# SUSY Gauge Theories &Opers

Talk at SUSY@50, Minneapolis, MN

05/20/2023

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# Early days of SUSY


[1] Yu.A. Golfand and E.P. Likhtman. Extension of the Algebra of Poincare Group Generators and Violation of p Invariance. *JETP Lett.*, 13:323–326, 1971.

[2] D.V. Volkov and V.P. Akulov. Possible universal neutrino interaction. *JETP Lett.*, 16:438–440, 1972.

[3] A. Neveu and J.H. Schwarz. Factorizable dual model of pions. *Nucl.Phys.*, B31:86–112, 1971.

[4] J. Wess and B. Zumino. Supergauge Transformations in Four-Dimensions. *Nucl.Phys.*, B70:39–50, 1974.

# Supersymmetry

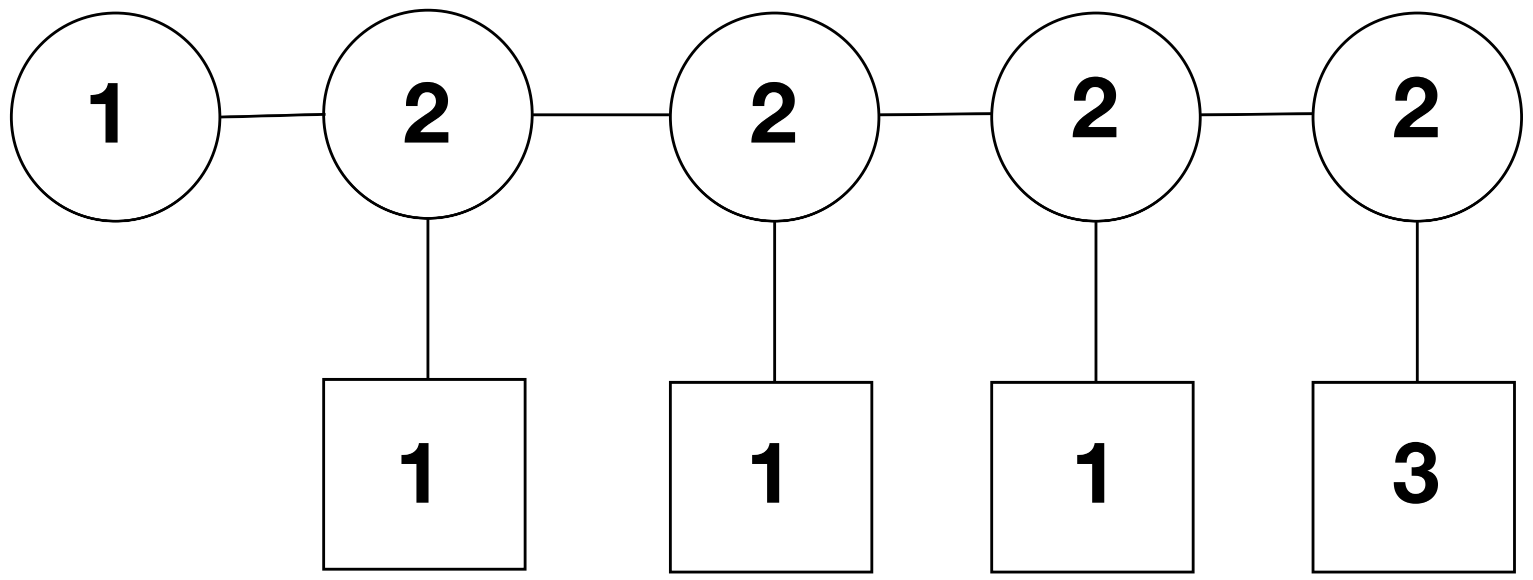
From my PhD thesis at University of Minnesota

Let us mention, however, that the status of supersymmetry as a branch of theoretical physics may soon completely change its form after new data on SUSY search will start coming from the Large Hadron Collider (LHC) in CERN. Astonishingly the Higgs boson discovery at $m_H \sim 126$ GeV has happened [5, 6] while the author was preparing the current thesis. Experimentalists in Geneva have done (and keep doing) an extraordinary job, and steadily we will know the answers to many questions about how real the supersymmetry is in the nearest future. As of the present day (Summer 2012) the perspectives of finding SUSY at LHC are not very optimistic (see e.g. [7]). In the worst case scenario, when the SUSY will be found not to be present, at least in the form we use to think about it, phenomenological studies in this direction will be virtually over and only abstract and formal aspects of supersymmetry will remain on the frontline of physics.

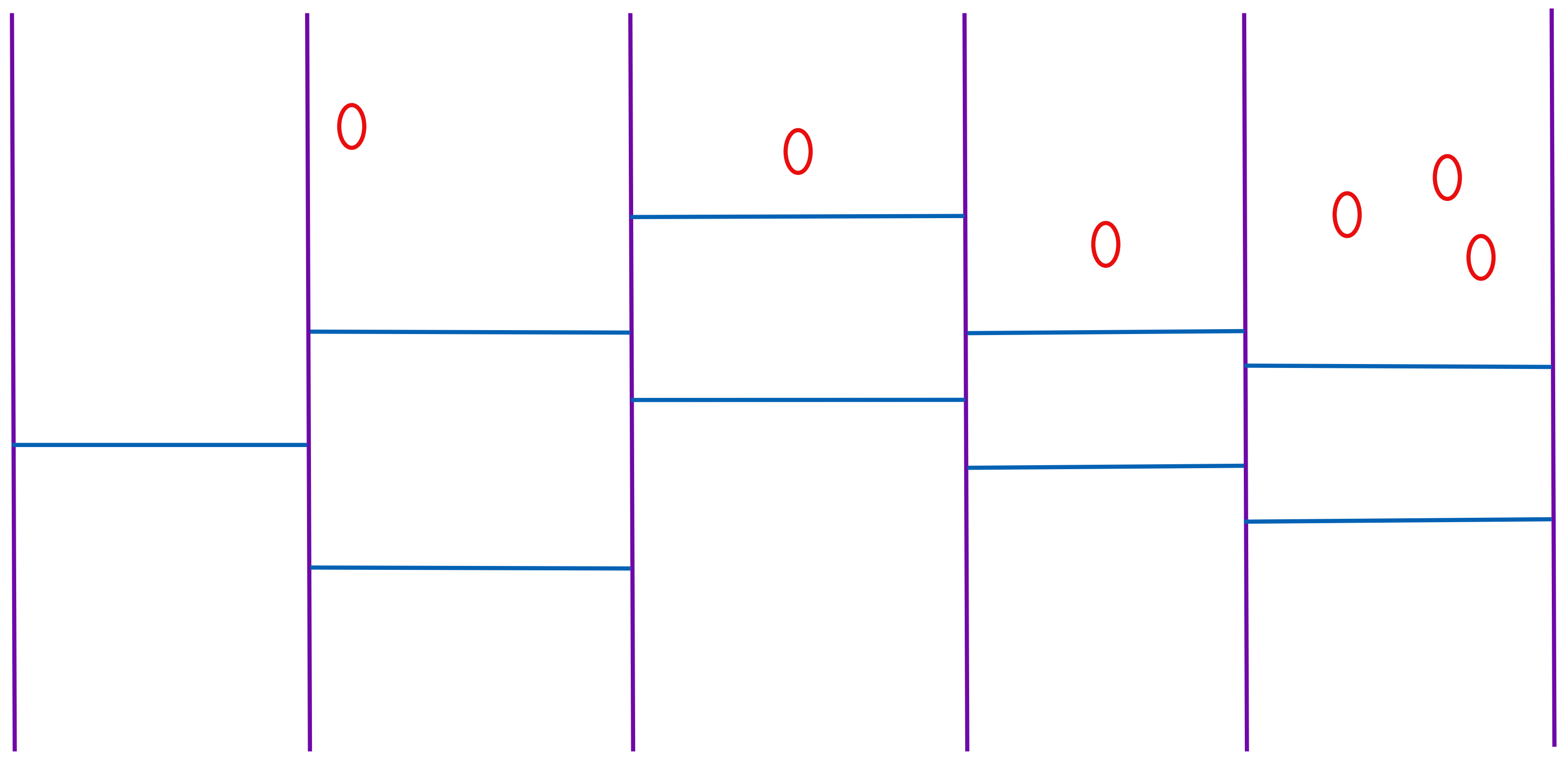
# The Success of SUSY

- LHC says — `sorry folks…' but:
- SUSY helps solve theories at strong coupling
- SUSY provides a plethora of exact computations (i.e. NSVZ, ADS, Gluino condensate, SW, etc.)
- SUSY is indispensable for String Theory
- SUSY gauge theories/String Theory provide powerful tools to approach mathematical problems (physical mathematics)

# $\mathcal{N} = 4$ 3d Gauge Theories



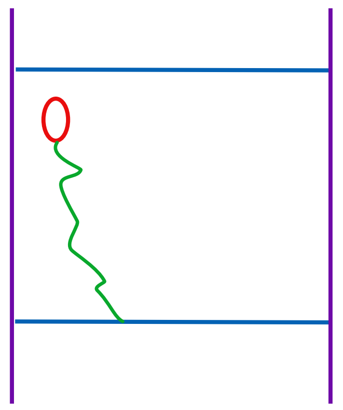
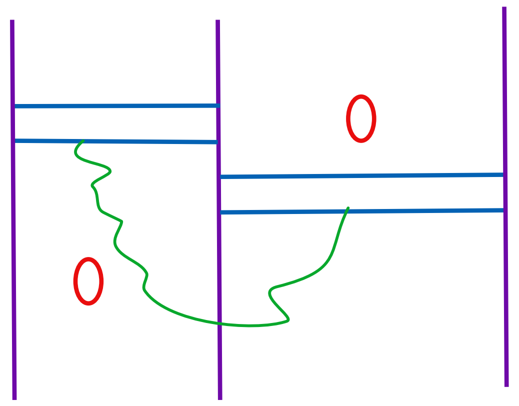
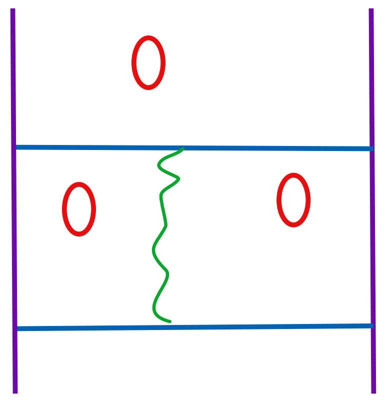

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| NS5 | x | x | x | | | | | x | x | x |
| D5 | x | x | x | | x | x | x | | | |
| D3 | x | x | x | x | | | | | | |

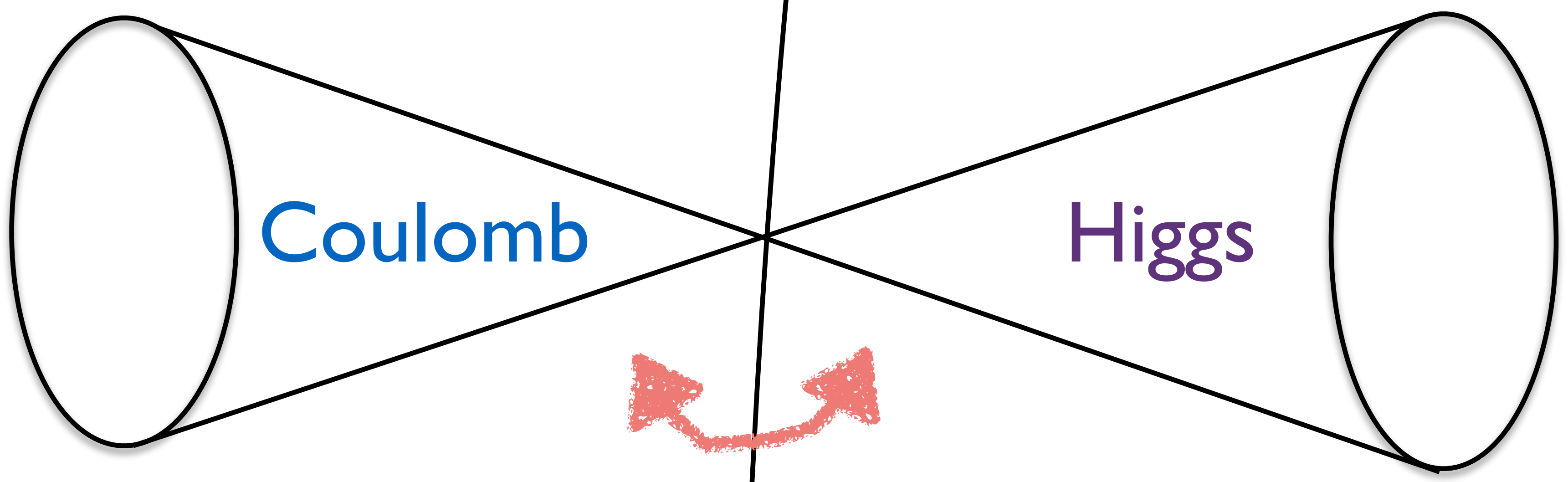


Gauge group $G = \prod_{i=1}^{\mathrm{rk}\mathfrak{g}} U(v_i)$

Flavor group $G_F = \prod_i U(w_i)$

Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

$\mathrm{Hom}(V_i, W_i)$  $\mathrm{Hom}(V_i, V_{i+1})$  $\mathrm{Hom}(V_i, V_i)$ 




**interchanged by 3d mirror**

$\mathcal{N}=2$ vectormultiplet inside $\mathcal{N}=4$ multiplet can be dualized to linear multiplet $\quad \phi_3$

Altogether give a complex scalar `**monopole**' operator charged under topological $U(1)$ symmetry

$$X = e^{\phi_3/g^2 + i\sigma}$$

$$J = *F \qquad d * J = 0$$

$\mathcal{N}=4$ vectormultiplet contains one more complex scalar

$$\phi = \phi_1 + i\phi_2 \qquad\qquad \mathrm{Tr}\phi^k$$

Classically Coulomb branch is $\quad \mathcal{M}_C \simeq (\mathbb{R}^3 \times S^1)^{\mathrm{rank}\, G}/W_G$

Monopole operators receive quantum corrections

Parameterized by VEVs of hypermultiplets modulo gauge transformations

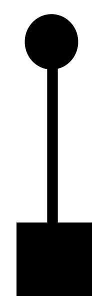
Moment map $\qquad \mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$

Quiver variety $\qquad X = \mu^{-1}(0)//_\theta G = \mu^{-1}(0)_{ss}/G$

Ex: $T^*Gr_{k,n}$

$\mathbf{v}_1 = k,\ \mathbf{w}_1 = n$

$V = \mathbb{C}^k$



$W = \mathbb{C}^n$

No quantum corrections

# $\mathcal{N} = 2$ Deformations

$\mathcal{N} = 4$ R-symmetry

$$SU(2)_C \times SU(2)_H$$

$\mathcal{N} = 2$ R-symmetry

$$U(1)_R \qquad \mathfrak{j}_C + \mathfrak{j}_H$$

$\mathcal{N} = 2^*$ Mass deformation

twisted mass $\quad U(1)_\epsilon$

for $\quad \mathfrak{j}_C - \mathfrak{j}_H$

Flavor symmetry

$$G_C \times G_H$$

$$SU(2)_C \times SU(2)_H$$

FI terms   masses

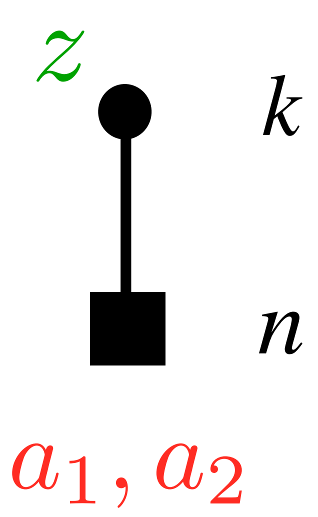
Circle compactificaiton

$$u^{2d} = u^{3d} - \frac{i}{R}\oint_{S^1} A^f$$

$$q = e^{R\epsilon}$$

The resulting theory is $\mathcal{N} = 2^*$ 3d theory on $\mathbb{R}^2 \times S^1$

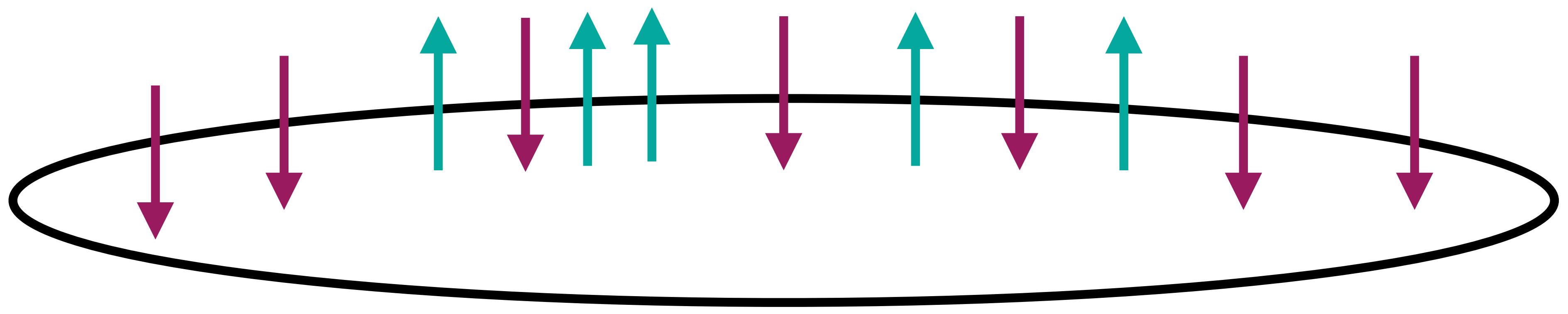
$U(k)$ SQCD with $n$ squarks

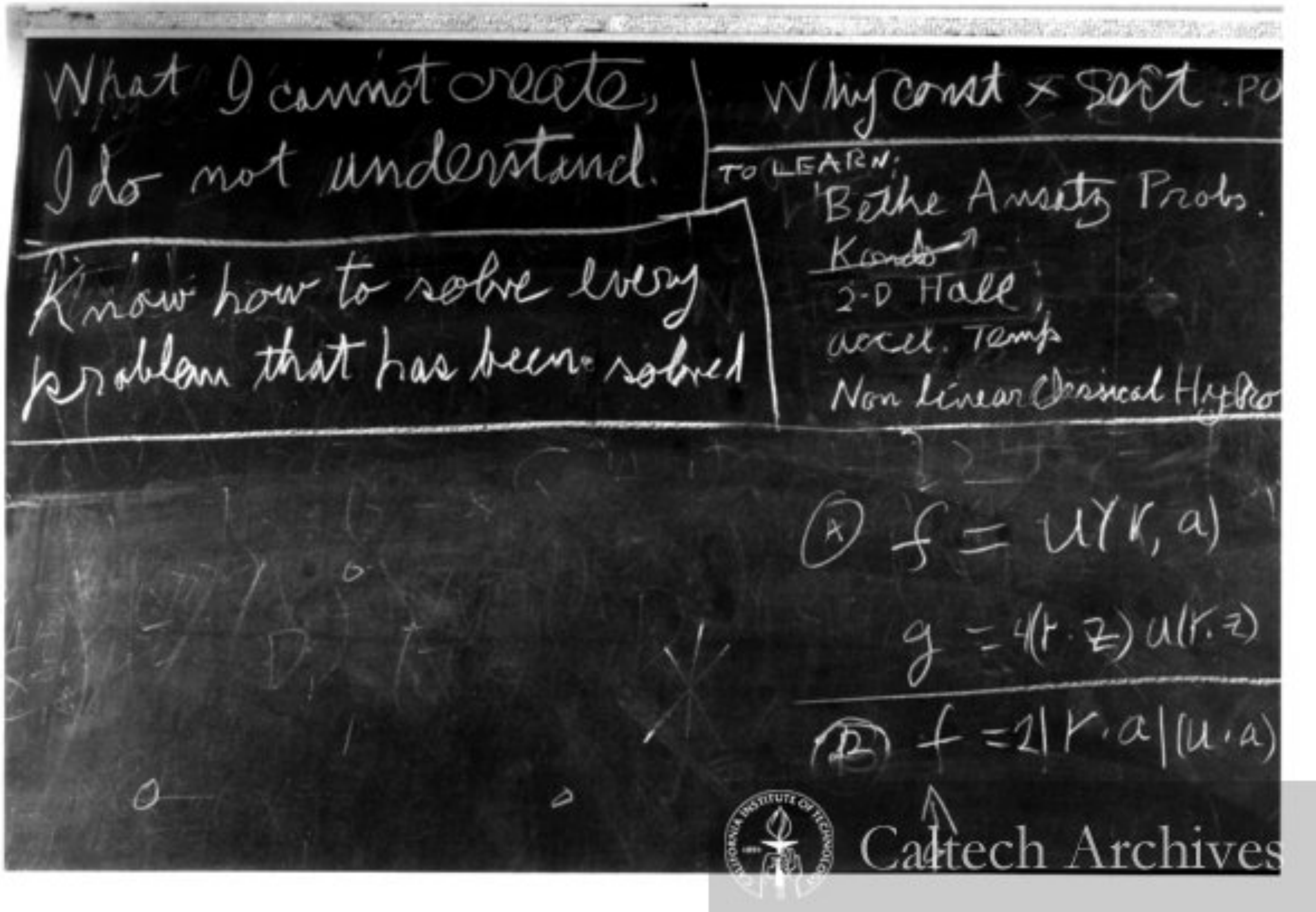


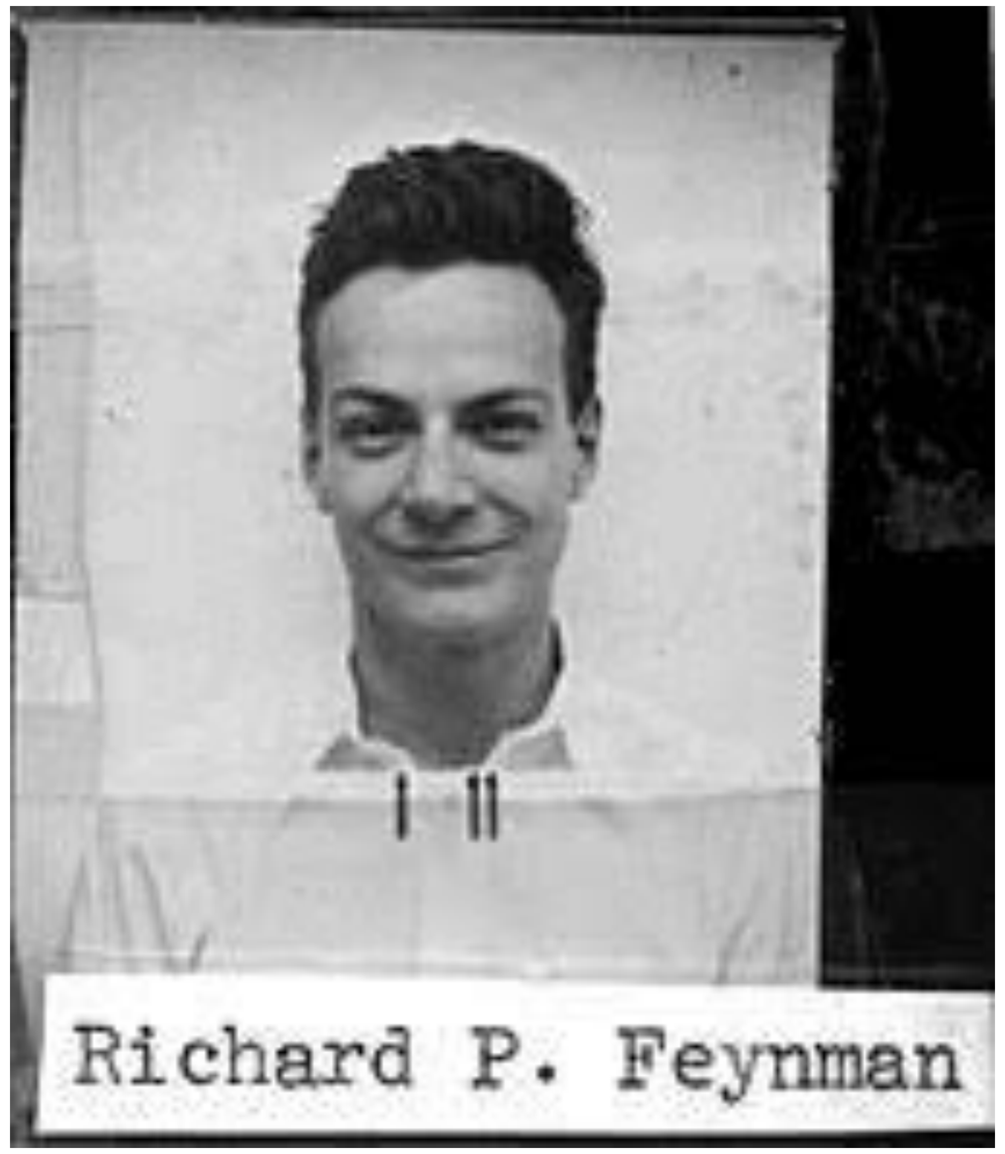

Space of massive vacua described by $SL(2)$ XXZ Bethe equations

$$\prod_{l=1}^{n} \frac{s_i - a_l}{s_i - q a_l} = z^2 q^k \prod_{j=1}^{k} \frac{q s_i - s_j}{s_i - q s_j}$$

$$i = 1 \cdots k.$$





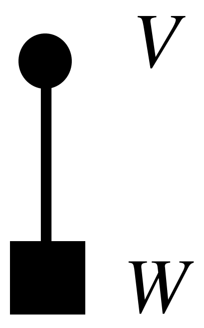




I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better.

# Seiberg-like Duality

There is a bijection between the spaces of vacua for $U(k)$ and $U(n-k)$ theories

These are different UV descriptions of the same IR fixed point




Short exact sequence of bundles

$$0 \to V \to W \to V^{\vee} \to 0$$

Introduce Baxter Q-functions

$$Q(u) = \prod_{i=1}^{k} (u - s_i) \qquad \widetilde{Q}(u)$$

Satisfy the QQ-relation

$$z\,\widetilde{Q}(qu)Q(u) - \widetilde{Q}(u)Q(qu) = \prod_{i=1}^{n} (u - a_i)$$

equivalent to the XXZ Bethe (SUSY vacua) equations

# The Juxtaposition of Duality Frames

Consider both duality frames simultaneously. Compose a vector in $\mathbb{C}^2$

$$s(u) = \begin{pmatrix} Q(u) \\ \widetilde{Q}(u) \end{pmatrix}$$

Consider its shifts by the $\mathcal{N} = 2^*$ mass $q = e^{R\epsilon}$

$$s(qu) = \begin{pmatrix} Q(qu) \\ \widetilde{Q}(qu) \end{pmatrix}$$

Calculate the determinant (q-Wronskian)

$$s(u) \wedge s(qu) = \begin{pmatrix} Q(u) & Q(qu) \\ \widetilde{Q}(u) & \widetilde{Q}(qu) \end{pmatrix}$$

Almost reproduces the left-hand side of the QQ-relation

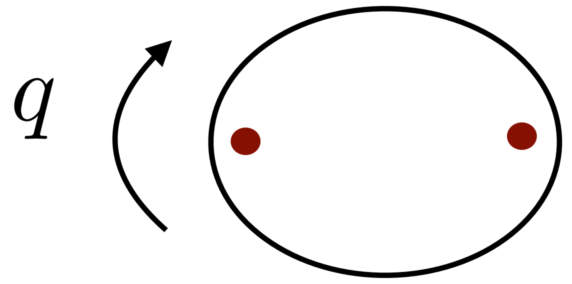
$$z\,\widetilde{Q}(qu)Q(u) - \widetilde{Q}(u)Q(qu) = \prod_{i=1}^{n}(u - a_i)$$

This brings us to a new geometric object - **Oper**

# q-Opers

Riemann sphere with multiplication

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$u \mapsto qu$$




Section $s(u)$

Connection $A(u) : E \to E^q$

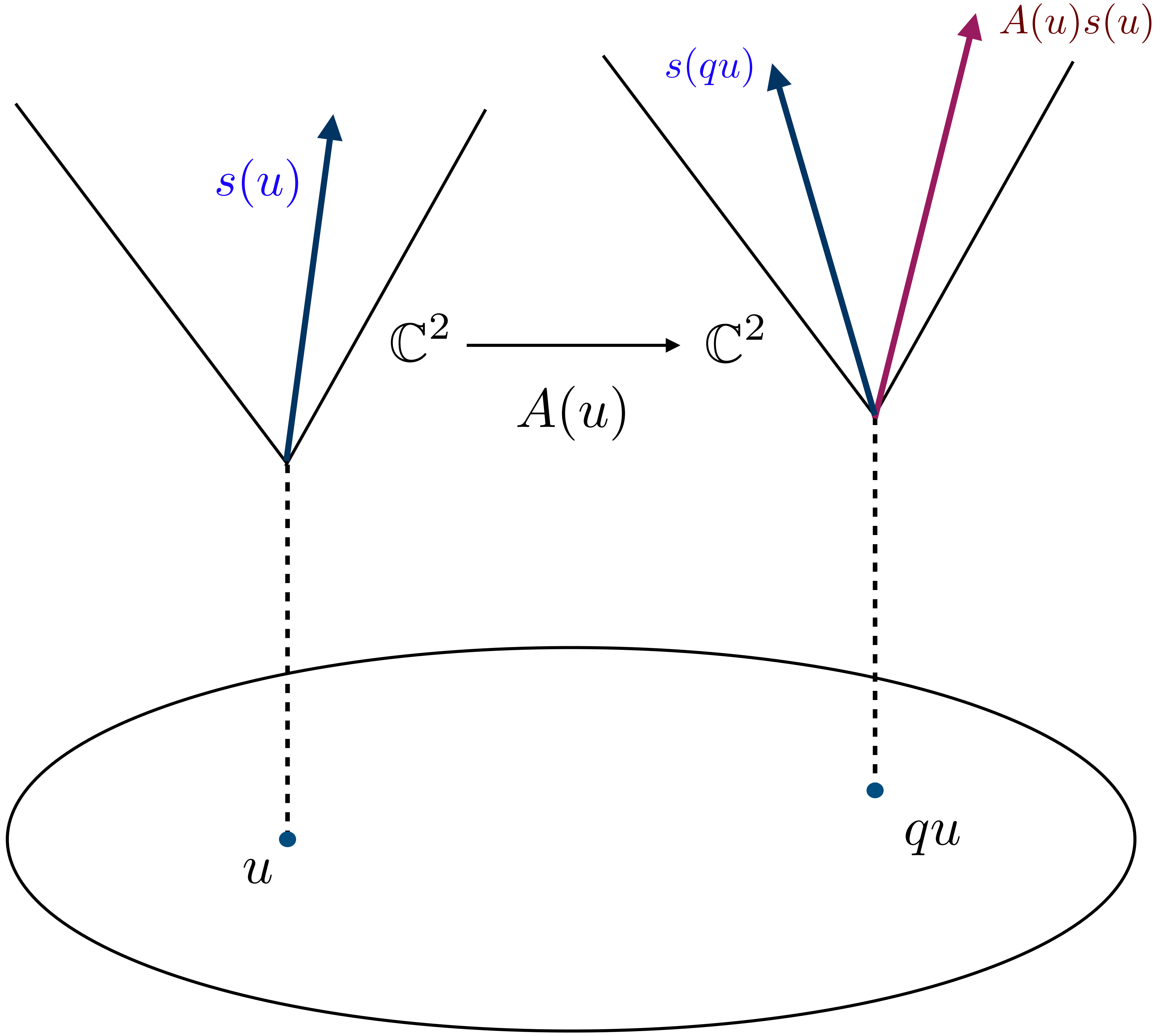
q-gauge transformation

$$A(u) \mapsto g(qu)A(u)g(u)^{-1}$$

(SL(2),q)-oper condition

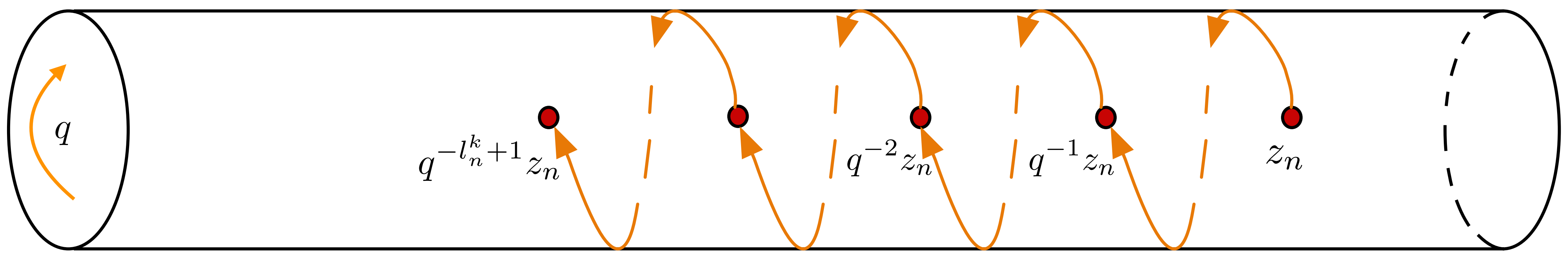
$$s(qu) \wedge A(u)s(u) \neq 0$$

Vector bundle $E$ of rank 2



# Singularities and Twists

Allow singularities $s(qu) \wedge A(u)s(u) = \Lambda(u)$ $\Lambda(u) = \prod_{l,j_l}(u - q^{j_l} a_l)$




Add Twists $Z = g(qu)A(u)g(u)^{-1}$

Section $s(u) = \begin{pmatrix} Q_+(u) \\ Q_-(u) \end{pmatrix}$ Twist element $Z = \mathrm{diag}(\zeta, \zeta^{-1})$ $z = \zeta^2$

q-Oper condition with $A(u) = Z - SL(2)$ QQ-system $\zeta^{-1} Q_+(u) Q_-(qu) - \zeta Q_+(qu) Q_-(u) = \Lambda(u)$

Difference Equation $D_q(s) = As$

Scalar difference operator $\left(D_q^2 - T(qu) D_q - \frac{\Lambda(qu)}{\Lambda(u)}\right) s_1 = 0$

# Trig Ruijsenaars-Schneider Hamiltonians

(SL(2),q)-oper condition

$$\det \begin{pmatrix} Q_+(u) & \zeta Q_+(qu) \\ Q_-(u) & \zeta^{-1} Q_-(qu) \end{pmatrix} = \Lambda(u)$$

Let $Q(u) = u - p_1$ $\widetilde{Q}(u) = u - p_2$

$$u^2 - u\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_1 + \frac{q\zeta - q\zeta^{-1}}{\zeta^{-1} - \zeta} p_2\right] + p_1 p_2 = (u - a_1)(u - a_2)$$

$T_1$ $T_2$

qOper condition yields
tRS Hamiltonians!

$$\det(u - T) = (u - a_1)(u - a_2)$$

# tRS Model with 2 Particles

Relativistic Hamiltonians

$$T_1 = \frac{\zeta_1 - q\zeta_2}{\zeta_1 - \zeta_2} p_1 + \frac{\zeta_2 - q\zeta_1}{\zeta_2 - \zeta_1} p_2$$

$$T_2 = p_1 p_2$$

Symplectic form

$$\Omega = \sum \frac{dp_i}{p_i} \wedge \frac{d\zeta_i}{\zeta_i}$$

Integrals of motion

$$T_i = E_i$$

Coordinates $\zeta_i$, momenta $p_i$ coupling constant $q$, energies $E_i$

Nonrelativistic limit

$$p_i = \exp \frac{P_i}{c} \qquad \zeta_i = \exp \frac{X_i}{c} \qquad T_{\text{Calogero}} = \lim_{c \to \infty} T_{\text{tRS}} - n\,mc^2$$

# q-Opers and q-Langlands

[Frenkel, PK, Zeitlin, Sage, **JEMS 2023**]

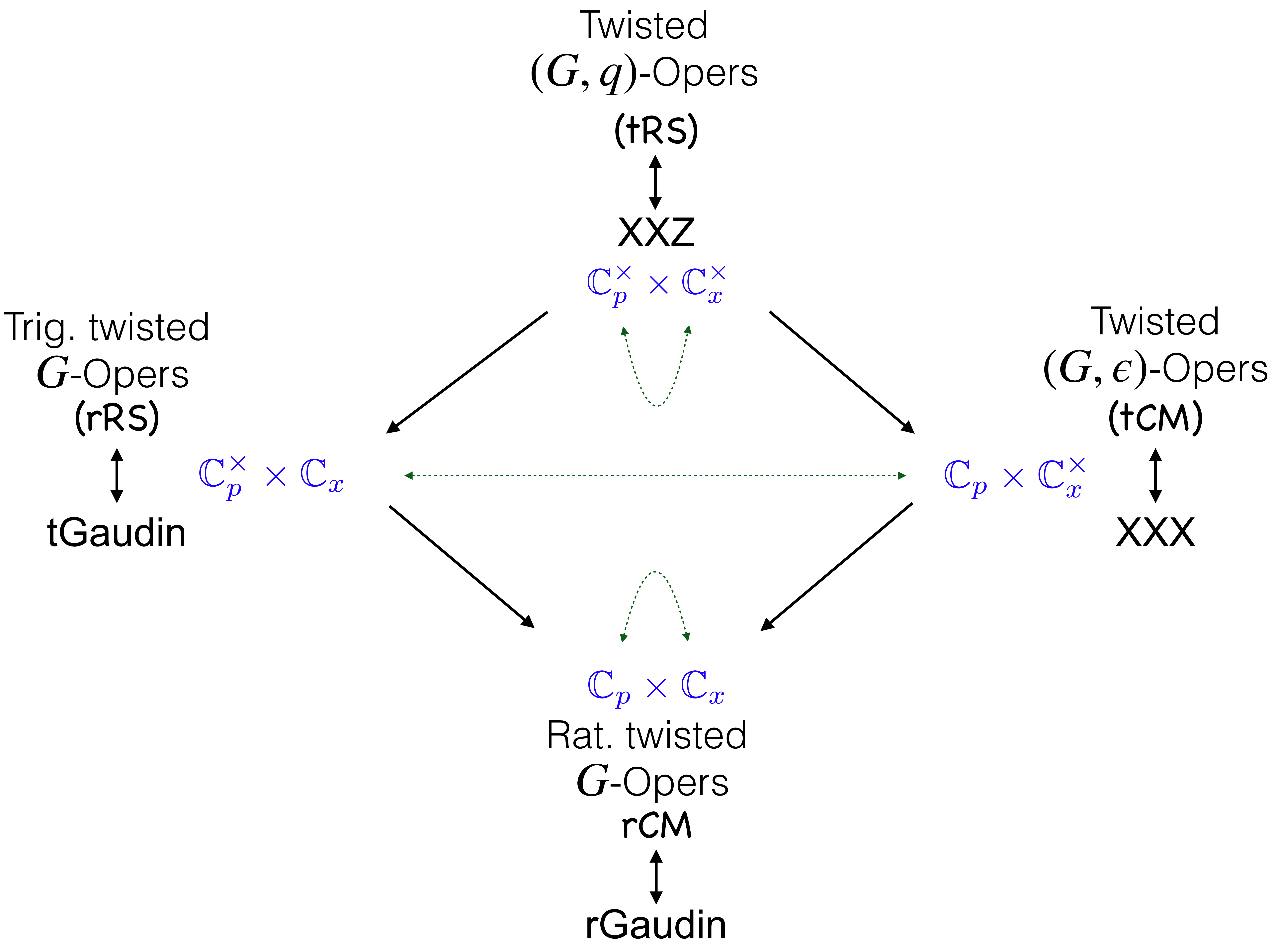
Miura $(G,q)$-oper with singularities $$A(u) = \prod_i \left(\zeta_i \frac{Q^i_+(qu)}{Q^i_+(u)}\right)^{\check{\alpha}_i} \exp \frac{\Lambda_i(u)}{g_i(u)} e_i$$

**Theorem:** There is a 1-to-1 correspondence between the set of nondegenerate $Z$-twisted $(G,q)$-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

$$\widetilde{\xi}_i\, Q^i_-(u)Q^i_+(qu) - \xi_i Q^i_-(qu)Q^i_+(u) = \Lambda_i(u) \prod_{j>i}\left[Q^j_+(qu)\right]^{-a_{ji}} \prod_{j<i}\left[Q^j_+(u)\right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

# Network of Dualities



# The Ubiquitous QQ-System

Bethe Ansatz equations for XXX, XXZ models — eigenvalues of Baxter operators

[Mukhin, Varchenko] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\hat{g})$

[Frenkel, Hernandez] ....

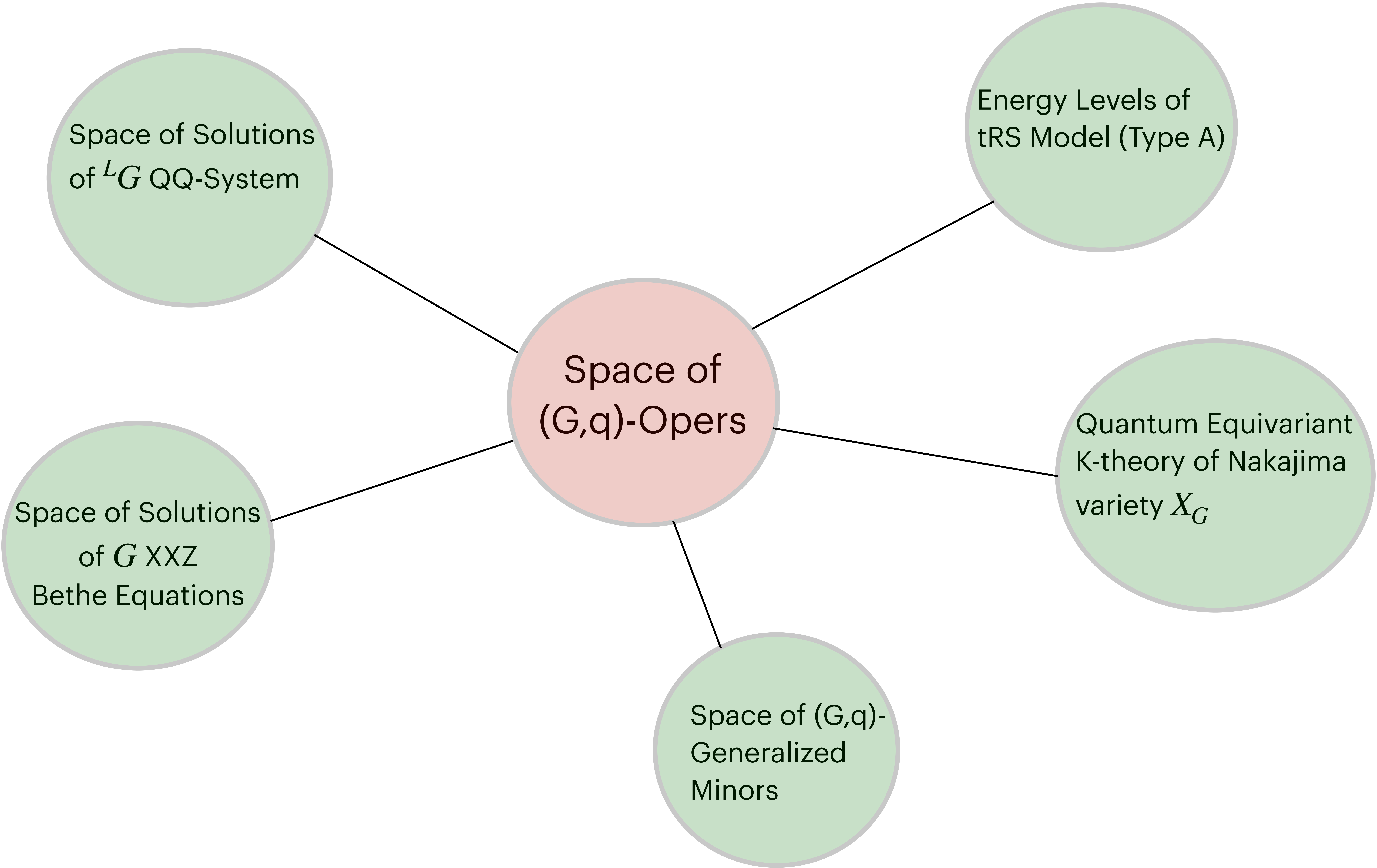
Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Nekrasov-Shatashvili] [Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

Spectral determinants in the QDE/IM Correspondence

[Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

**(G,q)-Opers**



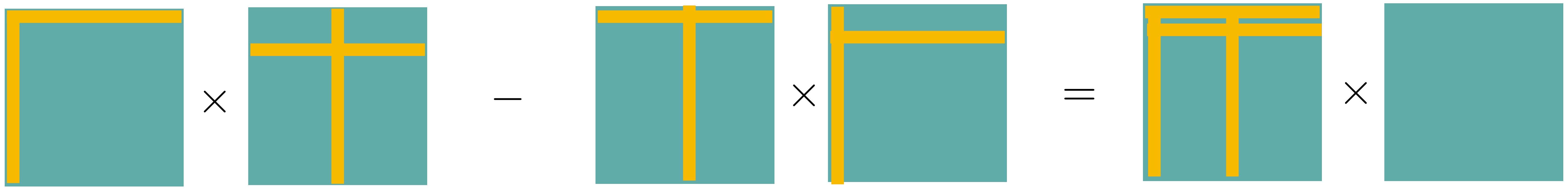
Space of Solutions of $^LG$ QQ-System
Energy Levels of tRS Model (Type A)
Space of (G,q)-Opers
Quantum Equivariant K-theory of Nakajima variety $X_G$
Space of Solutions of $G$ XXZ Bethe Equations
Space of (G,q)-Generalized Minors

# Cluster Algerbras

[PK, Zeitlin, 2022, **Crelle**]

The QQ-system $\xi_{i+1}\, Q^i_-(u)Q^i_+(u+\epsilon) - \xi_i Q^i_-(u+\epsilon)Q^i_+(u) = \Lambda_i(u)Q^{i+1}_+(u+\epsilon)Q^{i+1}_+(u)$

For $G = SL(n)$ obtain Lewis Carrol (Desnanot-Jacobi-Trudi) identity $M^1_1 M^2_i - M^1_i M^2_1 = M^{12}_{1i} M$



For general $G$ obtain relation on generalized minors $\Delta^{\omega_i}(v(u)) = Q^i_+(u)$ [Fomin Zelevinsky]

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}\Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i}\Delta_{u\cdot\omega_i, vw_i\cdot\omega_i} = \prod_{j\neq i}\Delta^{-a_{ji}}_{u\cdot\omega_j, v\cdot\omega_j},$$

$u, v \in W_G$