Q-opers — what they are and what are they good for?

Program: Geometric and Representation Theoretic Aspects of quantum integrability

Interactions between various branches of math/physics

- Quantum geometry (QK-theory of symplectic resolutions) [Givental et al] [Okounkov et al]
- Geom. rep. theory i.e. Geometric Langlands [AFO]
- BPS/CFT [Nekrasov Pestun Shatashvili]
- ODE/IM [BLZ DDT, ...]
- Integrable systems (both classical and quantum) and dualities between them

Opers will help us understand these connections and perhaps create some new.

Watch Anton's talk for recap.

① q-Opers: Consider vector bundle E over $\mathbb{P}^1$

Type A: $M_q: \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto qz$, $q \in \mathbb{C}^\times$, E^q - pullback of E via M_q

$G = GL(r+1)$
$G = SL(r+1)$

U - open dense subset of $\mathbb{P}^1$

Linear map $A: E \to E^q$

Upon choice of a trivialization $A(z) \in gl(N; \mathbb{C}(z))$

If you change a trivialization via $g(z) \in G(z)$ then

$$A(z) \longmapsto g(qz) A(z) g(z)^{-1}$$

$\forall$ linear map we have a difference equation $D_q(s) = A \cdot s$

Def: A meromorphic $(GL(r+1), q)$-oper on $\mathbb{P}^1$ is $(A, E, \mathcal{L}_\bullet)$, where E - rank $r+1$ vector bundle over $\mathbb{P}^1$ and $\mathcal{L}_\bullet$ is a complete flag of vector bundles

$$\underset{\text{line}}{\mathcal{L}_{r+1}} \subset \dots \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset \dots \subset \mathcal{L}_1 = E$$

so that the meromorphic $(GL(r+1), q)$-connection $A \in Hom_{\mathcal{O}_U}(E, E^q)$ s.t.

i) $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}^q$

ii) $\exists$ open dense set $U \subset \mathbb{P}^1$ s.t. the restriction of $A \in Hom(\mathcal{L}_\bullet, \mathcal{L}_\bullet^q)$ to $V = U \cap M_q^{-1}(V)$ is invertible and the induced maps

$$\overline{A}_i : \mathcal{L}_i / \mathcal{L}_{i+1} \xrightarrow{\sim} \left(\mathcal{L}_{i-1} / \mathcal{L}_i\right)^q \text{ are isomorphisms on } V$$

We have $(SL(r+1), q)$ oper if $\det A = 1$ on V.

In local form: Let $s(z)$ be section of $\mathcal{L}_{r+1}$. Consider

$$W_i(s)(z) = \left| s(z) \wedge A(z)\, s(M_q z) \wedge A(M_q z)\, s(M_q^2 z) \wedge \dots \wedge \prod_{j=0}^{i-2} A(M_q^{i-2-j} z)\, s(M_q^{i-1} z) \right|$$

$i = ?, \dots, r+1$

~~If you change a trivialization via~~ $g(z) \in G(z)$

oper conditions: $W_i \neq 0$. then $A(z) \mapsto g(qz)A(z)g(z)^{-1}$

Def: An $(SL(r+1),q)$-oper has regular singularities defined by polys $\{\Lambda_i(z)\}_{i=1..r}$ when $\bar{A}_i$ is an isomorphism except for roots of $\Lambda_i(z)$, $i=1,\dots,r$

$$W_k(s)(z) = P_1(z)\,P_2(M_q z)\dots P_k(M_q^{k-1}z), \qquad P_i = \Lambda_1(z)\dots\Lambda_i(z)$$

Def: The $(SL(r+1),q)$-oper is called Z-twisted if $A(z) = g(qz)\,Z\,g(z)^{-1}$, Z – diagonal element of $SL(r+1)$ $\quad Z = \mathrm{diag}(\xi_1,\dots,\xi_{r+1})$

Def: The Miura $(SL(r+1),q)$-oper is $(A, E, \mathcal{L}_\bullet, \hat{\mathcal{L}}_\bullet)$ where $(A,E,\mathcal{L}_\bullet)$ is $(SL(r+1),q)$-oper and flag $\hat{\mathcal{L}}_\bullet$ is preserved by the q-connection

Prop: There are exactly S_{r+1} worth Miura opers for a given Z-twisted q-oper if Z is regular semisimple.

In a standard basis $e_1, e_2, \dots, e_{r+1}$ in the space of sections relative position of $\mathcal{L}_\bullet$ and $\hat{\mathcal{L}}_\bullet$ can be expressed as follows:

$$\mathcal{D}_k(s) = e_1 \wedge \dots e_{r+1-k} \wedge s(z) \wedge Z s(M_q z) \wedge \dots \wedge Z^{k-1} s(M_q^{k-1} z)$$

D_k have a subset of zeros coincident w/ that of $W_k(s)(z)$. The rest of zeros correspond to points when flags are not in generic position: $(\mathcal{F}_{B_+,x} = aB_+,\ \mathcal{F}_{B_-,x} = bB_- \quad B\backslash G/B_+ = W_G,\ ab^{-1}=1)$

$$\det\left(\begin{array}{ccc|cccc} 1 & & & s_1(z) & \xi_1 s_1(M_q z) & \dots & \xi_1^{k-1} s_1(M_q^{k-1} z) \\ & \ddots & & \vdots & \vdots & & \vdots \\ & & 1 & s_k(z) & & & \\ & & & s_{k+1}(z) & & & \\ & 0 & & \vdots & \vdots & & \vdots \\ & & & s_{r+1}(z) & \xi_{r+1} s_{r+1}(M_q z) & \dots & \xi_{r+1}^{k-1} s_{r+1}(M_q^{k-1} z) \end{array}\right) = \beta_k W_k V_k$$

$$V_k(z) = \prod_{a=1}^{v_k}(z - v_{k,a}), \qquad D_{r+1}(s) = W_{r+1}(s), \qquad V_{r+1} = 1, \quad V_0 = 1$$

$$D_0 = e_1 \wedge \dots \wedge e_{r+1}$$

$$\det_{i,j}\left[\xi_i^{j-1}\, s^{(j-1)}_{r+1-k+i}(z)\right] = \beta_k W_k V_k, \qquad s_i^{(m)}(z) = s_i(M_q^m z)$$

Theorem: [KSZ] Polynomials $\{V_k(z)\}$, $k=1,\dots,r$ give the solution to the QQ-system.

$$\xi_{i+1}\, Q_i^+(M_q z)\,Q_i^-(z) - \xi_i Q_i^+(z)\,Q_i^-(M_q z) = (\xi_{i+1} - \xi_i)\,\Lambda_i(z)\,Q_{i-1}(M_q z)\,Q_{i+1}(z)$$

s.t. $V_i = Q_i^+(z)$. Moreover

$$Q^+_j(z) = \frac{1}{M_q^{i-r} W_{r-i}(z)} \frac{\det M_{1\ldots j}}{\det V_{1\ldots j}}, \qquad Q^-_j(z) = \frac{1}{M_q^{i-r} W_{r-i}(z)} \frac{\det M_{1\ldots j-1, j+1}}{\det V_{1\ldots j-1, j+1}},$$

where

$$M_{i_1 \ldots i_j} = \begin{pmatrix} s_{i_1} & \xi_{i_1} s_{i_1}^{(1)} & \cdots & \xi_{i_1}^{j-1} s_{i_1}^{(j-1)} \\ \vdots & \vdots & & \vdots \\ s_{i_j} & \xi_{i_j} s_{i_j}^{(1)} & \cdots & \xi_{i_j}^{j-1} s_{i_j}^{(j-1)} \end{pmatrix}, \qquad V_{i_1 \ldots i_j} = \begin{pmatrix} 1 & q\xi_{i_1} & \cdots & q^{j-1}\xi_{i_1}^{j-1} \\ \vdots & & & \\ 1 & q\xi_{i_j} & \cdots & q^{j-1}\xi_{i_j}^{j-1} \end{pmatrix}$$

Theorem: The solutions of nondegenerate QQ-system $\Longleftrightarrow$ Solutions of XXZ Bethe Ansatz equations

$$Q^+_j = \prod_{c=1}^{r_j} (z - s_{j,c}) \quad \text{Bethe roots}$$

$$v(z) = \begin{pmatrix} Q_+(z) & \frac{Q_-(z)}{Q_+(z)} \\ 0 & Q_+(z)^{-1} \end{pmatrix} \qquad D_q(s) = As$$

$D_q(s_1) = \Lambda s_2$

eliminate: $\left(D_q^2 - T(qz) D_q - \frac{\Lambda(qz)}{\Lambda(z)}\right) s_1 = 0$

<u>Ex:</u> $(SL(2), q)$-oper (+ tRS integrable system)

$$A(z) = \begin{pmatrix} \xi \frac{Q_+(qz)}{Q_+(z)} & \Lambda(z) \\ 0 & \xi^{-1} \frac{Q_+(z)}{Q_+(qz)} \end{pmatrix}$$

$s(z) \wedge Z\, s(qz) = \beta \Lambda(z)$

Let $s(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix}$, $\quad Z = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$, $\quad \Lambda(z) = \prod_{n=1}^{L} \prod_{e=0}^{K_n - 1} (z - q^e z_n)$

$$M(z) = \begin{pmatrix} Q_+(z) & \xi Q_+(qz) \\ Q_-(z) & \xi^{-1} Q_-(qz) \end{pmatrix} \qquad \det M = \beta \cdot \Lambda(z)$$

1) Calculate the determinant: QQ-system (is equivalent to $\Lambda(z) \sim Z$)

$$\xi^{-1} Q_+(z) Q_-(qz) - \xi Q_+(qz) Q_-(z) = (\xi^{-1} - \xi) \Lambda(z)$$

XXZ Bethe equations: $\quad Q_+ = \prod_{k=1}^{m} (z - w_k)$

$$\prod_{n=1}^{L} \frac{w_k - q z_n}{w_k - z_n} = \xi^2 \prod_{j=1}^{m} \frac{q w_m - w_j}{w_k - q w_j}, \qquad k = 1 \ldots m$$

2) <u>Def:</u> We call Z-twisted Miura $(SL(r+1), q)$-oper <u>canonical</u> if

1) Z-regular semisimple
2) $\deg D_k = k$
3) No other singularities except for the roots of $\Lambda(z) = D_{r+1}(z)$ which are distinct.

<u>Proposition:</u> $\{\xi_i(z)\}_{i=1,\ldots,r+1}$ are of degree 1.

Consider canonical $(SL(2), q)$-oper

$Q_+(z) = z - p_1 \qquad \Lambda_2(z) = (z - a_1)(z - a_2)$

$Q_-(z) = z - p_2,$

$$M(z) = \begin{pmatrix} z - p_1 & \xi(zq - p_1) \\ z - p_2 & \xi^{-1}(zq - p_2) \end{pmatrix} = z \begin{pmatrix} 1 & q\xi \\ 1 & q\xi^{-1} \end{pmatrix} - \begin{pmatrix} p_1 & p_1 \xi \\ p_2 & p_2 \xi^{-1} \end{pmatrix}$$

$$= z \cdot V + M(0)$$

$$\det M(z) = \det V \cdot \Lambda(z)$$

$$\det\left(M(z) V^{-1}\right) = \Lambda(z)$$

$$\det\left(z + M(0) \cdot V^{-1}\right) = \Lambda(z) \leftarrow \text{spectral curve for tRS!}$$

Claim: $L = -M(0)V^{-1} = \begin{pmatrix} \frac{\xi - q\xi^{-1}}{\xi - \xi^{-1}} p_1 & \frac{(q-1)\xi}{q(\xi - \xi^{-1})} p_1 \\ -\frac{(q-1)\xi p_2}{q(\xi - \xi^{-1})} & \frac{\xi q - \xi^{-1}}{\xi - \xi^{-1}} p_2 \end{pmatrix}$

is the Lax matrix for the tRS model

Works for $SL(N)$.

$$\det(z - L) = \sum_{k=0}^{r} z^k (-1)^k H^{tRS}_{n-k}$$

$$\mathrm{Fun}(q\mathcal{O}_{\mathbb{P}^1_2}) \cong \frac{\mathbb{C}(q, \xi_i, p_i, a_i)}{\left(H_k^{tRS} = e_k(a_1 \ldots a_{r+1})\right)} \cong \frac{\mathbb{C}(q, \xi_i, s_{k,\ell}, a_i)}{\text{Bethe}}$$

$$\cong \frac{\mathbb{C}(q, \xi_i, p_i, a_i)}{qW_r}$$

Quantum/Classical duality:

Symplectic form for tRS system: $\Omega = \sum_{i=1}^{r+1} \frac{dx_i}{x_i} \wedge \frac{dp_i}{p_i}$

H_k^{tRS} mutually commute w.r.t P.B. for Ω

$$\{H_k, H_\ell\}_\Omega = 0$$

Intersection of Lagrangian subvarieties $\mathcal{L}_1 \cap \mathcal{L}_2 = \frac{\mathbb{C}\{q, \xi_i, p_i a_i\}}{\left(H_k^{tRS} = e_k(a_1 \ldots a_{r+1})\right)}$

$$\mathcal{L}_1 = \{x_i = \xi_i\}, \quad \mathcal{L}_2 = \{H_i^{tRS} = e_i(a_i)\}$$

② The diamond. $\mathbb{P}^1$ has additive structure as well:

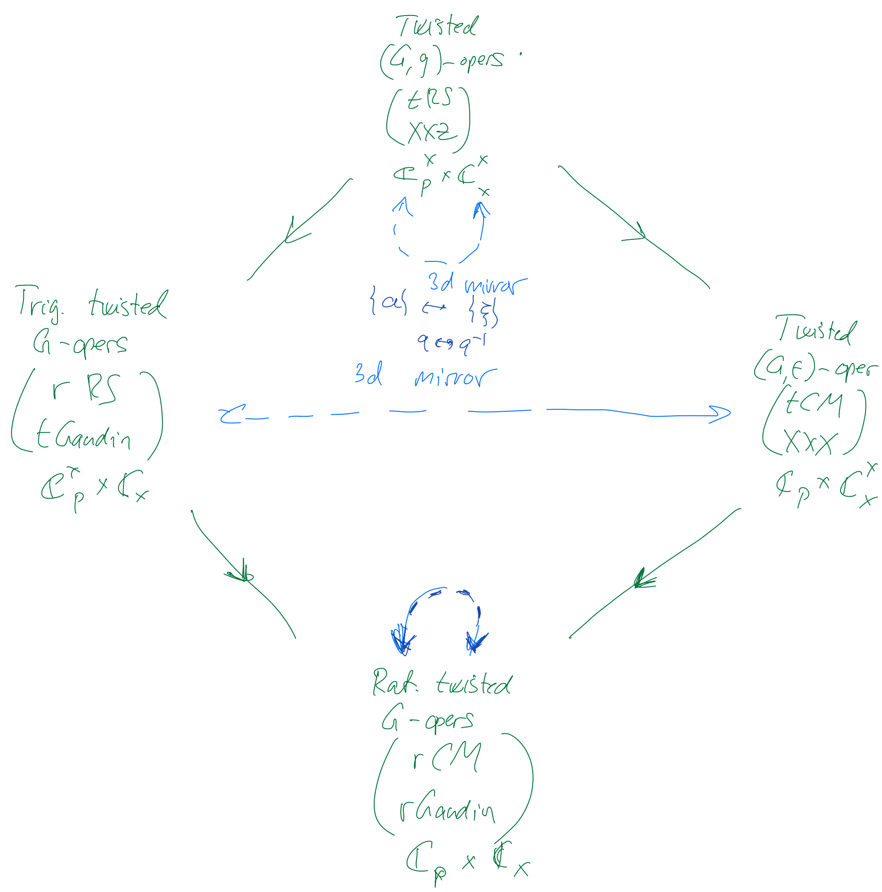
$M_\epsilon : z \mapsto z + \epsilon$, $\qquad E, \ E^\epsilon$ – pullback

By analogy can define (G, ϵ)-opers. Run the program, will end up with a different IS — ϵCM.

Opers: $\nabla : E \to E \otimes K$

$\mathcal{L}_\bullet \subset E \qquad \bar{\nabla}_i : \mathcal{L}_i / \mathcal{L}_{i+1} \xrightarrow{\sim} \mathcal{L}_{i-1} / \mathcal{L}_i \otimes K$

$\nabla_z = \partial_z + Z_1 \qquad W_i(s)(z) = \left(s(z) \wedge \nabla_z s(z) \wedge \ldots \wedge \nabla_z^{i-1} s(z)\right)_{\wedge^i \mathcal{L}_i}$



③ The Calogero Moser Space

Subset of $GL(N;\mathbb{C}) \times GL(N;\mathbb{C}) \times \mathbb{C}^{r+1} \times \mathbb{C}^{r+1}$ defined by
$\mathcal{M}'$

$$q\, MT - TM = u \otimes v^T$$

| 3d Mirror: $M \leftrightarrow T$, $q \leftrightarrow q^{-1}$

subject to group action

$$G: \; (M, T, u, v) \longmapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1}), \quad g \in GL(N;\mathbb{C})$$

Quotient of $\mathcal{M}'$ by G $\qquad \mathcal{M}'/G = \mathcal{M}$ – Calogero-Moser space.

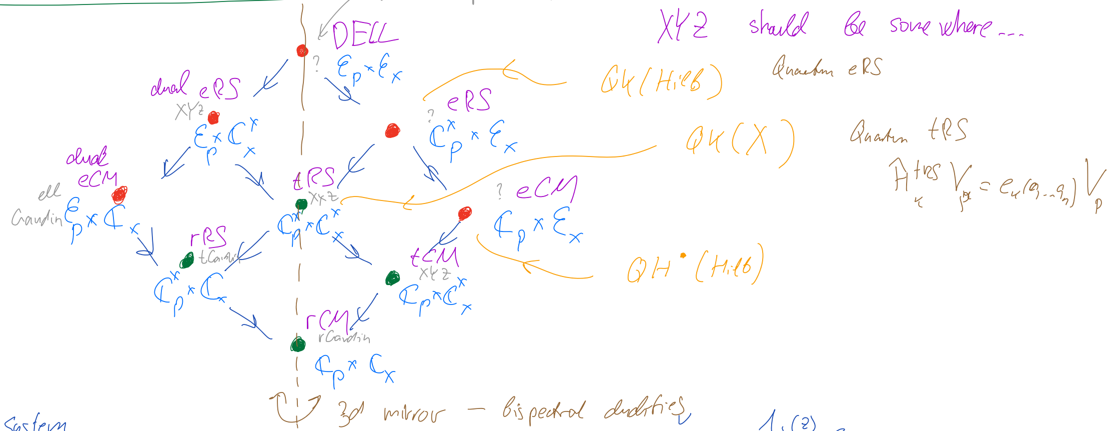
Let $M = \mathrm{diag}(\xi_1 \ldots \xi_{r+1}) \Rightarrow \quad T_{ij} = \dfrac{u_i v_j}{q\xi_i - \xi_j}$

define tRS momenta by rescaling:

$$p_i = -u_i v_i \frac{\prod_{k\neq i} (\xi_i - \xi_k)}{\prod_k (\xi_i - \xi_k q)}$$

$$T_{ji} = \frac{\prod_{k\neq i} (\xi_j - \xi_k q)}{\prod_{k\neq j} (\xi_j - \xi_k)}\, p_i$$

(4) Extension to Elliptic models



Extended QQ-system:

q-connection can be expressed as follows: $A(z) = \prod_i g(z)^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}$

$$g_i(z) = \xi_i \frac{Q_i^+(qz)}{Q_i^+(z)}$$

Bäcklund transformations.

Prop: $A \mapsto A^{(i)} = e^{\mu_i(qz) f_i} A(z) e^{-\mu_i(z) f_i}$, $\mu_i(z) = \dfrac{Q^+_{i-1}(z)\, Q^+_{i+1}(z)}{Q^i_+(z)\, Q^i_-(z)}$

then $A^{(i)}$ is obtained from A by substitution

$$Q_+^j(z) \mapsto Q_+^j(z), \quad j \neq i$$
$$Q_+^i(z) \mapsto Q_-^i(z), \quad Z \mapsto s_i(Z)$$

One can keep iterating

1 — 2 — 3 — • — • — … — • — r

to get $Q^-_{i,i+1,\ldots,i+n}(z)$

scale away $\Lambda_i(z)$

$$\xi_i Q_i^+(qz) Q_i^-(z) - \xi_{i+1} Q_i^+(z) Q_i^-(qz) = Q^+_{i-1}(qz) Q^+_{i+1}(z)$$
$$\xi_i Q^+_{i+1}(qz) Q^-_{i,i+1}(z) - \xi_{i+2} Q^+_{i+1}(z) Q^-_{i,i+1}(qz) = Q_i^-(qz) Q^+_{i+2}(z)$$
$$\vdots$$
$$\xi_1 Q^+_r(qz) Q^+_{1\ldots r}(z) - \xi_{r+1} Q^+_r(z) Q^-_{1\ldots r}(qz) = Q^-_{1,\ldots,r-1}(qz)$$

Prop: Let $v(z)$ be a diagonalizing gauge transformation for $(SL(r+1), q)$-oper.

Then

$$s(z) = \begin{pmatrix} Q_{1\ldots r}(z) \\ \vdots \\ Q^-_{r-1,r}(z) \\ Q_r(z) \\ Q^+_r(z) \end{pmatrix} \qquad v(z) = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

$$v(z) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Generalized minors G - simple, simply-connected complex Lie group

$G = N_- H N_+$ $\qquad$ V_i^+ – irrep of G w/ highest weight ω_i

$\mathfrak{g} = \mathfrak{n}_- \mathfrak{h}\, \mathfrak{n}_+$ $\qquad$ $h\, v^+_{\omega_i} = [h]^{\omega_i} v^+_{\omega_i}$

Principal minors [Fomin - Zelevinski]

$$\Delta : G \to \mathbb{C}^\times$$

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \qquad i = 1, \dots, r$$

Generalized minors for $u, v \in W_G$ ~ Weyl group

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g\, \tilde{v}) \qquad \text{~ lift to the loop group}$$

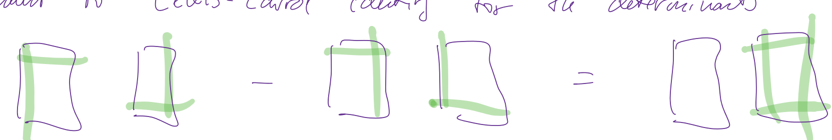
Proposition: Let $A(z) = v(qz)\, Z\, v(z)^{-1}$, $\quad v(z) \in B_-(z)$

$$\Delta_{w \cdot \omega_i, \omega_i}(v^{-1}(z)) = Q_+^{\omega, i}(z) \qquad \left(= s^i(z) \text{ for } G = SL(n) \right)$$

The QQ-system is equivalent to the following quadratic identity of Fomin and Zelevinski

$$\Delta_{u\omega_i, v\omega_i}\, \Delta_{u w_i \omega_i, v w_i \omega_i} - \Delta_{u w_i \omega_i, v\omega_i}\, \Delta_{u \cdot \omega_i, v w_i \omega_i} = \prod_{j \neq i} \Delta_{u\omega_j, v\omega_j}^{-a_{ji}}$$

equivalent to Lewis-Carrol identity for the determinants



Def: Generalized q-Wronskian $(\mathcal{F}_G, \mathcal{F}_{B_+}, \mathcal{G}, Z)$

$\exists\, V \subset \mathbb{P}^1$ – Zariski open s.t. for certain $\{v_i^+, v_{c_i}^+\}$, which are sections of line bundles $\{\mathcal{L}_i^+, \mathcal{L}_{c,i}^+\}_{i = 1..r}$ on $V = U \cap M_q^{-1}(U)$

meromorphic section $\mathcal{G}$ of $\mathcal{F}_G$ satisfies

$$\mathcal{G}^q \cdot v_i^+ = Z\, \mathcal{G}\, v^+_{c,i}$$

or $\quad \mathcal{G}(qz)\, v^+_{\omega_i} = Z\, \mathcal{G}(z)\, s^{-1} \cdot v^+_{\omega_i}$

$s = \prod_i s_i$ – lift of Coxeter element $c \in W_G$ to $G(z)$

Theorem: $\{\text{Gen } q\text{-Wronskians}\} \longleftrightarrow \left\{ \begin{matrix} Z\text{-twisted} \\ \text{Miura } (G,q)\text{-opers} \end{matrix} \right\}$

$\exists!\; \mathcal{W}(z) \in B_-(z)\, w_0\, B_-(z) \cap B_+(z)\, w_0\, B_+(z) \subset G(z)$

satisfying $\quad \mathcal{W}(q^{k+1} z)\, v^+_{\omega_i} = Z^k\, \mathcal{W}(z)\, s^{-1}(z)\, s^{-1}(qz) \dots s^{-1}(q^k z)\, v^+_{\omega_i}$

$i = 1, \dots, r \qquad k = 0, \dots, h-1.$

(5)

How to construct an oper for any group G?

Principal G-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$, $\mathcal{F}_G^q$ – pullback under M_q

A – section of $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$, U – Zariski open dense subset of $\mathbb{P}^1$

($\mathcal{F}_G|_U$ isomorphic to trivial G-bundle)

Let $V = U \cap M_q^{-1}(U)$

(G,q)-oper on $\mathbb{P}^1$ is $(\mathcal{F}_G, A, \mathcal{F}_{B_+})$, where A – meromorphic connection

$\mathcal{F}_{B_+}$ – reduction of $\mathcal{F}_G$ to B_+ s.t. restriction of $A: \mathcal{F}_G \to \mathcal{F}_G^q$ on V

takes values in $B_+(\mathbb{C}[V]) \subset B_+(\mathbb{C}[U])$

c – Coxeter element $c = \prod s_i$

Locally $A(z) = n'(z) \prod_i (\phi_i(z)^{\check{\alpha}_i} s_i)\, n(z)$, $\quad n, n'(z) \in N_+(z)$

Miura q-oper: $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$

$\mathcal{F}_{B_-}$ preserves q-connection

Th: Every Miura (G,q)-oper w/ regular singularities can be written as

$$A(z) = \prod_i g_i(z)^{-\check{\alpha}_i}\, e^{\frac{\Lambda_i(z)}{g_i(z)} f_i}, \quad g_i(z) \in \mathbb{C}(z)^\times$$

Miura-Plücker (G,q)-oper:

$v_{\omega_i} \downarrow f_i v_{\omega_i}$ $\Big\}$ W_i – 2d subspace spanned by $\{v_{\omega_i}, f_i \cdot v_{\omega_i}\}$ – B_+ invariant

v_{ω_i} – highest weight vector of $L_i \subset V_i$ – irrep of G with highest weight ω_i wrt $H \subset B_+$

Associated vector bundle $\mathcal{V}_i = \mathcal{F}_{B_+} \times_{B_+} V_i$

contains rank-2 subbundle $\mathcal{W}_i = \mathcal{F}_{B_+} \times_{B_+} W_i$

line subbundle $\mathcal{L}_i = \mathcal{F}_{B_+} \times_{B_+} L_i$

Associate to every $i = 1 \dots r$ an $(SL(2), q)$-oper

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ij}} \\ 0 & g_i(z)^{-1} \prod_{j \neq i} g_j(z)^{-a_{ij}} \end{pmatrix}$$

Def: Z-twisted Miura-Plücker oper:

$\exists\, v(z) \in B_+(z)$ s.t. $\forall i = 1 \dots r$

$$A_i(z) = v(qz)\, Z\, v(z)^{-1}\Big|_{W_i} = v_i(qz) Z_i v_i(z)^{-1},$$

$$v_i(z) = v(z)\big|_{W_i}, \quad Z_i = Z\big|_{W_i}$$

$\Downarrow$

QQ system

$\Downarrow$

every Miura-Plücker oper is Miura oper.