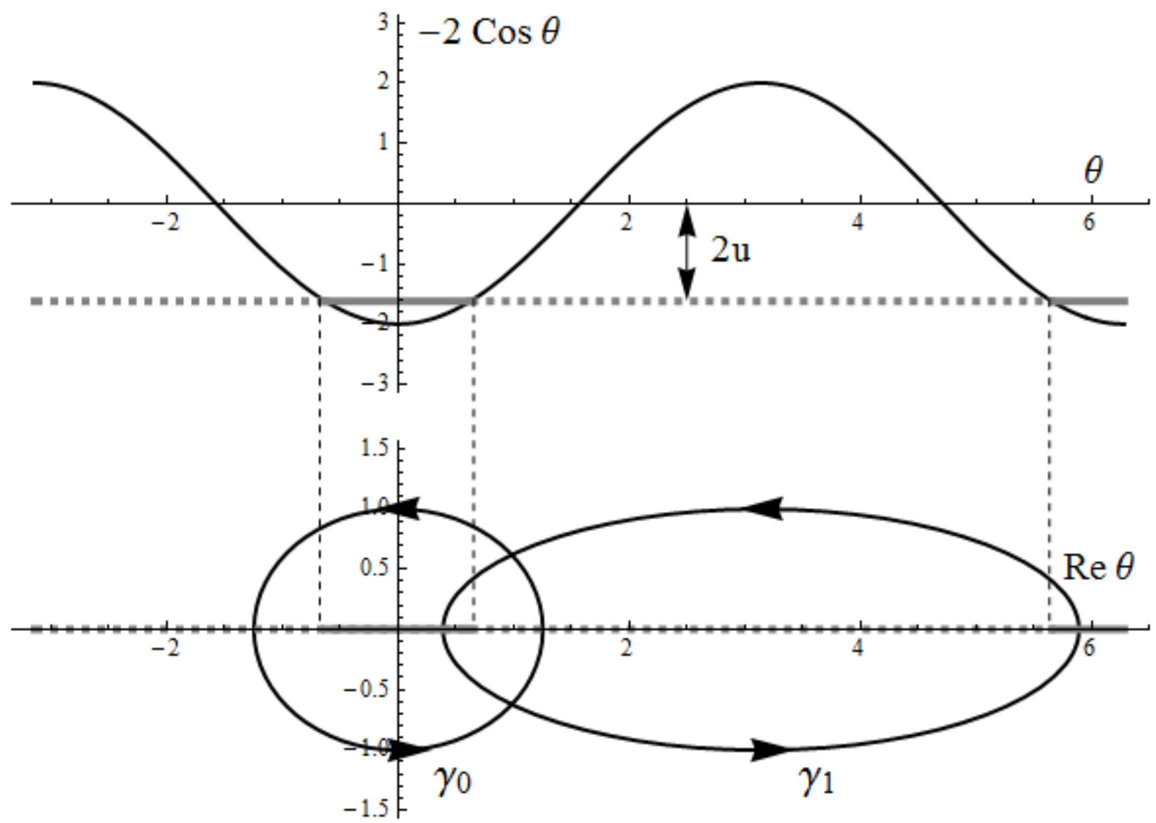




Cosine Potential

Energy equation (complexified)

$$2u = p^2 - 2\cos\theta$$



$$p \sim z^{-1/2}, \qquad (z \sim 0)$$

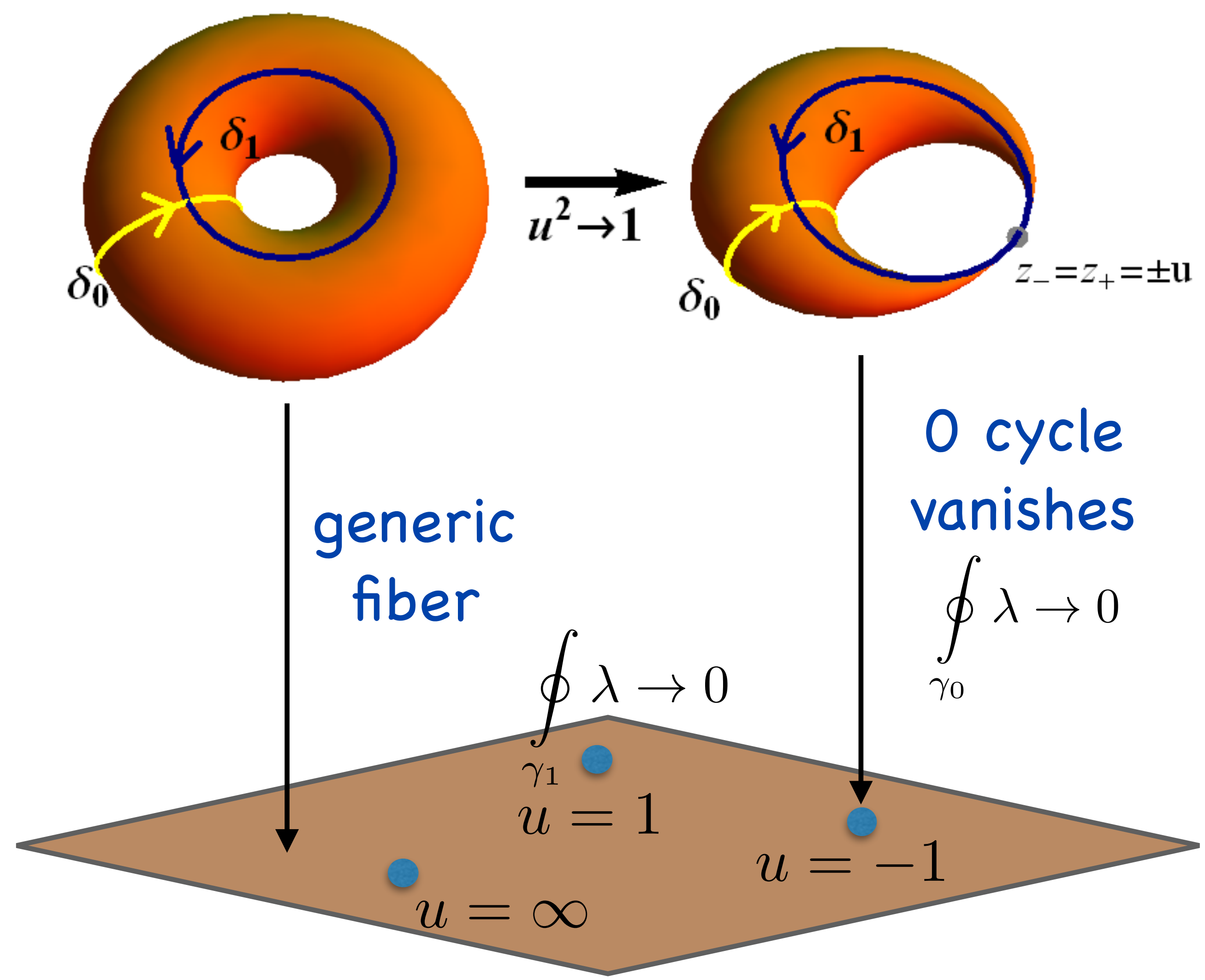
$$p \sim z^{1/2}, \qquad (z \sim \infty)$$

$$p \sim (z - z_\pm)^{1/2}, \qquad (z \sim z_\pm)$$

$$z_\pm = -u \pm i\sqrt{1-u^2}$$

Torus fibration over u-plane

$$S_j(u) = \oint_{\gamma_j} \lambda$$



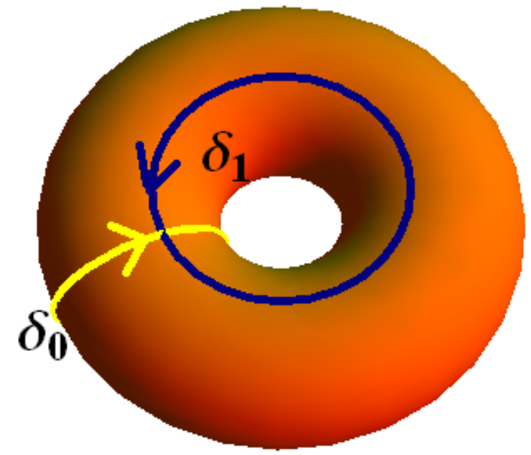
$$\lambda(u) = p(\theta)\, d\theta = p(z)\frac{dz}{iz} = \frac{(z^2 + 2uz + 1)^{1/2}}{iz^{3/2}}\, dz$$

A bit of Geometry

Homology

$$H_1(T^2;\mathbb{R}) = \mathbb{R}^2$$

$$\{\delta_0, \delta_1\}$$



Cohomology

$$H^1(T^2;\mathbb{R}) = \mathbb{R}^2$$

$$\{\lambda, \lambda'\}$$

Second derivative must be linearly dependent on λ, λ'

This results in **Picard-Fuchs** equation on period integrals

$$(u^2-1)S_j''(u) + \frac{1}{4}\,S_j(u) = 0$$

3 regular singularities: $u = \infty, \pm 1$

Two solutions

$$S(u) = A\,F_0(u^2) + B\,uF_1(u^2)$$

$$F_0(u^2) = {}_2F_1\left(-\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; u^2\right)$$

$$F_1(u^2) = {}_2F_1\left(+\frac{1}{4}, +\frac{1}{4}; \frac{3}{2}; u^2\right)$$

Period Integrals

Identify period integrals from boundary conditions

$$S_0(u) = C_{00} F_0(u^2) + C_{01} u F_1(u^2) \qquad C_{00} = e^{-i\pi/2} C_{10} = 8\pi^{-1/2}\Gamma(3/4)^2$$

$$S_1(u) = C_{10} F_0(u^2) + C_{11} u F_1(u^2) \qquad C_{01} = e^{+i\pi/2} C_{11} = \pi^{-1/2}\Gamma(1/4)^2.$$

Near $u = -1$ $\quad S_0(u) \propto (1+u)$

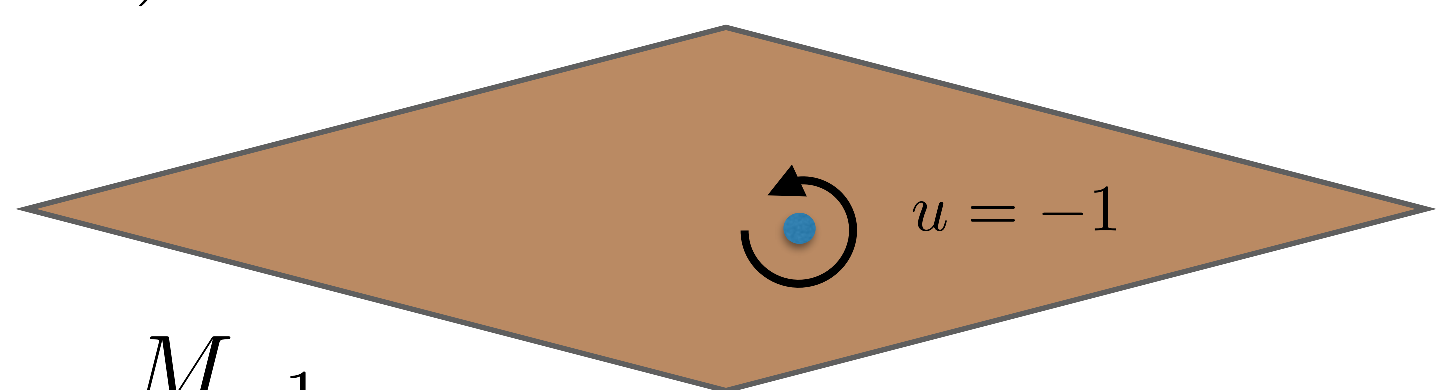
$$S_1(u) \propto \text{const} + (1+u)\ln(1+u)$$

Monodromy $\quad (u+1) \to (u+1)e^{2\pi i}$

$$\begin{pmatrix} S_0(u) \\ S_1(u) \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} S_0(u) \\ S_1(u) \end{pmatrix} = M_{-1} \begin{pmatrix} S_0(u) \\ S_1(u) \end{pmatrix}$$

resurgence

$$S_1(u) = Q_1(u) + \frac{i}{\pi} S_0(u)\ln(1+u)$$



M_{-1}

Monodromy

Near $u = -1$ $M_{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

Near $u = 1$ $M_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $S_0(u) = Q_0(u) - iS_1(u) \ln(1-u)/\pi$

Near $u = \infty$ (classical regime)

$$S_i(u) = u^{1/2} \left[V_i(u) + W_i(u) \ln u \right]$$

$$\begin{aligned} V_0(u) &= i\pi W_1(u) \mp V_1(u), \\ W_0(u) &= \mp W_1(u), \\ V_1(u) &= 4i\sqrt{2} \left[\ln\left(e^2/8\right) + 2/u \right] \\ W_1(u) &= -4i\sqrt{2} \left[1 - (4u)^{-2} \right], \end{aligned}$$

with little work

$$M_\infty = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$

Full monodromy is trivial

$$M_\infty = M_{-1} M_1$$

Spectrum

WKB quantization $\quad S_0(u_m) = 2\pi\alpha^{-1/2}(m+1/2)$

Near $\quad u=-1$

Energy levels $\quad \epsilon_m = 2\alpha u_m$

MFT *Debye-Huckel*

Pressure $\quad P = -eE_0\epsilon_0 = 2k_BTf - \sqrt{k_BTeE_0 f}.$

Bandwidth via instanton action $\quad (\Delta u)_m = \dfrac{\omega}{\pi\sqrt{\alpha}} e^{i\alpha^{1/2}S_1(u_m)/2}$

$$S_1(u_m) = 16i + \frac{2i}{\alpha^{1/2}}\left(m+\frac{1}{2}\right)\ln\left(\frac{m+1/2}{32e\alpha^{1/2}}\right)$$

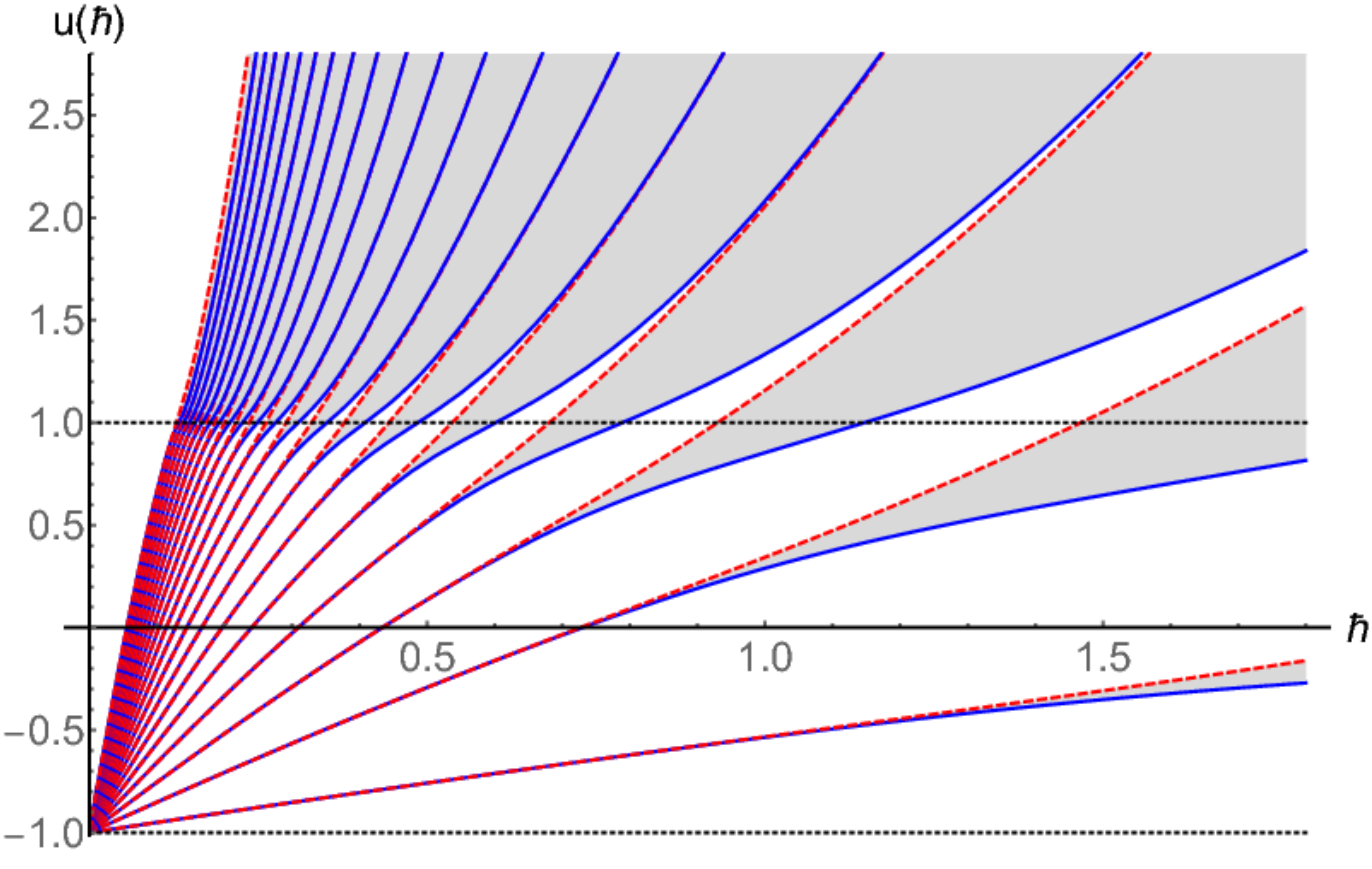
gives

$$\Delta\epsilon_m = \frac{4}{\pi}\left(\frac{32e}{m+1/2}\right)^{m+1/2} e^{-8\alpha^{1/2}+(m/2+3/4)\ln\alpha}$$

(cf. solutions of Mathieu equation, can do the same for Lame)

Band Spectrum of Mathieu Equation

[© Dunne Unsal]



(2,1) Gas

Energy equation $$\frac{3}{2}u = p^2 - \left(\frac{z^2}{2} + \frac{1}{z}\right)$$ also yields a torus

Picard-Fuchs equation $$(u^3+1)S_j''(u) + \frac{u}{4}S_j(u) = 0$$

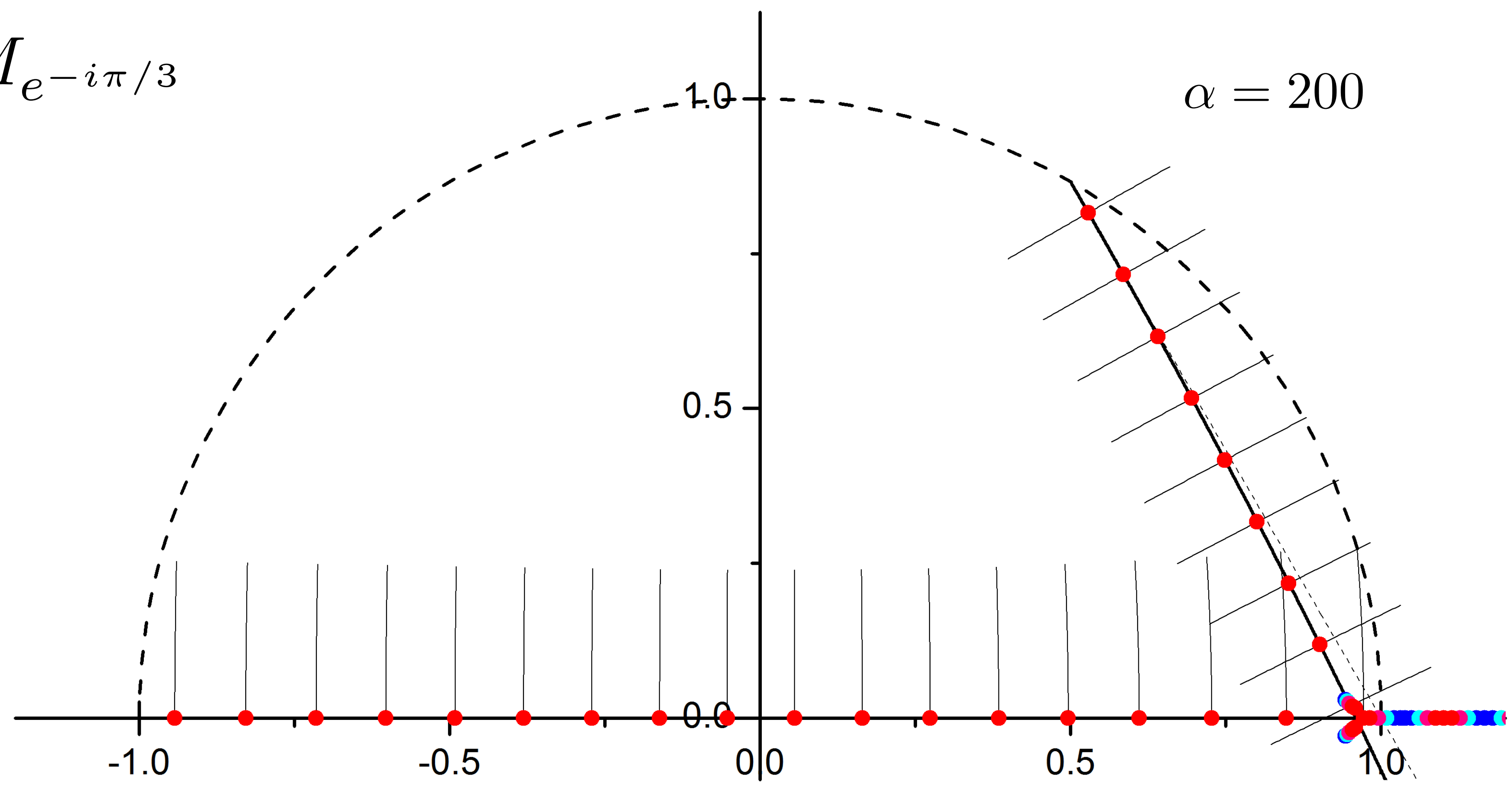
Solutions $$S_j(u) = C_{j0}F_0(u^3) + C_{j1}uF_1(u^3)$$

$$F_0(u^3) = {}_2F_1\left(-\frac{1}{6}, -\frac{1}{6}; \frac{2}{3}; -u^3\right)$$

$$F_1(u^3) = {}_2F_1\left(+\frac{1}{6}, +\frac{1}{6}; \frac{4}{3}; -u^3\right)$$

4 singularities in u-plane

$$M_\infty = M_{e^{i\pi/3}} \cdot M_{-1} \cdot M_{e^{-i\pi/3}}$$



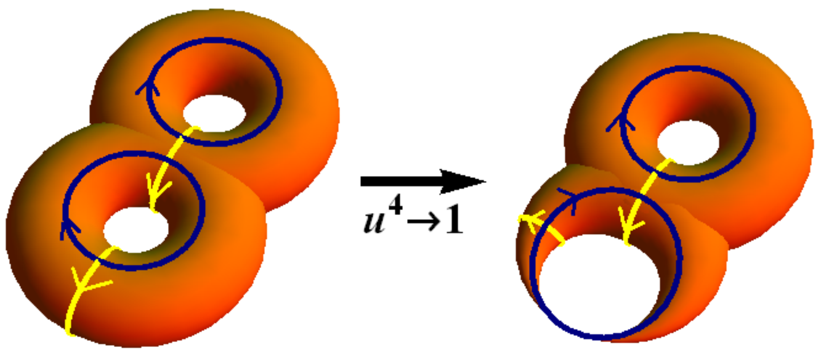
works great at strong coupling!

$$(\Delta\epsilon)_0 \approx 34.14\,\alpha^{3/4}\,e^{-7.35\sqrt{\alpha}}$$

(3,1) Gas

Energy equation describes genus 2 surface

$$\frac{4}{3}\,u = p^2 - \left(\frac{z^3}{3} + \frac{1}{z}\right)$$



Solutions of Picard Fuchs $\quad (u^4 - 1)S^{(4)} + 8u^3 S^{(3)} + \frac{217}{18}\,u^2 S'' + uS' + \frac{65}{144}\,S = 0$

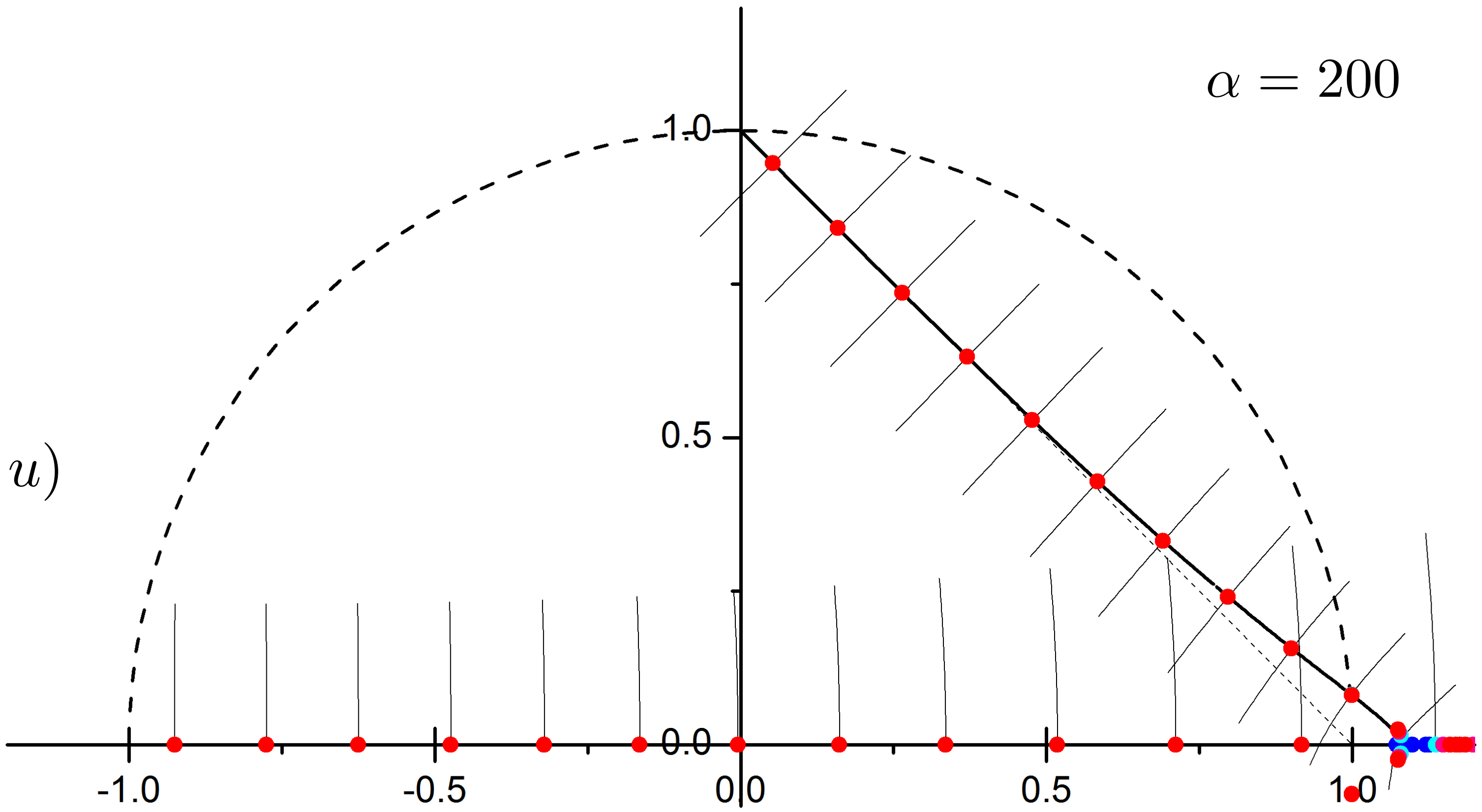
give four independent cycles

WKB near $u = -1$
gives great results!

$$S_0(u) \;=\; (\sqrt{8}\pi/3)(1+u) + \mathcal{O}(1+u)$$

$$S_{inst}(u) = Q_{inst}(u) + \frac{i}{\pi} S_0(u)\ln(1+u)$$

$$Q_{inst} \;=\; Q_3 - S_2 - Q_1$$



(N,M) Gases

For higher charges of ions

$$(4,1): \qquad \frac{5}{4}\,u = p^2 - \left(\frac{z^4}{4} + \frac{1}{z}\right),$$

$$(3,2): \qquad \frac{5}{6}\,u = p^2 - \left(\frac{z^3}{3} + \frac{1}{2z^2}\right)$$

degrees of PF grow

$$\begin{aligned}
(4,1): \quad & (u^5+1)S^{(4)}(u) + \frac{9u^5-1}{u}\,S^{(3)}(u) \\
& + \frac{235}{16}\,u^3 S''(u) + \frac{5}{4}\,u^2 S'(u) + \frac{39}{64}\,uS(u) = 0, \\
(3,2): \quad & (u^5+1)S^{(4)}(u) + \frac{9u^5-1}{u}\,S^{(3)}(u) \\
& + \frac{140}{9}\,u^3 S''(u) + \frac{5}{4}\,u^2 S'(u) + \frac{119}{144}\,uS(u) = 0.
\end{aligned}$$

all solutions have similar semiclassical behavior

$$S(u) \sim \sqrt{u},\ \sqrt{u}\log u$$

Resurgence

In a nutshell resurgence is a manifestation of an intimate connection between perturbative and nonperturbative phenomena in QFT/String theory

We have seen it in our QM examples $S_0(u) = e^{-i\pi/2} S_1(e^{i\pi} u)$

The classical action $[0]$ was related to the action on **instanton-antiinstanton trajectory** $[I, \bar{I}]$

Remarkably, this continues further…

$$[0] \sim [I, \bar{I}] \sim [II\bar{I}\bar{I}] \sim \dots$$

The reason of this is that perturbation series in QFT are **asymptotic**

Resurgence — 0+0d example

[A. Cherman, P. Koroteev, M. Unsal]

Integral

$$\mathcal{Z}(g) = \frac{1}{\sqrt{g}} \int_{-\pi/2}^{\pi/2} dx\, e^{-\frac{1}{2g}\sin(x)^2}$$

can be done to get a Bessel function

$$\mathcal{Z}(g) = \frac{\pi}{\sqrt{g}} e^{-\frac{1}{4g}} I_0\left(\frac{1}{4g}\right)$$

two saddles: pert at x = 0, S=0 and non-pert at x = pi/2, S = 1/2g

Taylor expand near x =0, integrate…

$$\mathcal{Z}(g) \stackrel{?}{\simeq} \sqrt{2\pi}\left[1 + \frac{g}{2} + \frac{9g^2}{8} + \frac{75g^3}{16} + \frac{3675g^4}{128} + \frac{59535g^5}{256} + \frac{2401245g^6}{1024} + \cdots\right]$$

coefficients grow factorially and needs to be **resumed**!

A well-defined answer should have this form

$$\mathcal{Z}(g, \sigma_0, \sigma_1) = \sigma_0 e^{-S_0} \sum_{k=0}^{\infty} p_{k,0} g^k + \sigma_1 e^{-S_1} \sum_{k=0}^{\infty} p_{k,1} g^k$$

Holomorphic Approach

Rewrite the partition function using inverse coupling as

$$\mathcal{Z}_*(\xi) = \int_{-\pi}^{+\pi} e^{\xi \cos\phi} d\phi = 2\pi I_0(\xi)$$

$$(2\xi)^{-1} = 2g$$

$$\mathcal{Z}(\xi) = \sqrt{\xi} e^{-\xi} \mathcal{Z}_*(\xi)$$

change variables $\cos\phi = -x$

$$\mathcal{Z}_*(\xi) = \int_{\mathcal{C}} \frac{e^{-\xi x}}{\sqrt{(1+x)(1-x)}} dx$$

We can interpret Z as a contour integral $\mathcal{Z}_*(\xi) = \int_A \lambda$

There must be a B-cycle. Let's try Picard-Fuchs $\xi \mathcal{Z}_*'' + \mathcal{Z}_*' - \xi \mathcal{Z}_* = 0$

which has 2 solutions! $\mathcal{Z}_* = C_1 I_0(\xi) + C_2 K_0(\xi)$

Near $\xi = 0$

$$K_0(\xi) = -(f(\xi) + \gamma_E) \log\left(\frac{\xi}{2}\right) + g(\xi)$$

$$I_0(\xi) = f(\xi)\,,$$

$$f(\xi) = -1 - \frac{\xi^2}{4} + O(\xi^4)$$

PF Monodromy

$$K_0(e^{\pi i}\xi) = f(\xi)\log\xi - \pi i f(\xi) = K_0(\xi) - \pi i I_0(\xi)$$

Define $\quad \mathcal{Z}_{*\,A}(\xi) = K_0(\xi)\,, \quad \mathcal{Z}_{*\,B} = \frac{1}{\pi i} I_0(\xi)$

Partition function $\quad \mathcal{Z}(\xi, c_1, c_2) = c_1 \mathcal{Z}_A(\xi) + c_2 \mathcal{Z}_B(\xi)$

Monodromy around the origin $\quad M_0 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$$\mathcal{Z}_A \to \mathcal{Z}_A - 2\mathcal{Z}_B$$

Perturbation series is divergent, however, if we take the series and subtract from it the same series after taking the monodromy, we'll get the `instanton' part. Factorially-divergent pieces disappear

The same can conclusion can be drawn from Borel resumation

Thanks!