On the quantum K-theory of Quiver Varieties at roots of Unity

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① Enumerative AG of quiver varieties is governed by quantum difference equations

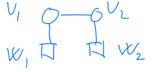
$$\Psi(zq^{\mathcal{L}}, a, q)\,\mathcal{L} = \mathbb{M}_{\mathcal{L}}(z,a,q)\,\Psi(z,a,q) \qquad \mathcal{L} \in \mathrm{Pic}(X) \simeq \mathbb{Z}^{\ell}$$

lattice of line bundles of X

$$\underset{\substack{\uparrow\\ \text{smooth}\\ \text{symplectic}}}{X} \ \underset{\substack{\uparrow\\ \text{open dense}\\ \text{set}}}{\subset}\ \mathfrak{X} = \left[\frac{\text{Reps of } G}{G}\right] \qquad G\text{-semisimple}/\mathbb{C}$$

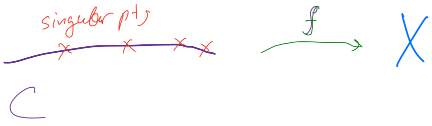
$\vec{z} = (z_1, \dots z_\ell)$ – kähler parameters

$\vec{a} = (a_1, \dots a_m)$ – equivariant par. (framing)



A quasimap from curve C to X

$C \overset{f}{\dashrightarrow} X$:= map $C \xrightarrow{f} \mathfrak{X}$ for all but finitely many points image lands in X.



Moduli space of QM of degree d

$$QM^{d}(X) = \{ \overset{\mathbb{P}^1}{\overset{\|}{C}} \overset{f}{\dashrightarrow} X \mid \deg f = d \} \qquad d \in H_2(X,\mathbb{Z})$$

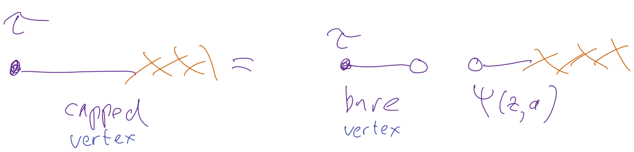
Shift of kähler parameters

$$zq^{\mathcal{L}} = (z_1 q^{c_1} \dots z_\ell q^{c_\ell}), \qquad \mathcal{L} = L_1^{c_1} \otimes \dots \otimes L_\ell^{c_\ell} \qquad L_i \text{ – taut. bundles}$$

The fundamental solution of q-difference equation

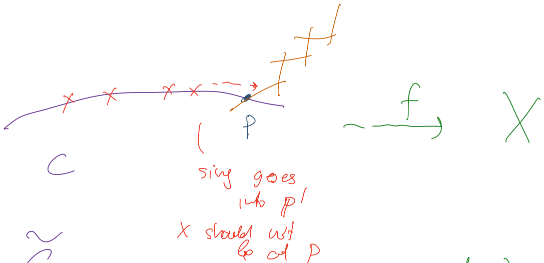
$$\Psi(z,a,q) = 1 + \sum_{d \in H_2(X,\mathbb{Z})_{\text{eff}}} \Psi_d(a,q)\, z^d \in K_T(X)[[z]]$$

Ψ – capping operator:



ev maps are not proper for bare vertex, pushforward not well-defined.

Compactify ⟹ relative quasimaps.

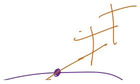


Allow f to sing + allow C to sing by adding bubbles

$$\begin{tikzcd} \tilde{C} \arrow[d, "\pi"'] \\ C \end{tikzcd}$$

$$QM^{d}(X)_{\text{rel}\,p} = \{ \tilde{C} \overset{f}{\dashrightarrow} X \}$$

↗ equipped w/ proper ev_p map to X. can do honest pushforwards



$- \quad p \quad \sim \quad \sim \quad \mathrm{ev}_{p,*}$ + sum over degrees

The difference equation works because for every coherent $\mathbb{C}^*_q$-equivariant sheaf $\mathcal{F}$ over $\mathbb{P}^1$

$$q^{\deg \mathcal{F}} = \det H^{\bullet}(\mathcal{F} \otimes (\mathcal{O}_0 - \mathcal{O}_\infty))$$

② **Integrability** is a consequence of the consistency of the QDE [Okounkov]

$$M_{\mathcal{L}_1 \mathcal{L}_2}(z) = M_{\mathcal{L}_2}(z q^{\mathcal{L}_1}) M_{\mathcal{L}_1}(z) = M_{\mathcal{L}_1}(z q^{\mathcal{L}_2}) M_{\mathcal{L}_2}(z)$$

$q \to 1$ limit
Integrability

$\mathcal{M}_{\mathcal{L}} = M_{\mathcal{L}}\big|_{q=1}$ - operator of quantum multiplication by $\mathcal{L}$
$\circledast \mathcal{L}$

$$[M_{\mathcal{L}_1}, M_{\mathcal{L}_2}] = 0$$

Q: eigenvalues of $\mathcal{M}_{\mathcal{L}}$? $\Rightarrow$ integrable spin chains
Bethe Ansatz

V_i - taut. bundles of X

$V_i = x_{i,1} + \ldots + x_{i,r_i}$, $r_i = \mathrm{rk}\, V_i$ - Grothendieck roots

ABA: Eigenvalues of $\mathcal{M}_{\mathcal{L}}(z)$ are given by $\lambda(z) = \prod_{i,j} x_{i,j}$, where $x_{i,j}$ solve Bethe equations.

③ Case $q^p = 1$. Let $p \in \mathbb{N}$ (not necessarily prime). Consider the following operator

$$M_{\mathcal{L}^p}(z) = M_{\mathcal{L}}(z) M_{\mathcal{L}}(z q^{\mathcal{L}}) M_{\mathcal{L}}(z q^{2\mathcal{L}}) \ldots M_{\mathcal{L}}(z q^{(p-1)\mathcal{L}})$$

ζ_p - primitive p-th root of unity

Let $\mathcal{M}_{\mathcal{L}, \zeta_p}(z) = M_{\mathcal{L}^p}(z)\big|_{q=\zeta_p}$

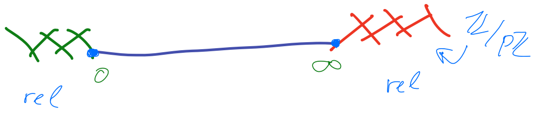
Similarly they commute

$$[\mathcal{M}_{\mathcal{L}_1, \zeta_p}, \mathcal{M}_{\mathcal{L}_2, \zeta_p}] = 0 \qquad \forall\, \mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(X)$$

Theorem 1: Let $\{\lambda_1(z,a), \lambda_2(z,a) \ldots\}$ be the set of eigenvalues of $\mathcal{M}_{\mathcal{L}}(z,a)$. Then the eigenvalues of $\mathcal{M}_{\mathcal{L}, \zeta_p}(z,a)$ are $\{\lambda_1(z^p, a^p), \lambda_2(z^p, a^p), \ldots\}$. They do not depend on the choice of ζ_p.

Theorem 2: (k-th lift of Frobenius intertwiner)

The operator $F(z,a,\zeta_p) := \Psi(z,a,q) \cdot \Psi(z^p, a^p, q^{p^2})^{-1}\Big|_{q \to \zeta_p}$ is well defined.



$$F(z)\, \mathcal{M}_{\mathcal{L}, \zeta_p}(z)\, F(z)^{-1} = \mathcal{M}_{\mathcal{L}}(z^p)$$

Theorem 3: p-curvature and Frobenius (see below)

Part of proof of Th 1:

As a part of calculation of the series for $\Psi(z,a,q)$ we get expressions like

$$\Phi(z) = \frac{(qw)_\infty}{(\hbar w)_\infty} = \prod_{i=0}^{\infty} \frac{1 - q w q^i}{1 - \hbar w q^i} \longrightarrow \exp\left(- \sum_{m=1}^{\infty} \frac{(q^m - \hbar^m) w^m}{m(1-q^m)} \right)$$

Character of the $\hbar$

$(1-q)(1+q+\ldots+q^{m-1})$

filter v

vertex: $V_i^{(\tau)}(z) = \int_{\gamma_i} \Phi(z,x) \cdot e(x,z) . \tau(x) \prod_{a,b} \frac{dx_{a,b}}{x_{a,b}}$

As $q \to 1$ $\longrightarrow \exp\left(-\frac{1}{1-q} \sum \frac{(1-\hbar^m) w^m}{m^2}\right) = \exp\left(-\frac{Li_2(w) - Li_2(\hbar w)}{1-q}\right)$

$\xrightarrow[q\to 1]{} \exp\left(-\frac{Y(x,z)}{1-q}\right)(1+\dots)$

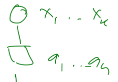
$Y(x,z)$ - Yang-Yang function

$\exp \frac{\partial Y}{\partial x} = 1 \Leftrightarrow$ Bethe eqns

as $q \to \zeta_p$ $\quad \Phi(z) \to \exp\left(-\frac{1}{1-q^p} \sum \frac{(1-\hbar^{pm}) w^{pm}}{m^2}\right)$

$\frac{1}{1-q^{p^2}} \rightleftarrows \frac{1}{(1-q^p)(1+q^p+\dots+q^{p(p-1)})} \longrightarrow \frac{1}{(1-q^p)p}$

Ex: $T^* Gr_{k,n}$



$$\prod_{j=1}^{n} \frac{x_m - a_j}{a_j - \hbar x_m} \prod_{i=1}^{k} \frac{x_j - \hbar x_m}{\hbar x_j - x_m} = z\hbar^{-k/2}, \qquad m = 1..k$$

④ p-curvature and Frobenius

K-th $\Rightarrow$ quantum cohomology : $\quad q = e^{\epsilon} = 1+\epsilon+\dots$, $a_i = q^{s_{k_i}} = 1 + s\epsilon a_i + \dots$

$M_{\mathcal{L}_i}(z,a) = 1 + \epsilon s C_i(z,a) + \dots$ $\qquad C_i(z)$ - quant. mult. by divisor in $QH^{\bullet}(X)$

$q^{z_i \frac{\partial}{\partial z_i}} = 1 + \epsilon z_i \frac{\partial}{\partial z_i}$

Cohomological limit over $\mathbb{Q}_p$. Now let p be prime, $\mathbb{Q}_p$ - field of p-adic numbers

$\mathbb{Z}_p \subset \mathbb{Q}_p$, p-adic norm $|p|_p = \frac{1}{p}$ $\qquad (x = a_0 + a_1 p + a_2 p^2 + \dots)$ $a_i \in \mathbb{Z}_p$

Field extension:

$\mathbb{Q}_p(\pi)$ s.t. $\underline{\pi^{p-1} = -p}$ $\qquad |\pi|_p = \frac{1}{p^{\frac{1}{p-1}}} < 1$

Now this field contains ζ_p:

$$\zeta_p = 1 + b\pi + O(\pi^2), \qquad b = 9 1 \dots, p-1$$

$\mathbb{Z}_p[\pi] \subseteq \mathbb{Q}_p(\pi)$, the ideal generated by π is maximal

$$\mathbb{Z}_p[\pi]/(\pi) = \mathbb{F}_p$$

Lemma: $\quad (1+\pi\alpha+\pi^2\beta)^p = 1 + \pi^p(\alpha^p - \alpha) + O(\pi^{p+1})$

Proof: $\quad (1+\pi\alpha+\pi^2\beta)^p = \sum_{k=0}^{p} \binom{p}{k} (\pi\alpha+\pi^2\beta)^k = 1 + \underset{-\pi^{p-1}}{\overset{\shortparallel}{p}}(\pi\alpha+\pi^2\beta) + \frac{p(p-1)}{2}(\quad)^2$

$+ \dots + (\pi\alpha+\pi^2\beta)^p = 1 - \pi^p\alpha + \pi^p\alpha^p + O(\pi^{p+1})$

p-curvature: $C_p(\nabla_i) = \nabla_i^p - \nabla_i \mod p$ $\qquad \nabla_i = z_i \frac{\partial}{\partial z_i} - s\, C_i(z)$

Lemma: $(\nabla_i)^p - \nabla_i \equiv z_i^p \tilde{\nabla}_i^p \pmod p$ $\qquad \tilde{\nabla}_i = \frac{\partial}{\partial z_i} - \frac{s}{z_i} C_i(z)$

Reduction to $\mathbb{F}_p$: Let $q \in \mathbb{Q}_p(\pi)$ $\qquad q = 1 + \pi + O(\pi^c)$

Consider iterated product $M_{\mathcal{L}_i, \xi_p}(z, a, q) = \left(M_{\mathcal{L}_i}(z, a, q)\, q^{z_i \frac{\partial}{\partial z_i}} \right)^p$

$$M_{\mathcal{L}_i}(z, a, q)\, q^{z_i \frac{\partial}{\partial z_i}} = 1 + \pi \nabla_i(z) + \dots$$

$$\left(M_{\mathcal{L}_i}(z, a, q)\, q^{z_i \frac{\partial}{\partial z_i}} \right)^p = 1 + \pi^p (\nabla_i^p - \nabla_i) + \dots$$

Finally:

$$\frac{M_{\mathcal{L}_i, \xi_p}(z) - 1}{\pi^p} \equiv \nabla_i^p - \nabla_i \mod \pi \equiv C_p(\nabla_i) \mod p$$

Frobenius action: $M_{\mathcal{L}_i}(z^p, a^p, q^p) = 1 + C_i(z^p, a^p)\, \pi^p (s^p - s) + O(\pi^{p+1})$

or

$$\frac{M_{\mathcal{L}_i}(z^p, a^p, q^p) - 1}{\pi^p} \equiv (s^p - s)\, C_i(z^p, a^p) \mod \pi$$

The Isospectrality theorem, [Etingof–Varchenko]

Operators $C_p(\nabla_i)$ and $(s^p - s) C_p(\nabla) \circ F_p$ are isospectral modulo p.