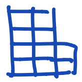
Elliptic stable envelope for $\mathrm{Hilb}^n(\mathbb{C}^2)$

$$X = \mathrm{Hilb}^n(\mathbb{C}^2) := \{ J \subset \mathbb{C}[x,y] : \dim_{\mathbb{C}} \mathbb{C}[x,y]/J = n \}$$

Torus action $T \circlearrowright X$: $(x,y) \longrightarrow (x t_1, y t_2)$

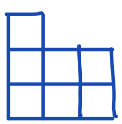
More convenient choice of param. $a = t_1/t_2$, $\hbar = t_1 t_2$.

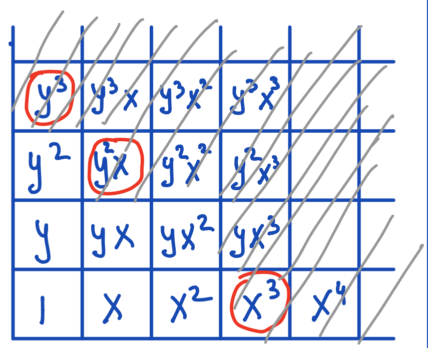
Torus fixed points

$X^T = \{ \lambda =$  $, |\lambda| = n \}$

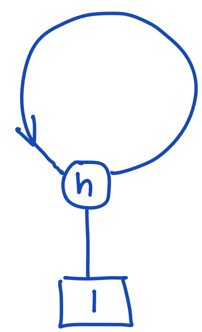
$J = \langle y^3, x y^2, x^3 \rangle$

$\|$

$\lambda =$ 



X is a Nakajima's quiver variety

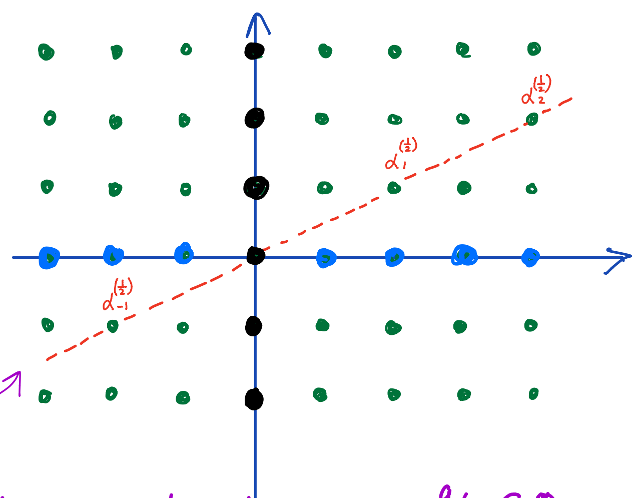


Fock space representation of quan. toroidal $U_{t_1,t_2}(\widehat{\widehat{gl_1}})$

Maulik – Okounkov theory gives:

(1) Action of $U_{t_1,t_2}(\widehat{\widehat{gl_1}}) \circlearrowright$ Fock

$$\text{Fock} = \bigoplus_{n=1}^{\infty} K_T(\mathrm{Hilb}^n(\mathbb{C}^2)) \simeq \mathbb{Q}[p_1, p_2 \ldots] \otimes \mathbb{Q}(t_1, t_2)$$



Ex: Slope 0-subalg.

$$\alpha_n^{(0)} = \begin{cases} \frac{\partial}{\partial p_n} & ; \ n > 0 \\ p_n & n < 0 \end{cases}$$

Slope ∞ -subalgebra

$\alpha_n^{(\infty)}$ = n-th Macdonald operator.

For each slope $s = a/b \in \mathbb{Q}$

$[\alpha_n^{(s)}, \alpha_m^{(s)}] = n^{(s)} \delta_{n+m}$ = Heisenberg H^s subalgebra in $U_{t_1,t_2}(\widehat{\widehat{gl_1}})$

K-theoretic Stable bases

For each $s = a/b \overset{MO}{\Longrightarrow}$ basis $S_\lambda^{(a/b)}$ of Fock

in which Heisenberg $H_{a/b}$ acts in "simplest way"

Ex: $S_\lambda^{(0)}$ – Schur polynomials (Pieri Rules)

$S_\lambda^{(\infty)}$ – Macdonald polynomials

$S_\lambda^{(a/b)}$ – "Rational" Schur polyn.

For action of $H_{a/b}$ in basis $S_\lambda^{(a/b)}$

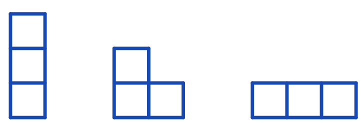
see A. Negut "The a/b Pieri Rule"

Change of stable basis

In "Infinitesimal change of stable basis" E. Gorsky and A. Neguț studied transition matrices

$$S_\lambda^{(a/b+\epsilon)} = \sum_{|\mu|=n} T_{\lambda,\mu}^{(a/b)} \, S_\mu^{(a/b-\epsilon)} \qquad 0<\epsilon<<1$$

__Ex.__

$n=3 \Rightarrow$  , $s=\frac{1}{2}$

$$T^{(1/2)} = \begin{bmatrix} 1, & 0, & a^3\hbar^{-1}(\hbar^2-1) \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}$$

$\exists$ diagonal matrix γ_s of monom. in a such that

$\gamma_s T^{(s)} \gamma_s^{-1}$ – Laurent polyn in $\hbar$.

Depends only on denominator of $s=a/b$.

Leclerc – Thibon involution for $U_\hbar(\widehat{gl_b})$

There exists unique involution of Fock module of $U_\hbar(\widehat{gl_b})$ such that

(1) $\overline{a(\hbar)\, x + b(\hbar)\, y} = a(\hbar^{-1})\, \overline{x} + b(\hbar^{-1})\, \overline{y}$

(2) $\overline{|\emptyset\rangle} = |\emptyset\rangle$

(3) $\overline{f_i\, v} = f_i\, \overline{v}$

Transition matrix between standard basis $|\lambda\rangle$ and $\overline{|\lambda\rangle}$ is a matrix

$$T^{L-T}_{gl_b}$$

Conjectures

(1) $T^{(a/b)} = T^{L-T}_{gl_b}$

(2) Action of $H_{a/b}$ extends to a geometric action of $U_\hbar(\widehat{gl_b})$

In this talk

- All bases $S_\lambda^{(a/b)}$ appear as limits of "elliptic stable envelope"

- The elliptic stable envelope nicely describes "3D-mirror symmetry"

$$X \longleftrightarrow X^! \quad \left(\begin{array}{c}\text{Symplectic}\\ \text{duality}\end{array}\right)$$

- Conjectures of GN are obvious from 3D-mirror symmetry of ell. stable envelope.

K-theory vs Elliptic cohomology

Assume $X^T = \{p_1, \ldots, p_n\}$ - finite set.
Fix $q \in \mathbb{C}^\times$ and $E = \mathbb{C}^\times / q^{\mathbb{Z}}$ - elliptic curve

K-theory class $\mathcal{F} \in K_T(X)$

is $\mathcal{F} = (\mathcal{F}|_{p_1}, \mathcal{F}|_{p_2} \ldots \mathcal{F}|_{p_n})$

$\mathcal{F}|_{p_i}$ - functions on $\mathrm{Spec}(K_T(pt)) = T$

which "glue" to global function on $\mathrm{Spec}(K_T(X))$

Elliptic cohom. class $\mathcal{F}$

$\mathcal{F} = (\mathcal{F}|_{p_1}, \mathcal{F}|_{p_2} \ldots \mathcal{F}|_{p_n})$

$\mathcal{F}|_{p_i}$ - sections of line bundles over $\mathrm{Ell}_T(pt) = T/q^{\mathrm{cochar}(T)} = E^{\mathrm{rk}(T)}$

which glue to a function on $\mathrm{Ell}_T(X)$

$\mathcal{F}|_{p_i}$ - "quasiperiodic functions"

$$\mathrm{Spec}(K_T(X)) \xleftarrow{q=0} \mathrm{Ell}_T(X)$$

$$\begin{array}{ccc} \mathrm{Spec}(K_T(X)) & & \\ \Psi & & \\ \mathcal{F} & \xleftarrow{q=0} & \text{section } \mathcal{F} \end{array}$$

Extended elliptic cohomology

$$E_T(X) = \mathrm{Ell}_T(X) \times \underbrace{E^{\mathrm{rk}(\mathrm{Pic}_X)}}$$

"Kähler parameters"
"dynamical parameters"

For the Hilbert scheme $\mathrm{Pic}(X) = \mathbb{Z}$

$\Rightarrow$ $f|_p$ - sections of line bundles over $E^3 = E_{t_1} \times E_{t_2} \times E_z$

Elliptic stable envelopes

For every $\lambda \in X^T$ Aganagic-Okounkov (1604.00423)
construct a class

$$\mathrm{Stab}(\lambda) = \text{section of l.b. over } E_T(X)$$

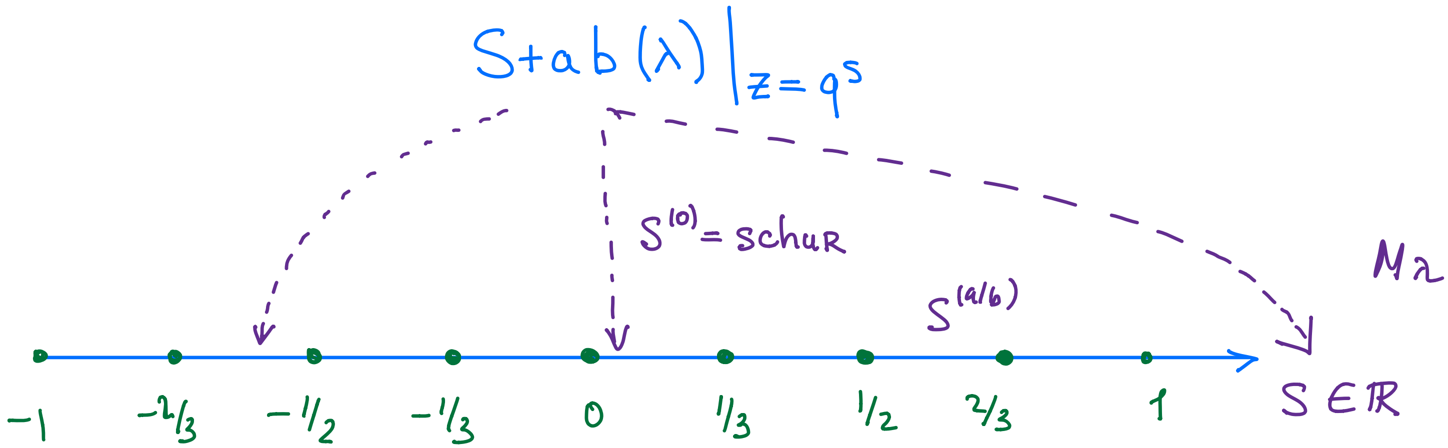
such that in K-theory limit, and generic $s \in \mathbb{Q}$

$$\lim_{q\to 0} \mathrm{Stab}(\lambda)_{z=q^s} = S^{(s)}_\lambda$$

slope s basis of Fock.

Limits of elliptic = K-theoretic

Ex: $n = 3$



$$\text{Walls} = \left\{ \frac{a}{b} \in \mathbb{Q} : \quad |b| \leq n \right\}$$

Farey sequence of level n.

Ex: $n=2$, $\left\{ \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array},\ \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \right\}$

Let $f = \mathrm{Stab}\left(\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\right)$, Then the components are

$$f\Big|_{\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} = \vartheta(t_2)\vartheta(t_2^2)$$

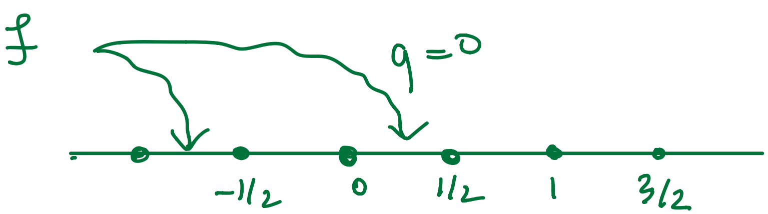
$$f_{\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}} = \frac{\vartheta(t_2)^2\vartheta(t_1t_2)\vartheta(\frac{t_2}{t_1}z)}{\vartheta(t_1)\vartheta(z)} + \frac{\vartheta(t_2)\vartheta(t_1/t_2)\vartheta(t_1t_2)\vartheta(z^2t_2)\vartheta(t_1t_2z)}{\vartheta(t_1)\vartheta(z^2t_1t_2)\vartheta(z)}$$

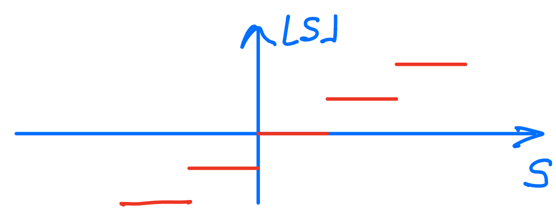
Theta function $\vartheta(x) = (x^{1/2}-x^{-1/2})\prod_{i=1}^{\infty}(1-xq^i)(1-q^i/x)$

In K-theory limit $q\to 0$:

integral part of s

$$\vartheta(x) \to x^{1/2}-x^{-1/2}\ ; \quad \lim_{q\to 0}\frac{\vartheta(xz)}{\vartheta(z)}\Big|_{z=q^s} = x^{-\lfloor s\rfloor - 1/2}$$





3D-mirror symmetry

Define $T_{\lambda,\mu}(a,\hbar,z) = \mathrm{Stab}(\lambda)\big|_{\mu}$

Conjecture: "$T(a,\hbar,z) = T(z, \frac{1}{\hbar}, a)^{-1}$"

Elliptic stable envelope of X

Elliptic stable envelope of $X^! \simeq X$

a - equivariant
z - kähler

z - equivariant
a - kähler

Note: In K-theory limit we lose z

so, we lose the symmetry $a \longleftrightarrow z$.

K-theory limit to a wall

Instead of $\lim\limits_{q\to 0} \mathrm{Stab}\big|_{z=q^s}$ consider $\boxed{\lim\limits_{q\to 0} \mathrm{Stab}\big|_{z=zq^s}}$

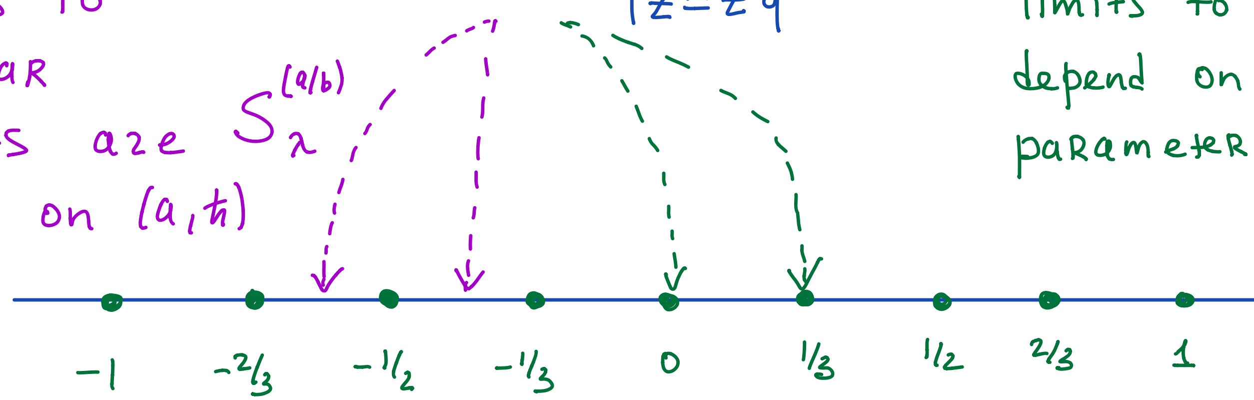
$$\lim_{q\to 0} \frac{\vartheta(azq^s)}{\vartheta(zq^s)} = \begin{cases} a^{-\lfloor s\rfloor - 1/2}, & s \notin \mathbb{Z} \\ \dfrac{1-az}{1-z}\, a^{-s-1/2}, & s \in \mathbb{Z} \end{cases}$$

Same as before

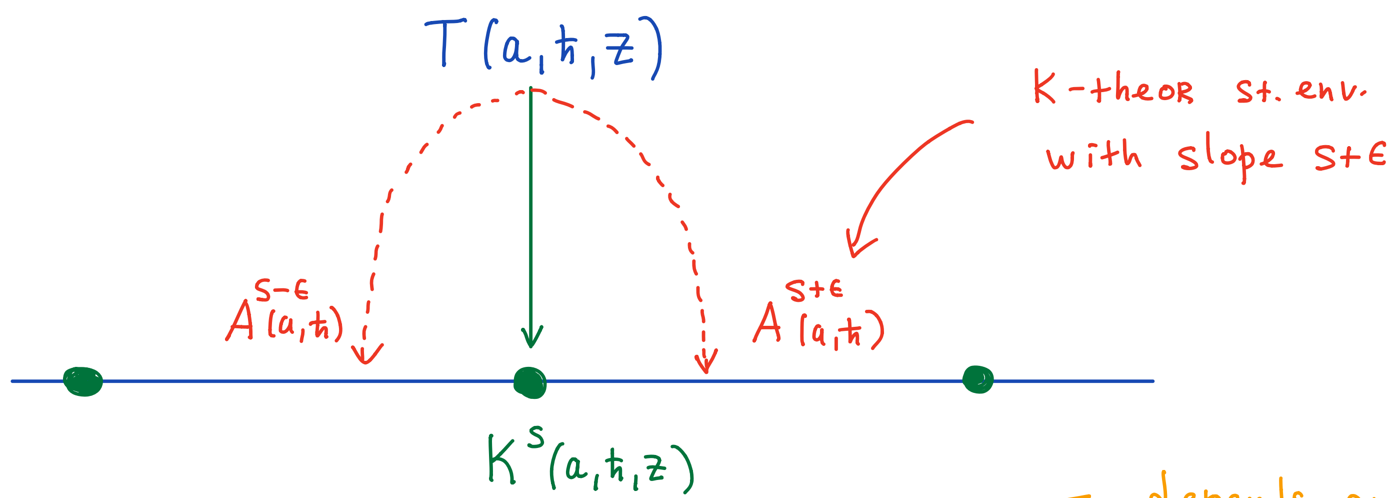
$\mathrm{Stab}\big|_{z=zq^s}$

limits to regular points are $S_\lambda^{(a/b)}$ depend on $(a,\hbar)$

limits to walls depend on all parameters $a, z, \hbar$



K-theory limit to a wall



Key observation is factorization:

depends on $z,\hbar$ only

$$K^s(a,\hbar,z)\, A^{s\pm\epsilon}(a,\hbar)^{-1} = \gamma_s \, Z^{\pm}(z,\hbar)\, \gamma_s^{-1}$$

same diag. matrix as in G.N.

Theorem [Y. Kononov – S.]

Let $s = a/b$ and $K^s(a,\hbar,z) = \lim_{q\to 0} T(a,\hbar,z q^s)$

Then

$$K^s(a,\hbar,z) = \gamma_s Z^+(z,\hbar)\gamma_s^{-1} A^{s+\epsilon}(a,\hbar) = \gamma_s Z^-(z,\hbar)\gamma_s^{-1} A^{s-\epsilon}(a,\hbar)$$

- $A^{s\pm\epsilon}$ – matrices of K-th stable envelopes of X.
- $Z^{\pm}(z,\hbar)$ – matrices of K-th stable envelopes of $Y_s \subset X^!$ with slopes $\pm\epsilon$.
- $$Y_s = (X^!)^{\mu_s} \subset X^!$$

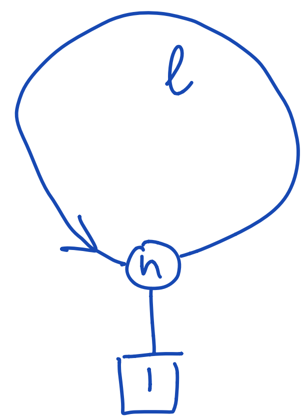
$\mu_s = \langle e^{2\pi i \cdot s} \rangle = \mathbb{Z}_b$ acting on $X^!$ via $z \to e^{2\pi i s} \cdot z$

Note: Walls $= \{ s \in \mathbb{Q}$ such that $Y_s \neq X^T \}$

Proof of GN conjectures

For a wall $s = a/b$, where the action of $U_\hbar(\widehat{\mathfrak{gl}}_b)$ comes from?

$$X \circlearrowright \mu_s = \langle e^{2\pi i s} \rangle \iff$$



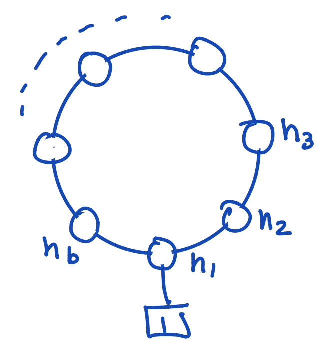
$\ell \to e^{2\pi i s} \cdot \ell$

μ_s - Rotates the loop in the quiver

$\Rightarrow$

$$Y_s = X^{\mu_s} = \bigsqcup_{n_1+n_2+\dots+n_b=n} X(n_1, n_2, \dots, n_b) \quad \text{with}$$

$X(n_1 \dots n_b) =$



$$\bigoplus_s K_T(Y_s) = \text{Fock module of } U_\hbar(\widehat{\mathfrak{gl}}_b)$$

by Maulik-Okounkov construction.

Proof of GN conjectures

Why the Leclerc-Thibon matrices coincide with transition matrices $\{S_\lambda^{a/b-\epsilon}\} \longrightarrow \{S^{a/b+\epsilon}\}$?

Factorization at a wall $s=a/b$ gives:

$$\gamma_s Z^+(z,\hbar)\gamma_s^{-1} A^{s+\epsilon}(a,\hbar) = \gamma_s Z^-(z,\hbar)\gamma_s^{-1} A^{s-\epsilon}(a,\hbar)$$

$$\Rightarrow \underbrace{A^{s+\epsilon}(a,\hbar)\cdot A^{s-\epsilon}(a,\hbar)}_{\text{Gorsky-Negut transition matrix } \{S_\lambda^{a/b-\epsilon}\} \longrightarrow \{S^{a/b+\epsilon}\}} = \gamma_s \underbrace{Z^+(z,\hbar)^{-1} Z^-(z,\hbar)}_{\text{Transition matrix from } |\lambda\rangle \text{ to } \overline{|\lambda\rangle} \text{ in Fock-module of } U_\hbar(\widehat{gl}_b)} \gamma_s^{-1}$$

Gorsky-Negut transition matrix

$\{S_\lambda^{a/b-\epsilon}\} \longrightarrow \{S^{a/b+\epsilon}\}$

Transition matrix from $|\lambda\rangle$ to $\overline{|\lambda\rangle}$ in Fock-module of $U_\hbar(\widehat{gl}_b)$.

For small slopes $\pm\epsilon$ is independent of z.

Concluding Remarks

- These ideas work for all X where Stab can be defined

e.g. $X = T^* G/B \Longleftrightarrow X^! = T^*(G/B)^L$

- A lot of potential applications to enumerative geometry, quantum differential equations for X $\longleftrightarrow$ KZ connections for $X^!$