Partitions in Math and Physics

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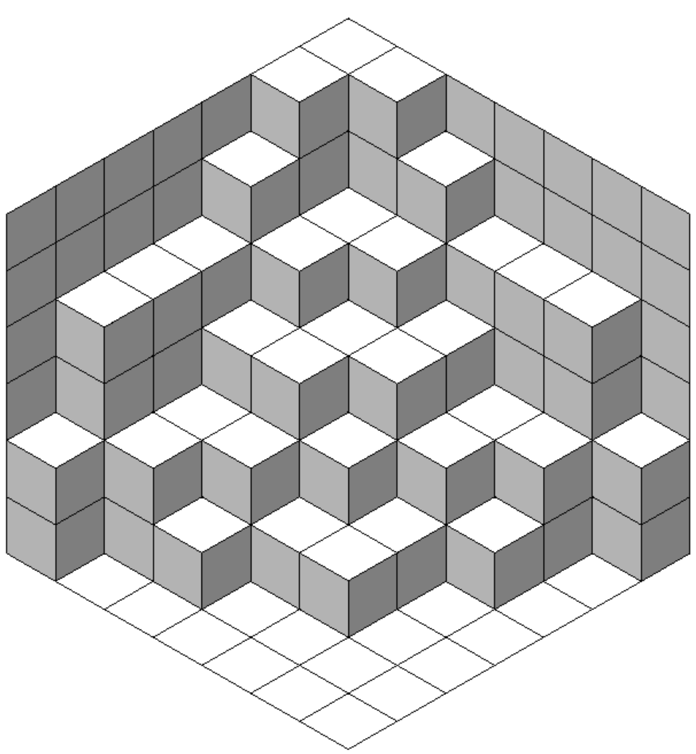
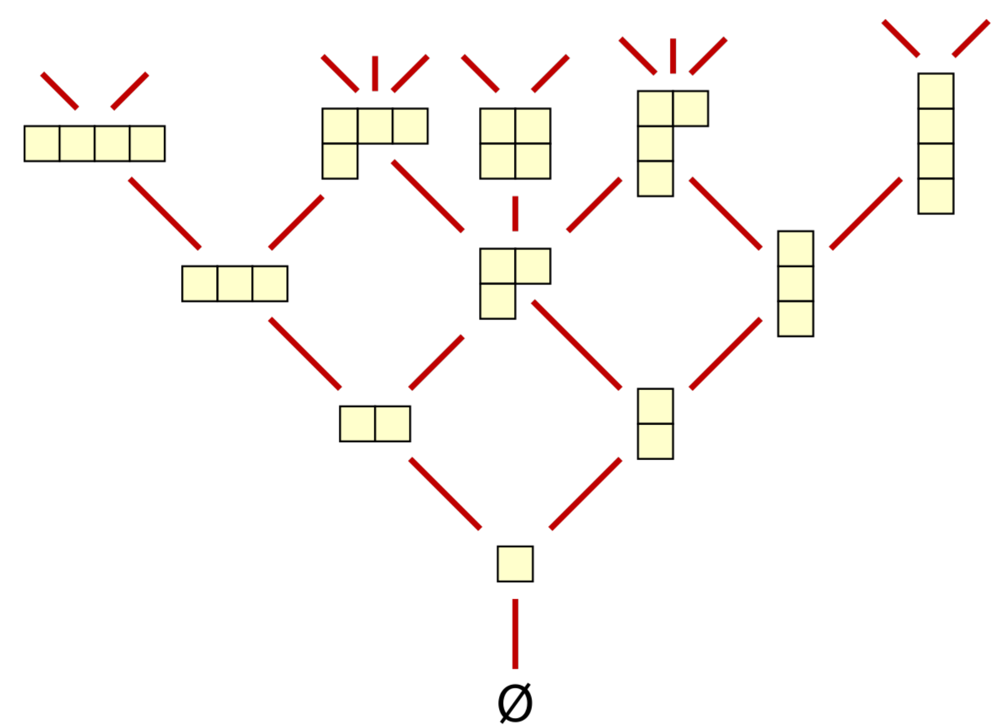
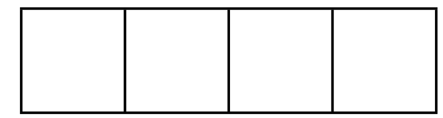




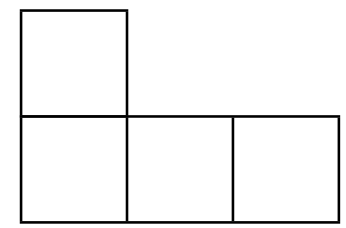
Partitions

There are several ways to decompose an integer into sums of smaller integers

4=1+1+1+1

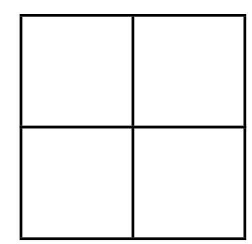


4=2+1+1

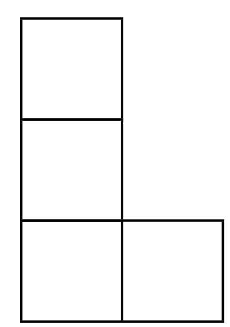


Young diagrams

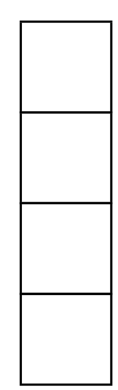
4=2+2



4=3+1



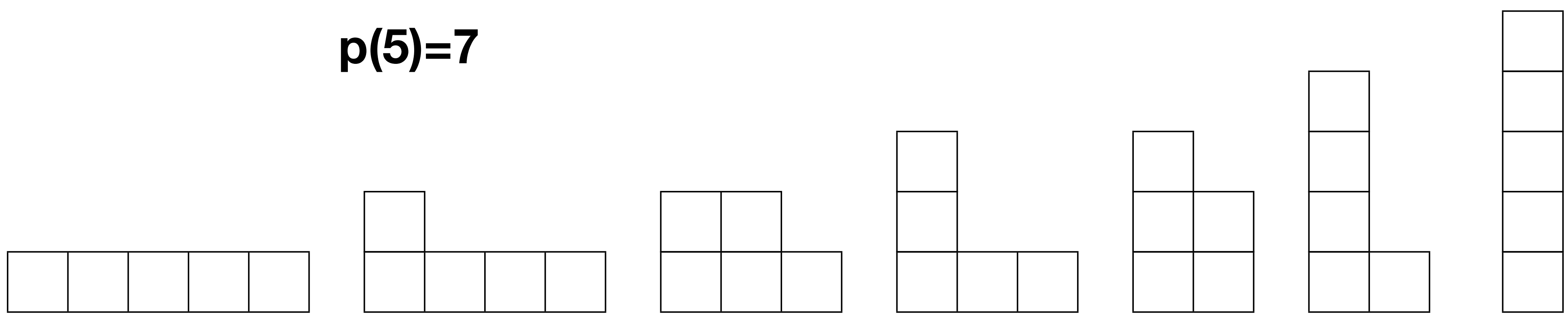
4=4+0



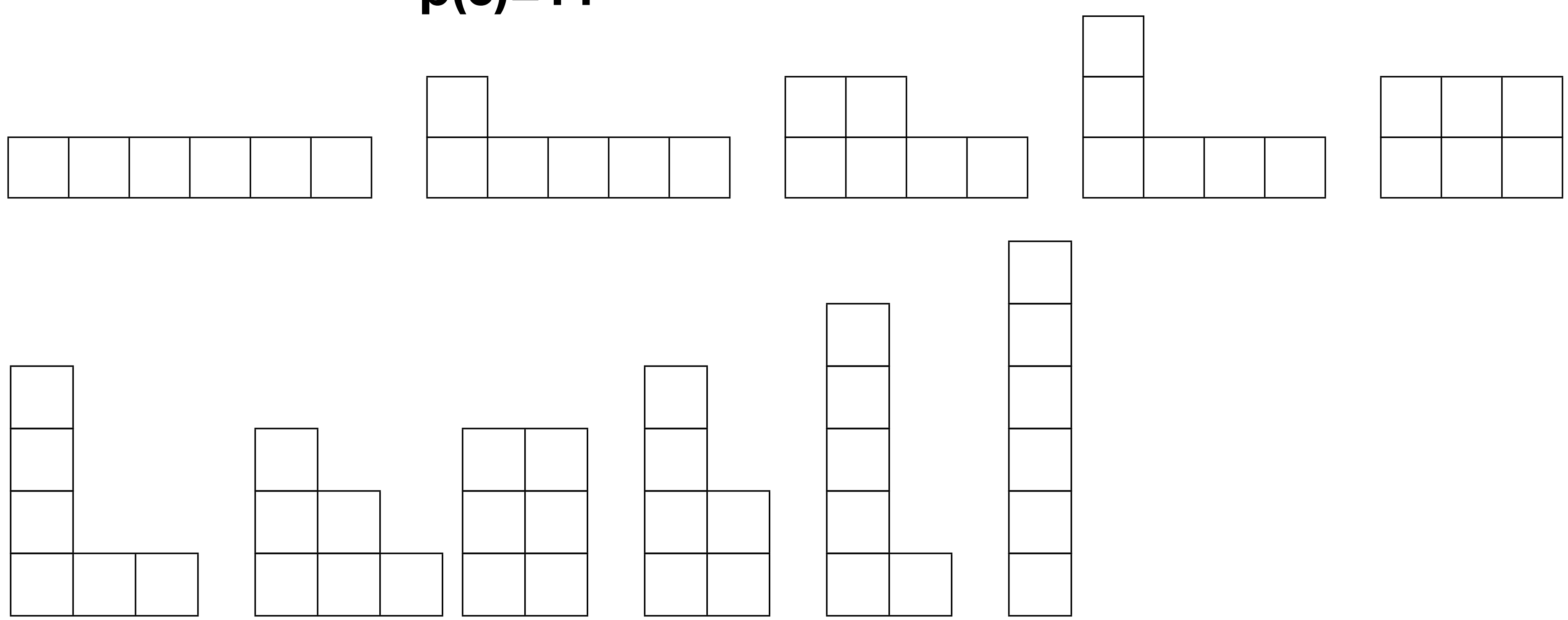
Partitions

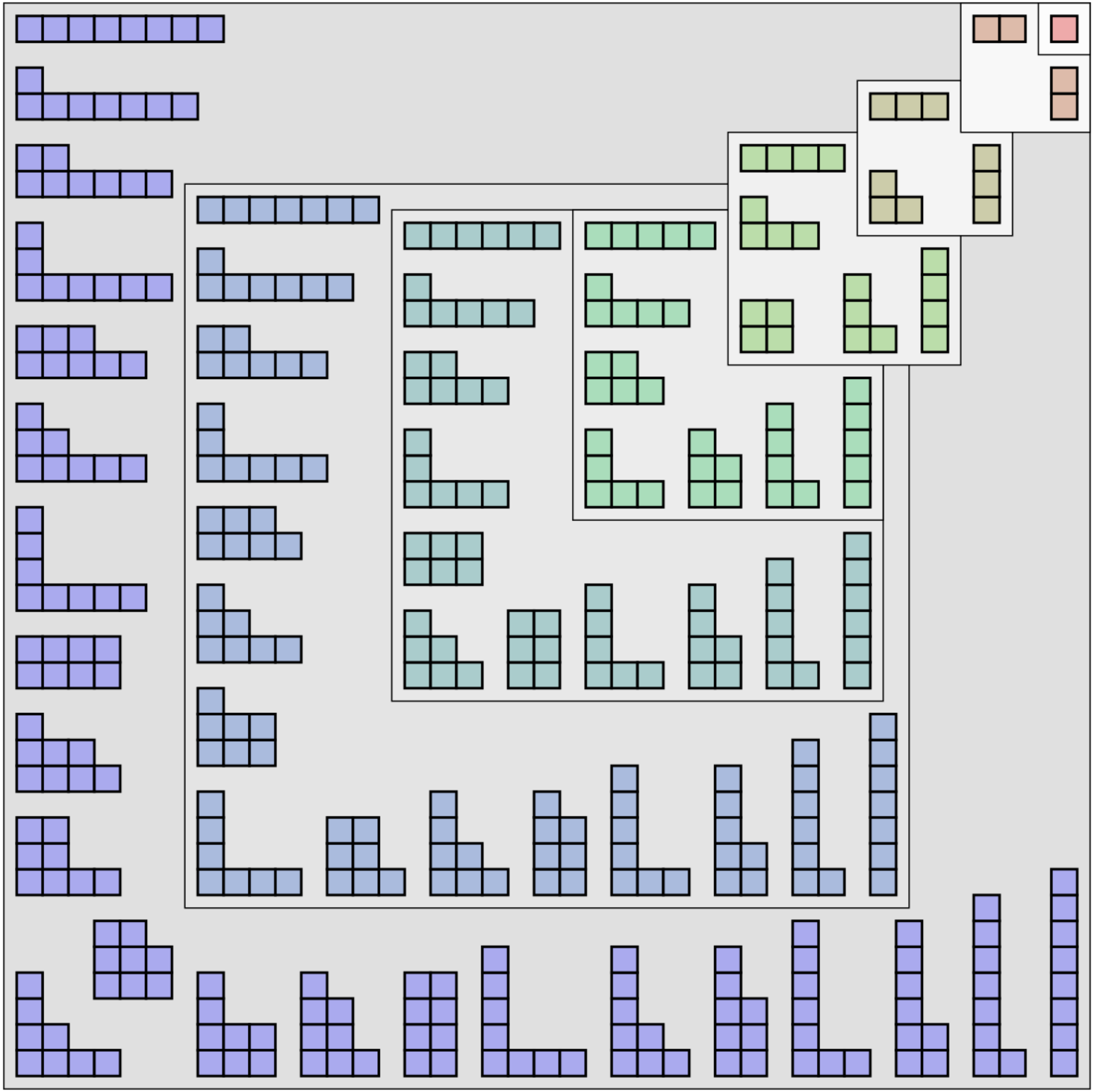
Problem: Find all partitions of numbers 1,2,3,4,5,6,7,8,9 together with their Young diagrams

n=5 **p(5)=7**



n=6 **p(6)=11**





Partitions

n=7 **p(7)=15**

{7}, {6, 1}, {5, 2}, {5, 1, 1}, {4, 3}, {4, 2, 1}, {4, 1, 1, 1}, {3, 3, 1}, {3, 2, 2}, {3, 2, 1, 1},
{3, 1, 1, 1, 1}, {2, 2, 2, 1}, {2, 2, 1, 1, 1}, {2, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}

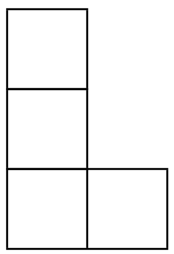
n=8 **p(8)=22**

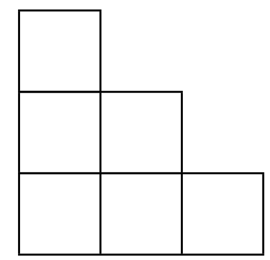
{8}, {7, 1}, {6, 2}, {6, 1, 1}, {5, 3}, {5, 2, 1}, {5, 1, 1, 1}, {4, 4}, {4, 3, 1}, {4, 2, 2}, {4, 2, 1, 1},
{4, 1, 1, 1, 1}, {3, 3, 2}, {3, 3, 1, 1}, {3, 2, 2, 1}, {3, 2, 1, 1, 1}, {3, 1, 1, 1, 1, 1}, {2, 2, 2, 2},
{2, 2, 2, 1, 1}, {2, 2, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}

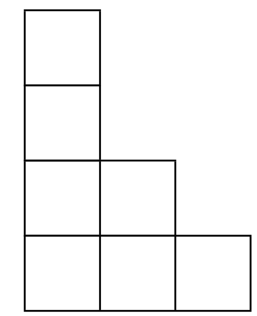
n=9 **p(9)=30**

{9}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2}, {5, 2, 1, 1},
{5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1}, {4, 1, 1, 1, 1, 1},
{3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1}, {3, 2, 1, 1, 1, 1},
{3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1},
{1, 1, 1, 1, 1, 1, 1, 1, 1}

Odd & Distinct Parts







Problem (a): Count the number of partitions with ***odd parts*** from the previous examples

Problem (b): Count the number of partitions with ***distinct parts*** from the previous examples

n=7 **p(7)=15**

{7}, {6, 1}, {5, 2}, {5, 1, 1}, {4, 3}, {4, 2, 1}, {4, 1, 1, 1}, {3, 3, 1}, {3, 2, 2}, {3, 2, 1, 1}, {3, 1, 1, 1, 1}, {2, 2, 2, 1}, {2, 2, 1, 1, 1}, {2, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1}

n=8 **p(8)=22**

{8}, {7, 1}, {6, 2}, {6, 1, 1}, {5, 3}, {5, 2, 1}, {5, 1, 1, 1}, {4, 4}, {4, 3, 1}, {4, 2, 2}, {4, 2, 1, 1}, {4, 1, 1, 1, 1}, {3, 3, 2}, {3, 3, 1, 1}, {3, 2, 2, 1}, {3, 2, 1, 1, 1}, {3, 1, 1, 1, 1, 1}, {2, 2, 2, 2}, {2, 2, 2, 1, 1}, {2, 2, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1}

n=9 **p(9)=30**

{9}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2}, {5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1}, {4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1}, {3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}

Odd & Distinct Partitions

Find **odd and distinct** partitions for n = 1,2,...,11

n=9, p(9)=30

$$\{9\}, \{8,1\}, \{7,2\}, \{7,1,1\}, \{6,3\}, \{6,2,1\}, \{6,1,1,1\}, \{5,4\}, \{5,3,1\}, \{5,2,2\}, \{5,2,1,1\},$$
$$\{5,1,1,1,1\}, \{4,4,1\}, \{4,3,2\}, \{4,3,1,1\}, \{4,2,2,1\}, \{4,2,1,1,1\}, \{4,1,1,1,1,1\}, \{3,3,3\},$$
$$\{3,3,2,1\}, \{3,3,1,1,1\}, \{3,2,2,2\}, \{3,2,2,1,1\}, \{3,2,1,1,1,1\}, \{3,1,1,1,1,1,1\}, \{2,2,2,2,1\}$$
$$\{2,2,2,1,1,1\}, \{2,2,1,1,1,1,1\}, \{2,1,1,1,1,1,1,1\}, \{1,1,1,1,1,1,1,1,1\}$$

n=10, p(10)=42

$$\{\{10\}, \{9,1\}, \{8,2\}, \{8,1,1\}, \{7,3\}, \{7,2,1\}, \{7,1,1,1\}, \{6,4\}, \{6,3,1\}, \{6,2,2\}, \{6,2,1,1\},$$
$$\{6,1,1,1,1\}, \{5,5\}, \{5,4,1\}, \{5,3,2\}, \{5,3,1,1\}, \{5,2,2,1\}, \{5,2,1,1,1\}, \{5,1,1,1,1,1\},$$
$$\{4,4,2\}, \{4,4,1,1\}, \{4,3,3\}, \{4,3,2,1\}, \{4,3,1,1,1\}, \{4,2,2,2\}, \{4,2,2,1,1\}, \{4,2,1,1,1,1\},$$
$$\{4,1,1,1,1,1,1\}, \{3,3,3,1\}, \{3,3,2,2\}, \{3,3,2,1,1\}, \{3,3,1,1,1,1\}, \{3,2,2,2,1\}, \{3,2,2,1,1,1\}$$
$$\{3,2,1,1,1,1,1\}, \{3,1,1,1,1,1,1,1\}, \{2,2,2,2,2\}, \{2,2,2,2,1,1\}, \{2,2,2,1,1,1,1\},$$
$$\{2,2,1,1,1,1,1,1\}, \{2,1,1,1,1,1,1,1,1\}, \{1,1,1,1,1,1,1,1,1,1\}\}$$

n=11, p(11)=56

$\{\{11\},\{10,1\},\{9,2\},\{9,1,1\},\{8,3\},\{8,2,1\},\{8,1,1,1\},\{7,4\},\{7,3,1\},\{7,2,2\},\{7,2,1,1\},$

$\{7,1,1,1,1\},\{6,5\},\{6,4,1\},\{6,3,2\},\{6,3,1,1\},\{6,2,2,1\},\{6,2,1,1,1\},\{6,1,1,1,1,1\},\{5,5,1\},$

$\{5,4,2\},\{5,4,1,1\},\{5,3,3\},\{5,3,2,1\},\{5,3,1,1,1\},\{5,2,2,2\},\{5,2,2,1,1\},\{5,2,1,1,1,1\},$

$\{5,1,1,1,1,1,1\},\{4,4,3\},\{4,4,2,1\},\{4,4,1,1,1\},\{4,3,3,1\},\{4,3,2,2\},\{4,3,2,1,1\},\{4,3,1,1,1,1\},$

$\{4,2,2,2,1\},\{4,2,2,1,1,1\},\{4,2,1,1,1,1,1\},\{4,1,1,1,1,1,1,1\},\{3,3,3,2\},\{3,3,3,1,1\},\{3,3,2,2,1\}$

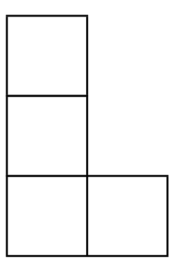
$\{3,3,2,1,1,1\},\{3,3,1,1,1,1,1\},\{3,2,2,2,2\},\{3,2,2,2,1,1\},\{3,2,2,1,1,1,1\},\{3,2,1,1,1,1,1,1\},$

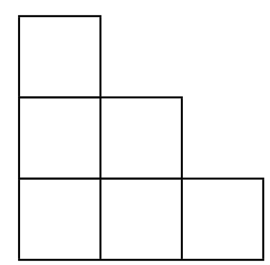
$\{3,1,1,1,1,1,1,1,1\},\{2,2,2,2,2,1\},\{2,2,2,2,1,1,1\},\{2,2,2,1,1,1,1,1\},\{2,2,1,1,1,1,1,1,1\},$

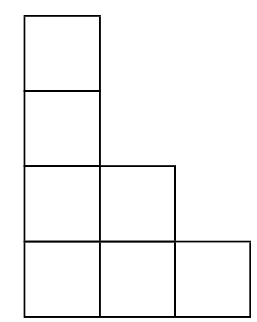
$\{2,1,1,1,1,1,1,1,1,1\},\{1,1,1,1,1,1,1,1,1,1,1\}\}$

Odd vs. Distinct

n	1	2	3	4	5	6	7	8	9
p(n)	1	2	3	5	7	11	15	22	30
# odd	1	1	2	2	3	4	5	6	8
# dist.	1	1	2	2	3	4	5	6	8







Problem: Why is the number of these partitions is the same for every n?

odd	distinct
5	5
3,1,1	4,1
1,1,1,1,1	3,2

odd	distinct
5,1	6
3,3	5,1
3,1,1,1	4,2
1,1,1,1,1,1	3,2,1

odd	distinct
7	7
5,1,1	6,1
3,3,1	5,2
3,1,1,1,1	4,3
1,1,1,1,1,1,1	4,2,1

odd	distinct
7,1	8
5,3	7,1
5,1,1,1	6,2
3,3,1,1	5,3
3,1,1,1,1,1	5,2,1
1,1,1,1,1,1,1,1	4,3,1

odd	distinct
9	9
7,1,1	8,1
5,1,1,1	7,2
5,3,1	6,3
3,3,3	6,2,1
3,3,1,1,1	5,4
3,1,1,1,1,1,1	5,3,1
1,1,1,1,1,1,1,1,1	4,3,2

Matching

From Distinct to Odd.

$$2 \to 1, 1$$
$$4 \to 1, 1, 1, 1$$
$$6 \to 3, 3$$
$$8 \to 1, 1, 1, 1, 1, 1, 1, 1$$

$$8 \to 1, 1, 1, 1, 1, 1, 1, 1$$
$$7, 1 \to 7, 1$$
$$6, 2 \to 3, 3, 1, 1$$
$$5, 3 \to 5, 3$$
$$5, 2, 1 \to 5, 1, 1, 1$$
$$4, 3, 1 \to 3, 1, 1, 1, 1, 1$$

From Odd to Distinct.

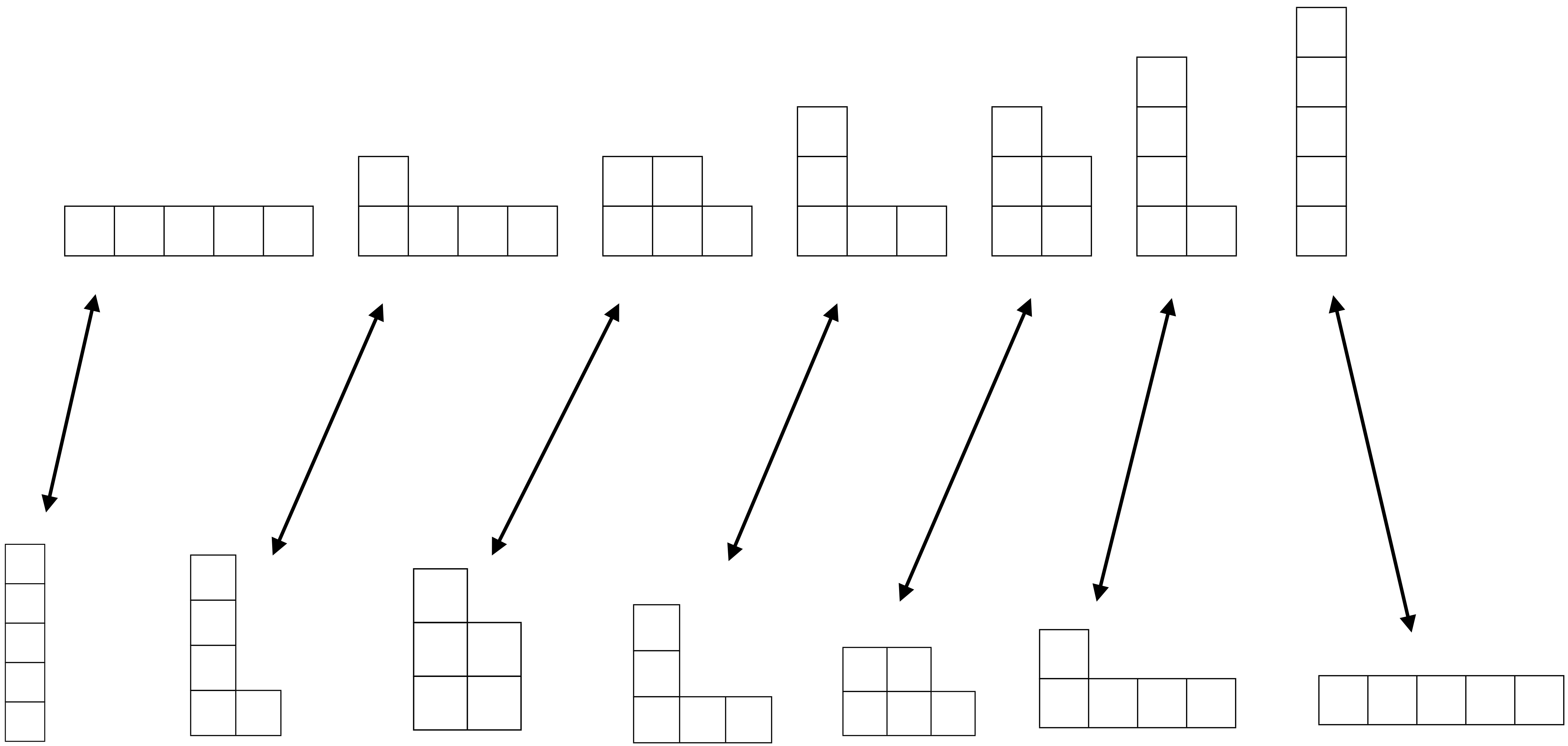
$$1 \to 1$$
$$1, 1 \to 2$$
$$1, 1, 1 \to 2, 1$$
$$1, 1, 1, 1 \to 4$$
$$1, 1, 1, 1, 1 \to 4, 1$$
$$1, 1, 1, 1, 1, 1 \to 4, 2$$
$$1, 1, 1, 1, 1, 1, 1 \to 4, 2, 1$$
$$1, 1, 1, 1, 1, 1, 1, 1 \to 8$$
$$1, 1, 1, 1, 1, 1, 1, 1, 1 \to 8, 1$$

Binary presentation

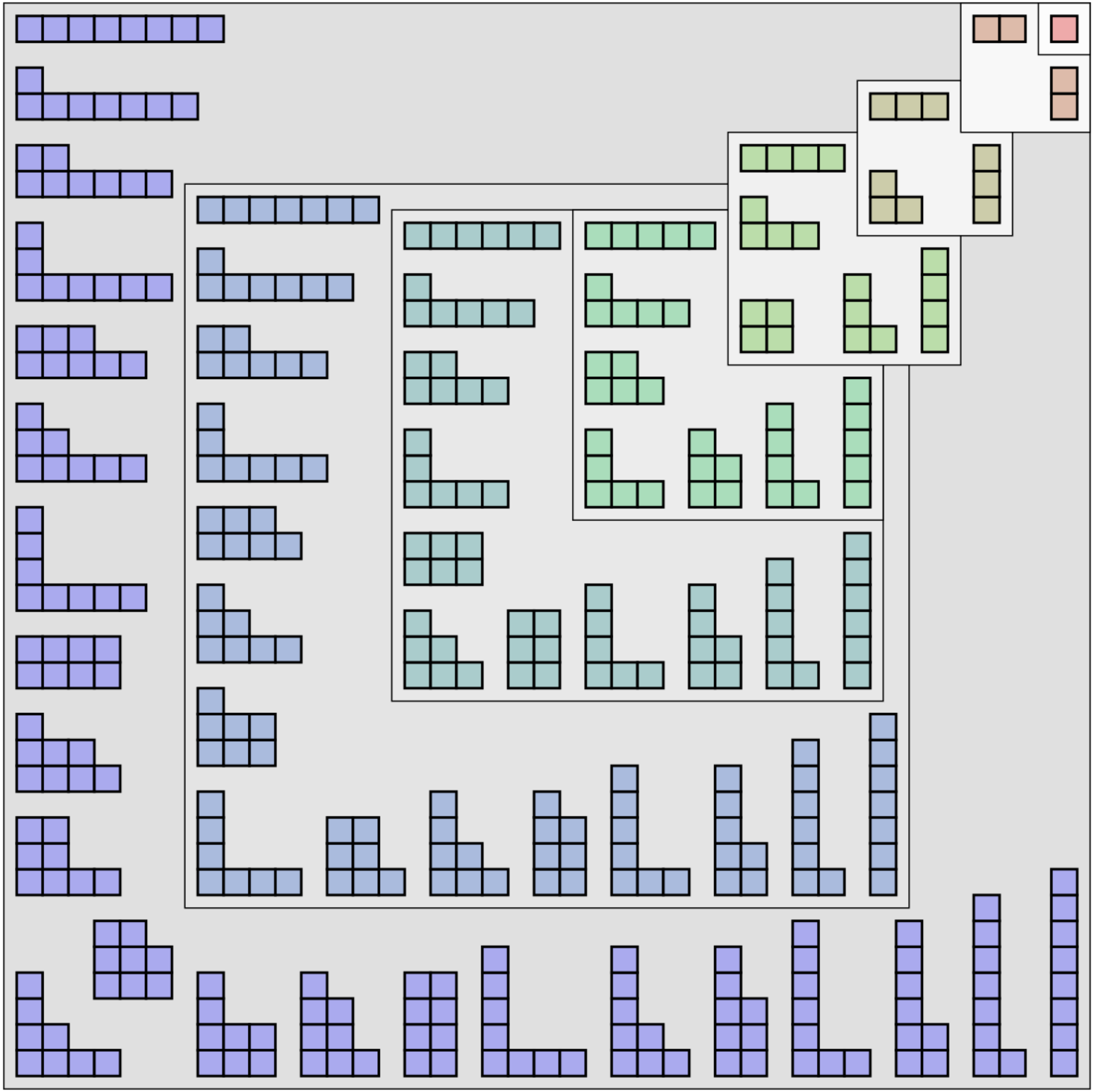
$$3 = 2^1 + 2^0\,, \qquad 5 = 2^2 + 2^0\,, \qquad 8 = 2^3$$

Conjugated Partitions

Flip Young diagrams over the diagonal



Problem: how many self-conjugated (symmetric) partitions are there?



p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

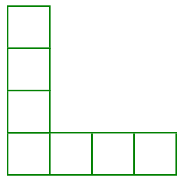
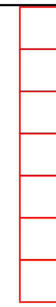
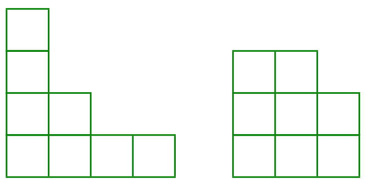
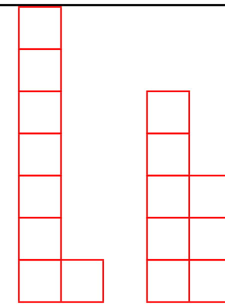
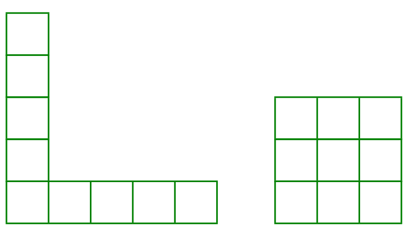
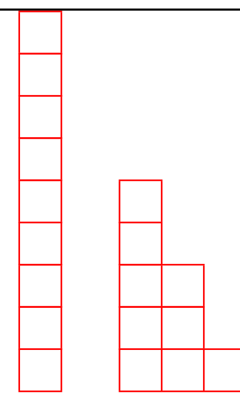
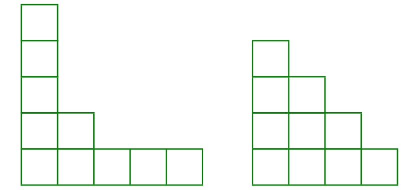
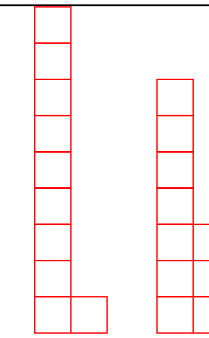
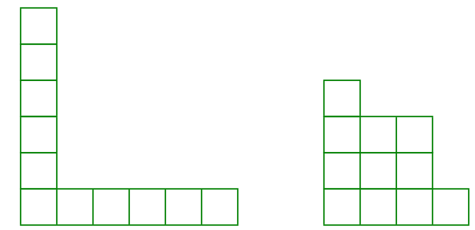
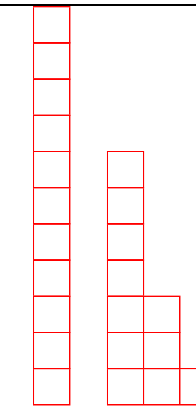
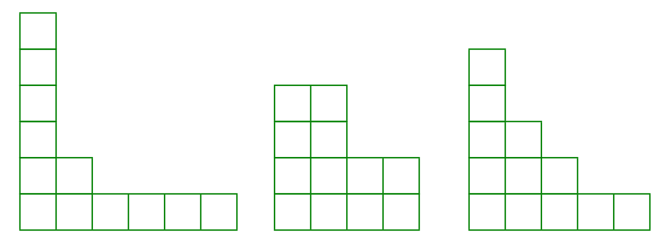
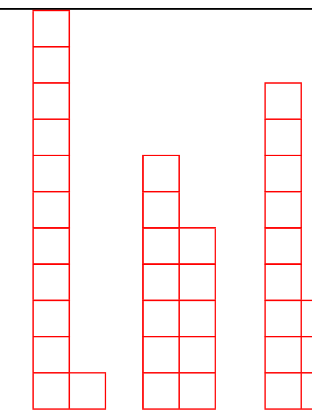
p(11)

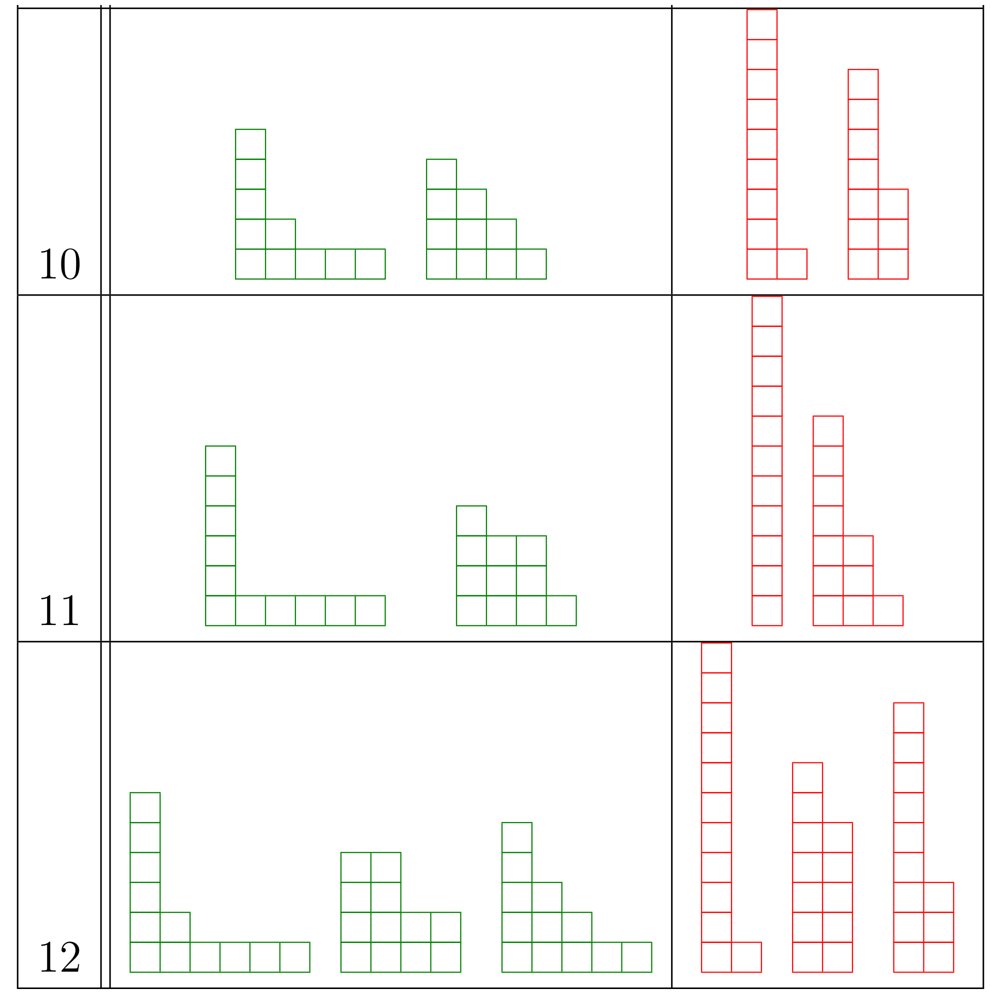
{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

Odd, Distinct, Symmetric

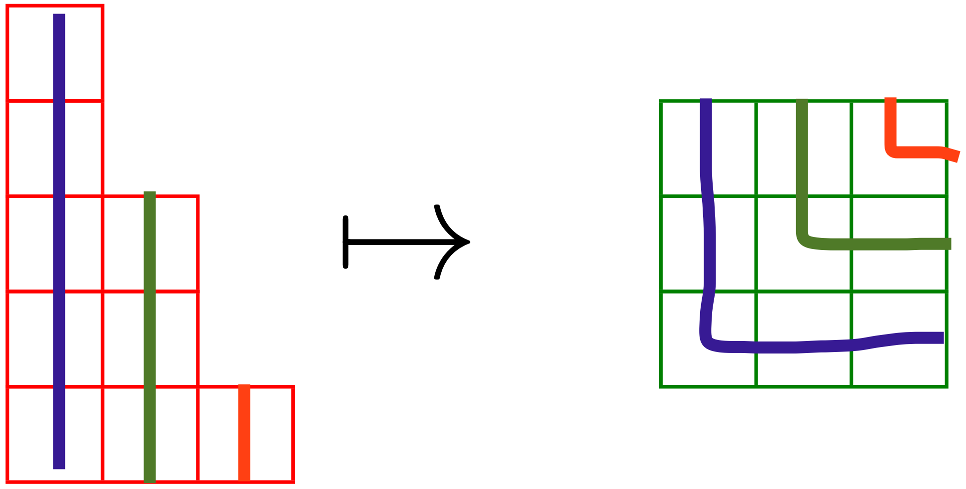
n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Problem: Show that Symmetric = Odd&Distinct

n	Symmetric	Odd&Distinct
7		
8		
9		
10		
11		
12		

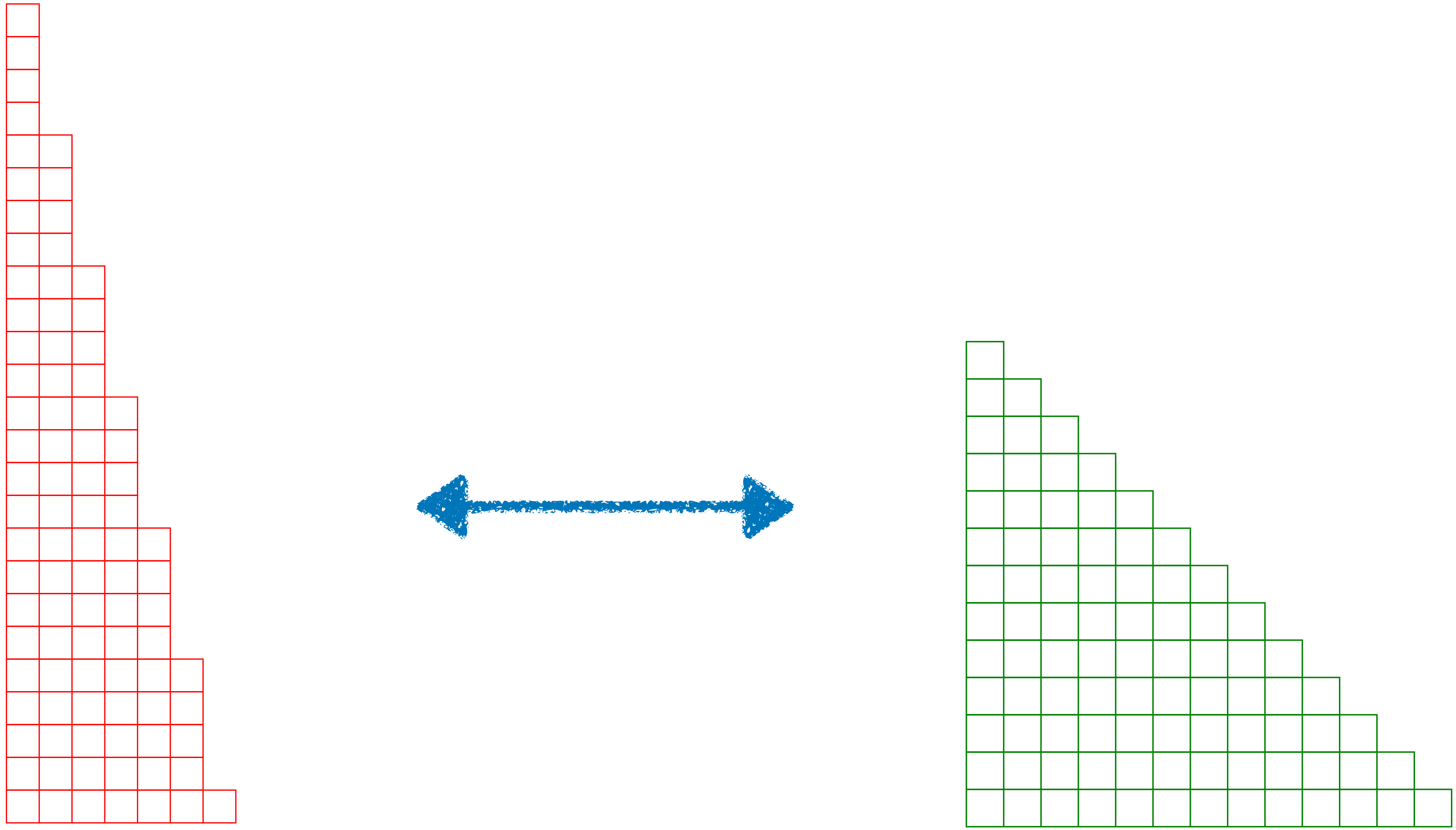

10
11
12

Matching



Odd&Distinct vs Symmetric

Triangular numbers

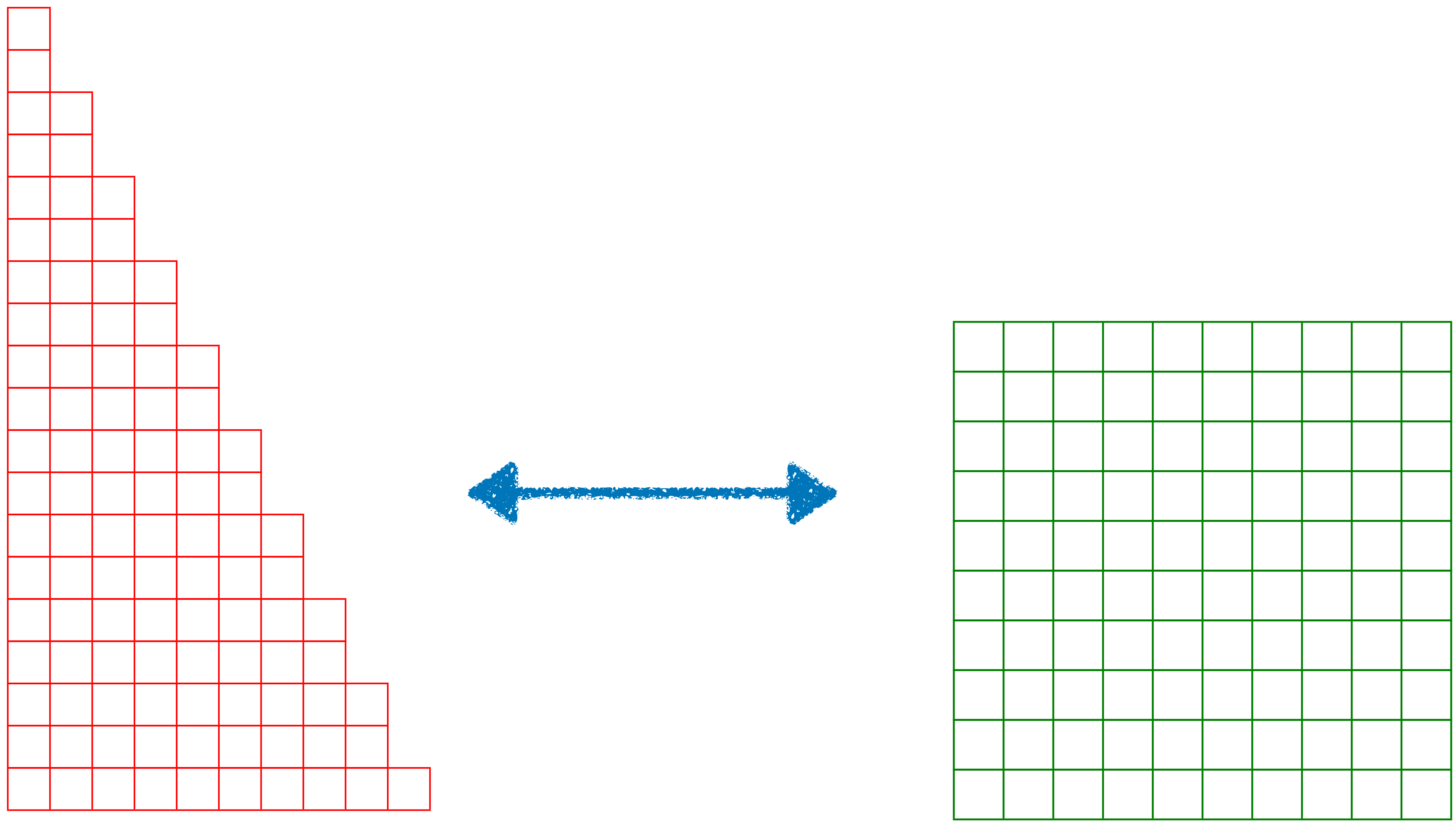


$\{25,21,17,13,9,5,1\}$

$\{13,12,11,10,9,8,7,6,5,4,3,2,1$

Odd&Distinct vs Symmetric

Square numbers



$\{19, 17, 15, 13, 11, 9, 7, 5, 3, 1\}$

$\{10, 10, 10, 10, 10, 10, 10, 10, 10, 10\}$

Computing Sums

Compute the sum. How many terms are in the sum?

- $1 + 3 + 5 + \ldots + 117 + 119 = ?$
- $1 + 5 + 9 + 13 + 17 + 21 + 25 = ?$
- $1 + 5 + 9 + 13 + \ldots + 81 = ?$

Divisibility by 3

Consider partitions of n whose parts are not divisible by **3**

Compare those with partitions of n in which **each part** is not repeated **3 or more times**

Divisibility by 4

Consider partitions of n whose parts are not divisible by **4**

Compare those with partitions of n in which **each part** is not repeated **4 or more times**

Divisibility by n

Consider partitions of n whose parts are not divisible by **n**

Compare those with partitions of n in which **each part** is not repeated **n** **or more times**

Restricted Partitions

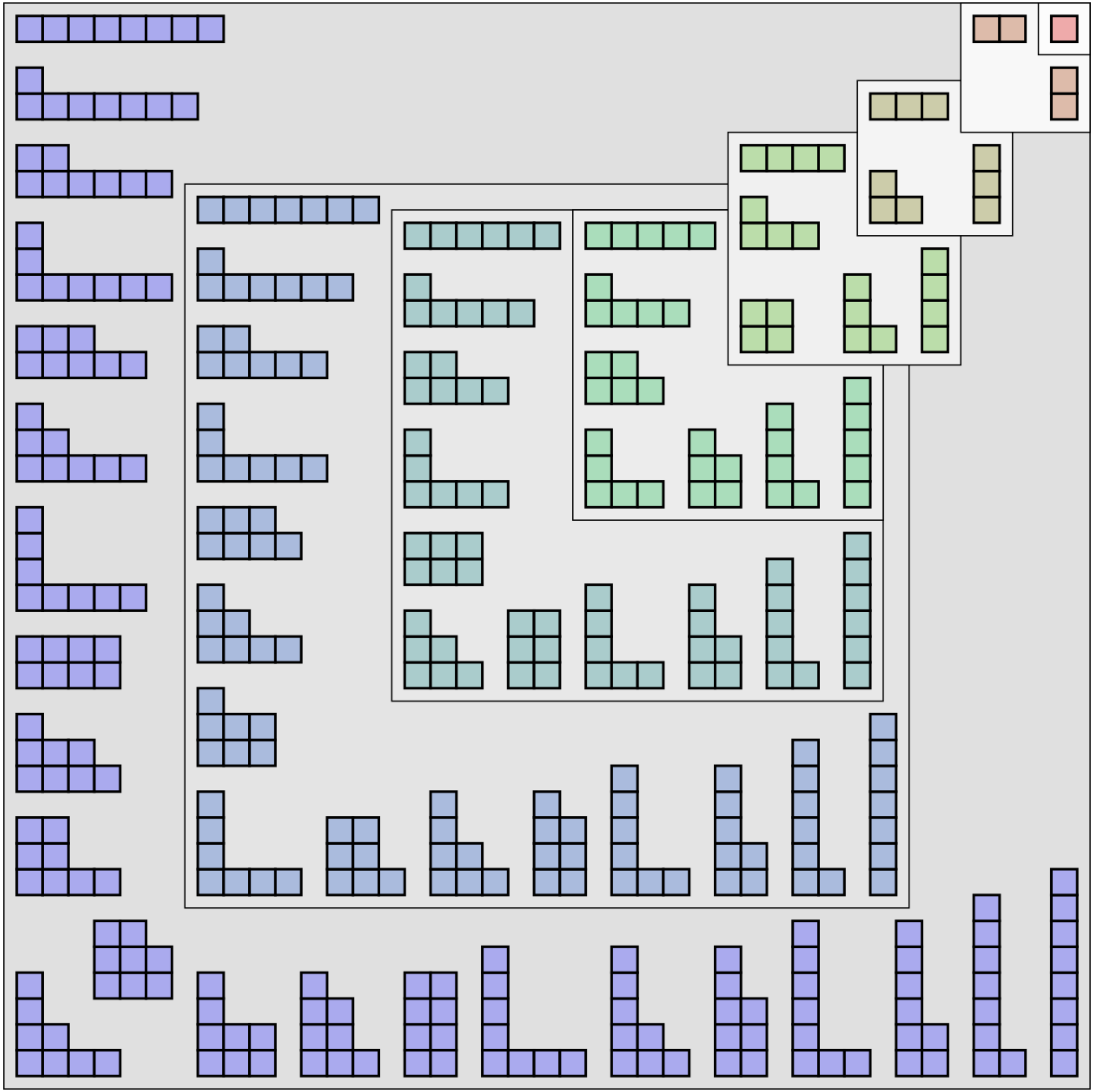
Restricted Partitions. Let us now look at integer partitions of n which have *exactly* 4 parts. From the list of partitions of 7

$$\{\{7\},\{6,1\},\{5,2\},\{5,1,1\},\{4,3\},\{4,2,1\},\{4,1,1,1\},\{3,3,1\},\{3,2,2\},$$
$$\{3,2,1,1\},\{3,1,1,1,1\},\{2,2,2,1\},\{2,2,1,1,1\},\{2,1,1,1,1,1\},\{1,1,1,1,1,1,1\}\}$$

Only these qualify

$$\{\{4,1,1,1\},\{3,2,1,1\},\{2,2,2,1\}\}$$

Make lists for $n = 4, 5, 6$ with all possible restricted parts, i.e. all partitions of 4 with 1, with 2, with 3 parts, etc., same for $n = 5$ and $n = 6$. Count each number of partitions, call it $p_k(n)$. Do you see any pattern?



	1	2	3	4	5	6	7	8
p(1)	1							
p(2)	1	1						
p(3)	1	1	1					
p(4)	1	2	1	1				
p(5)	1	2	2	1	1			
p(6)	1	3	3	2	1	1		
p(7)	1	3	4	3	2	1	1	
p(8)	1	4	5	5	3	2	1	1

p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2}, {5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1}, {4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1}, {3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1}, {2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(11)

{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

	1	2	3	4	5	6	7	8	9	10	11	12
p(1)	1											
p(2)	1	1										
p(3)	1	1	1									
p(4)	1	2	1	1								
p(5)	1	2	2	1	1							
p(6)	1	3	3	2	1	1						
p(7)	1	3	4	3	2	1	1					
p(8)	1	4	5	5	3	2	1	1				
p(9)	1	4	7	6	5	3	2	1	1			
p(10)	1	5	8	9	7	5	3	2	1	1		
p(11)												
p(12)												

Partitions of n

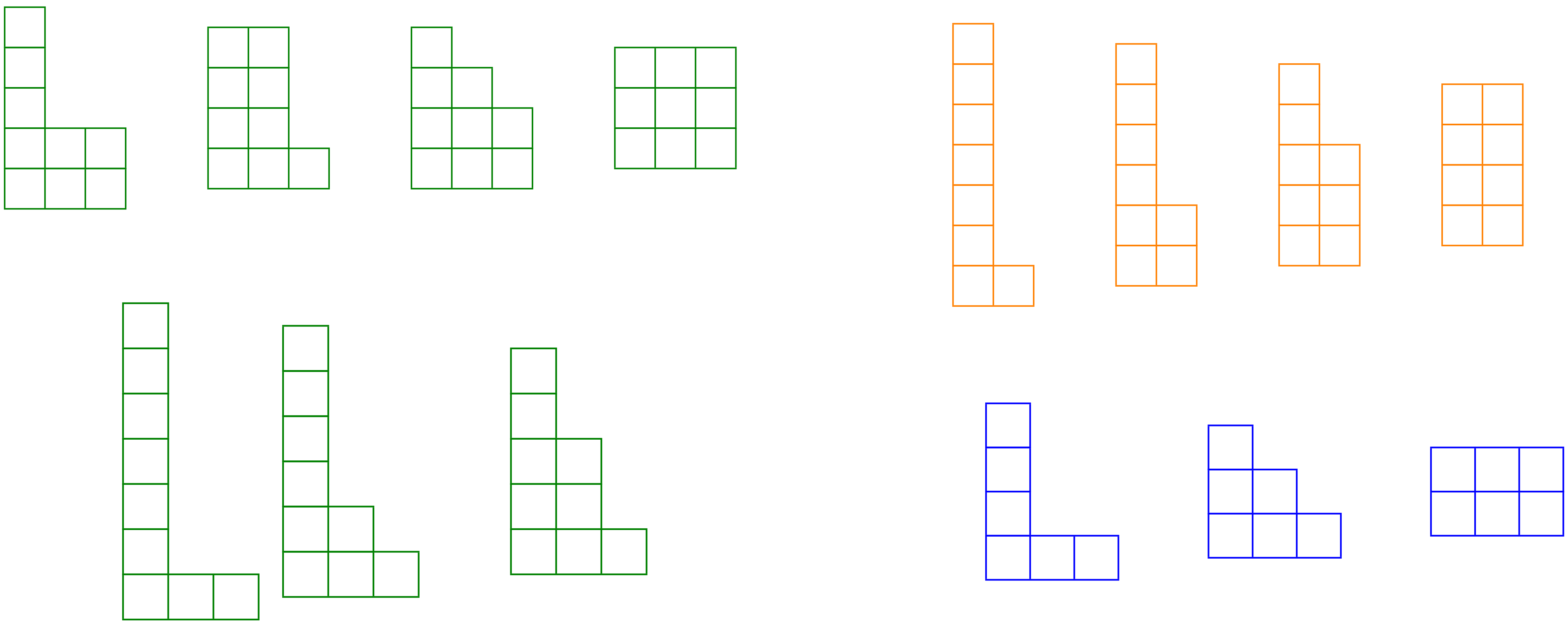
n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Recurrent Formula

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$

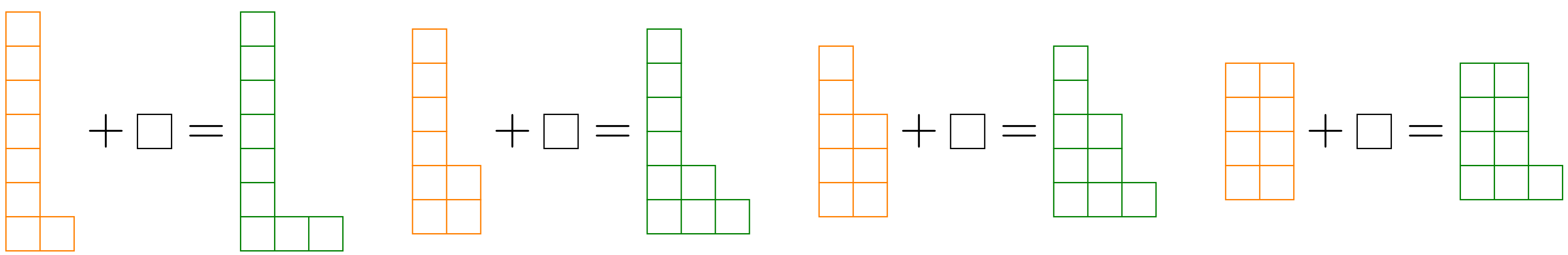
$$p_3(9) = p_2(8) + p_3(6)$$

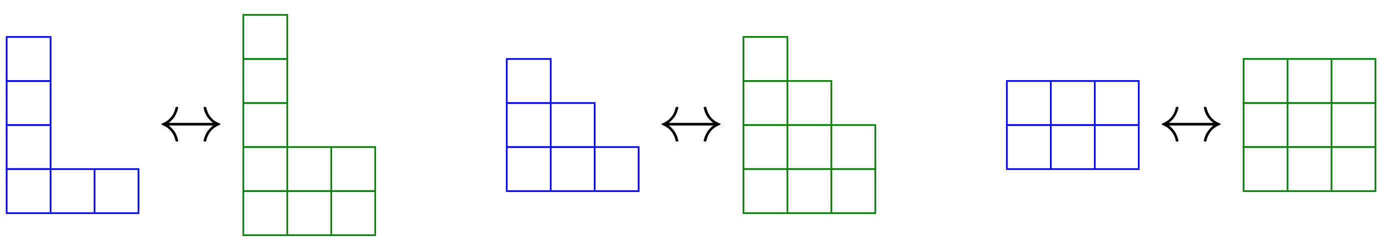
$$7 \qquad 4 \qquad 3$$



Matching

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$$





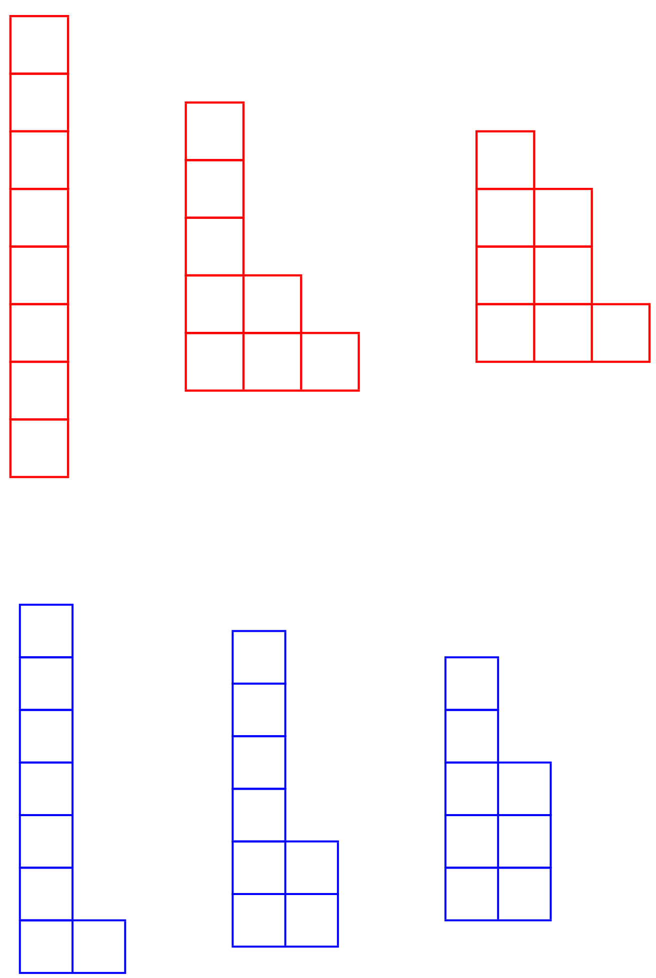
Understanding p(n)

n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

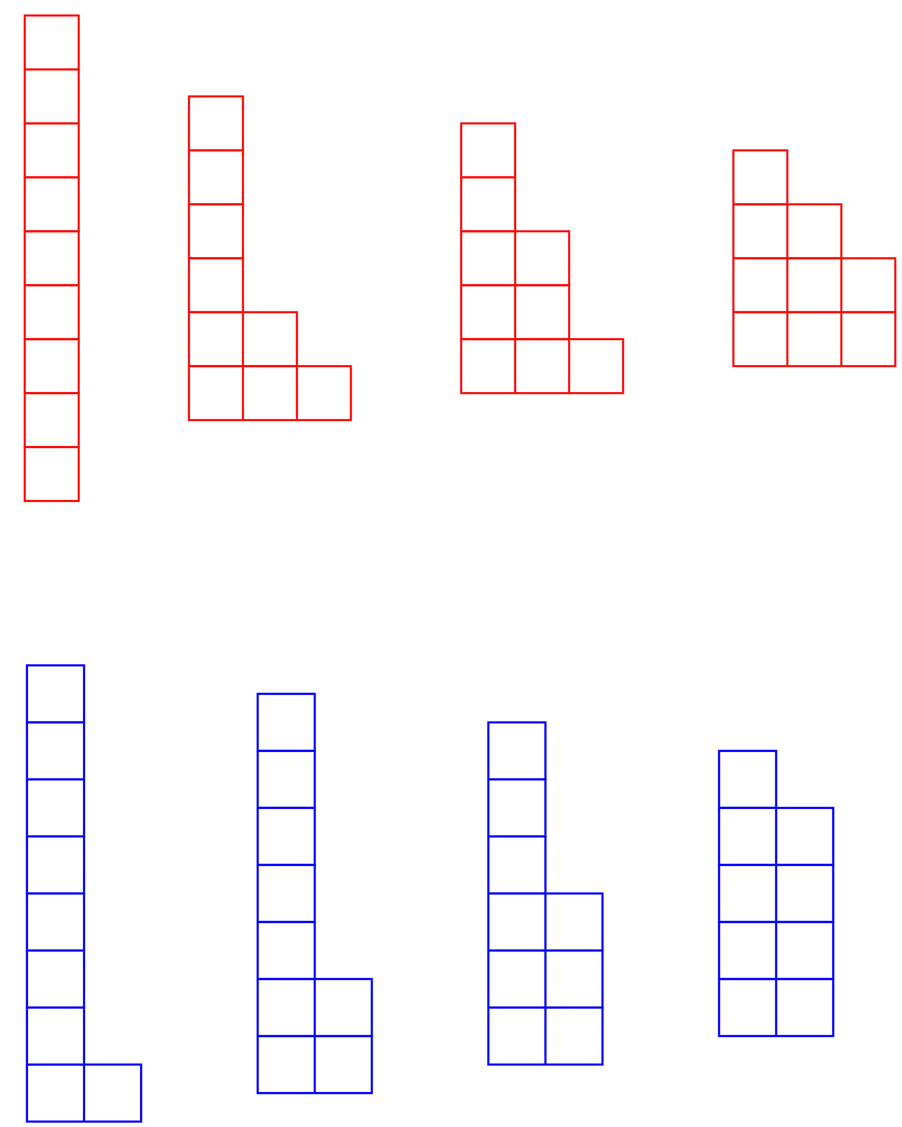
Odd and Even Number of parts

For each **n** count *distinct partitions* with **odd** and **even** *number of parts*

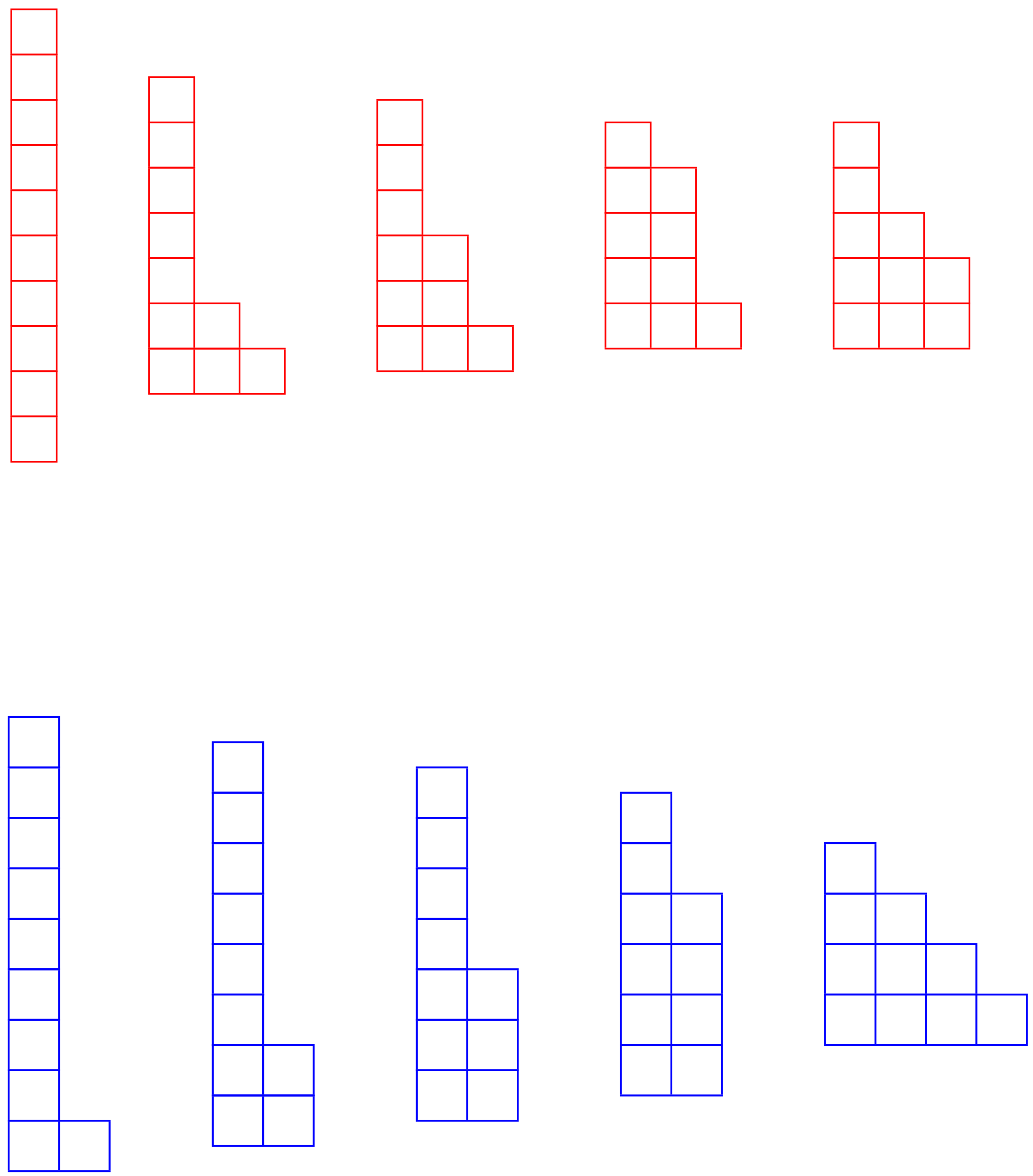
n=8



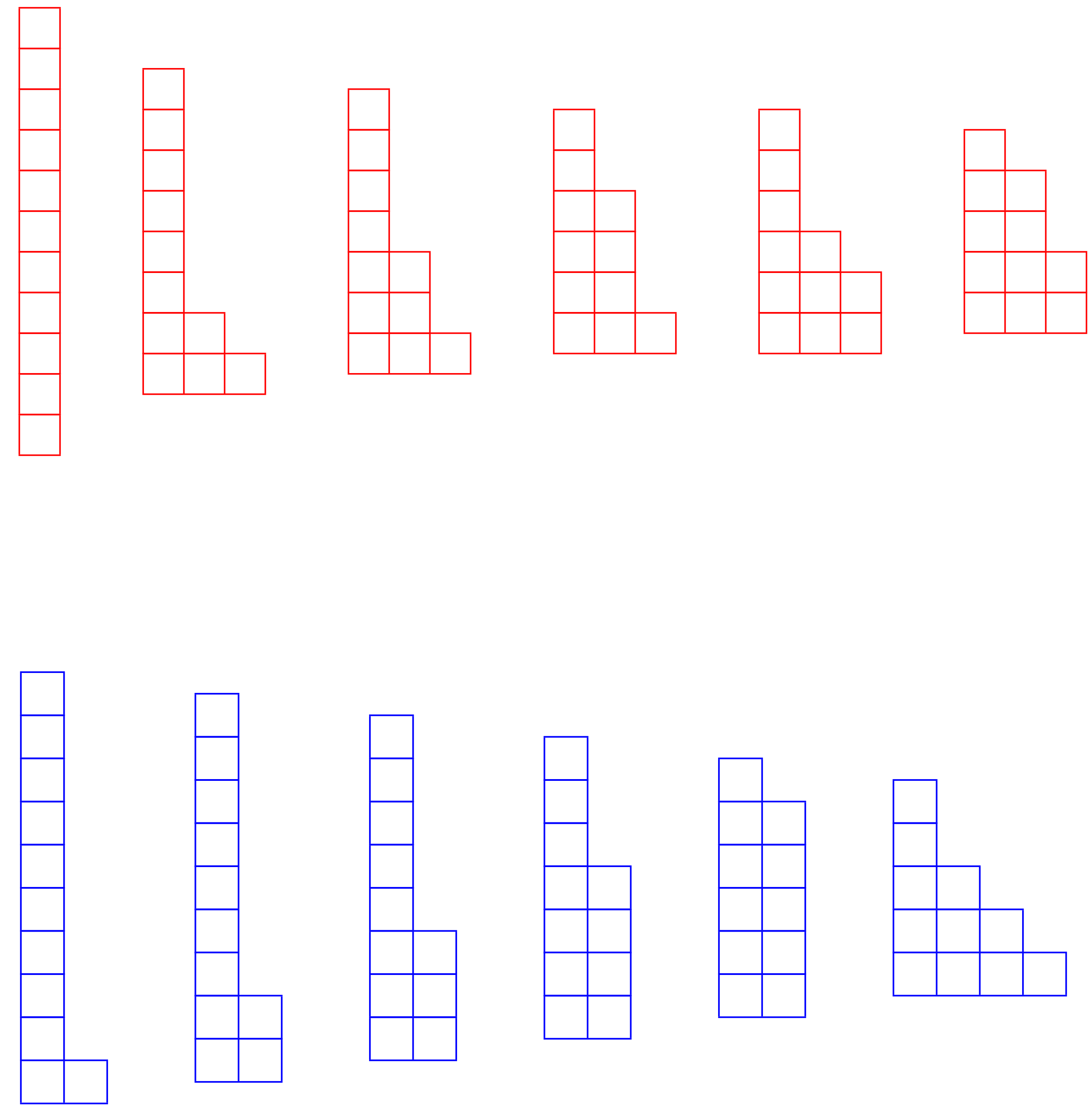
n=9



n=10

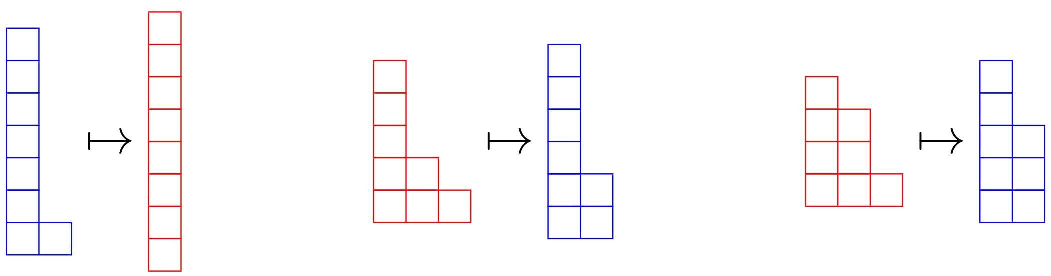


n=11

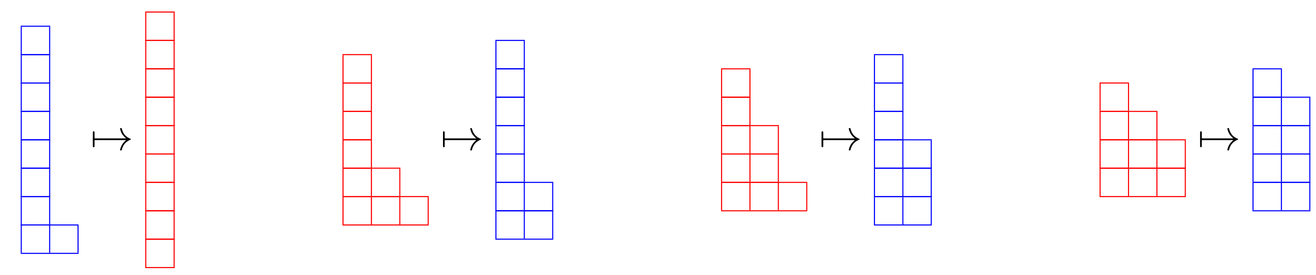


Matching

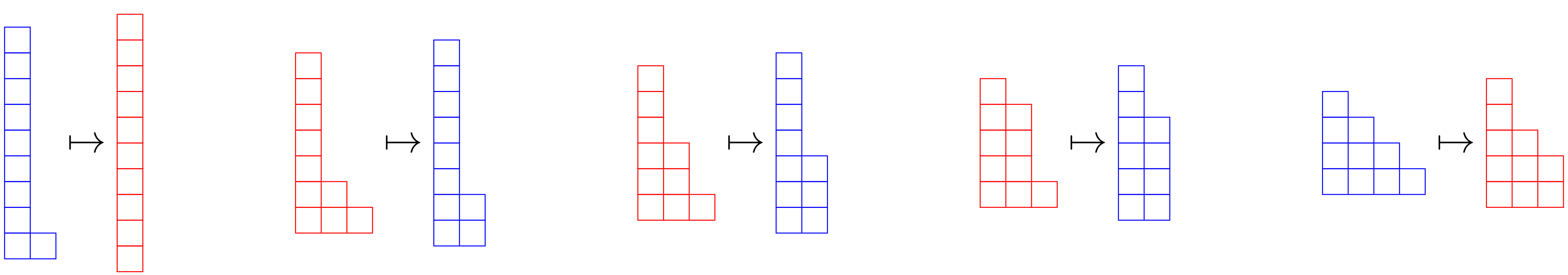
n=8



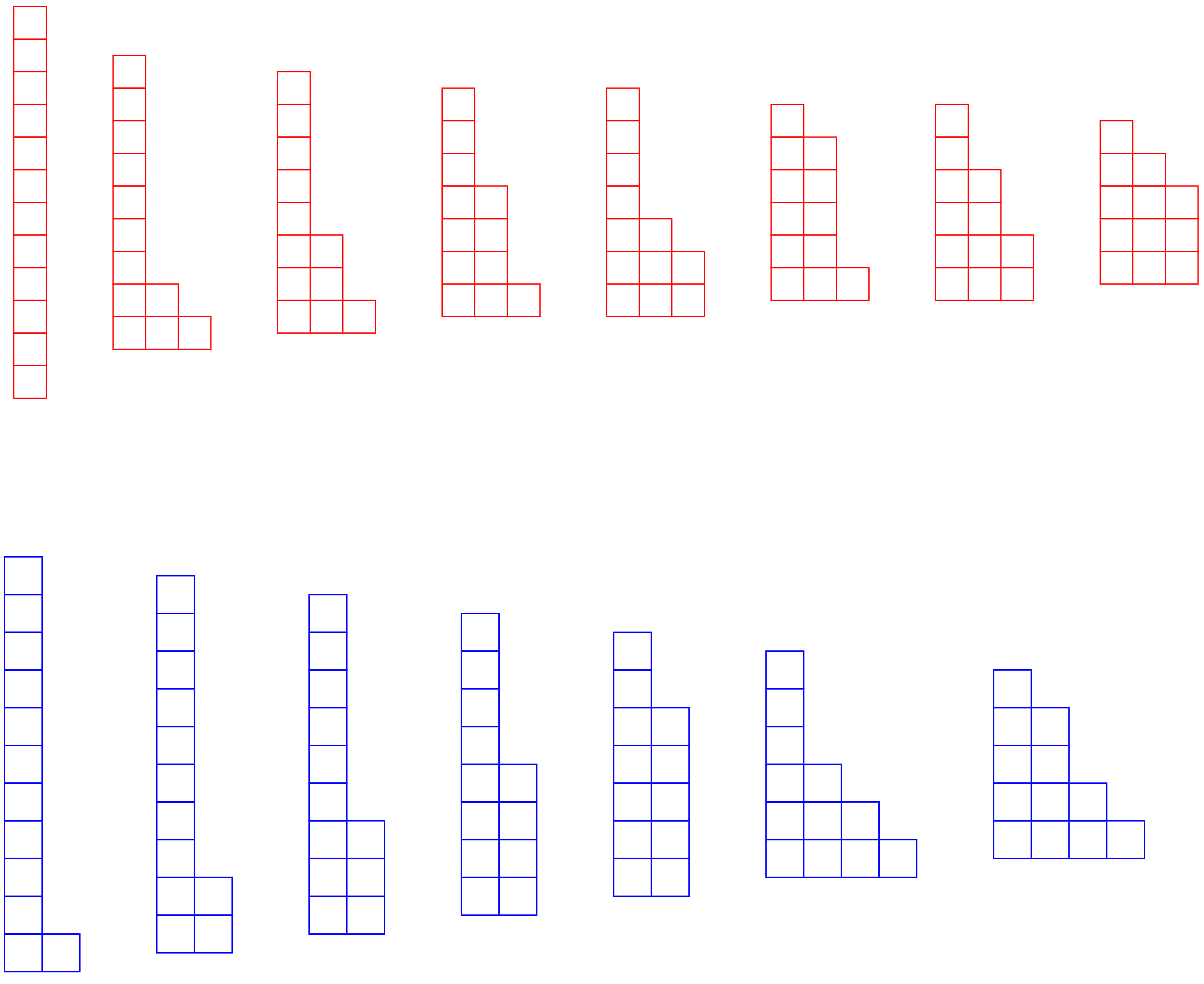
n=9



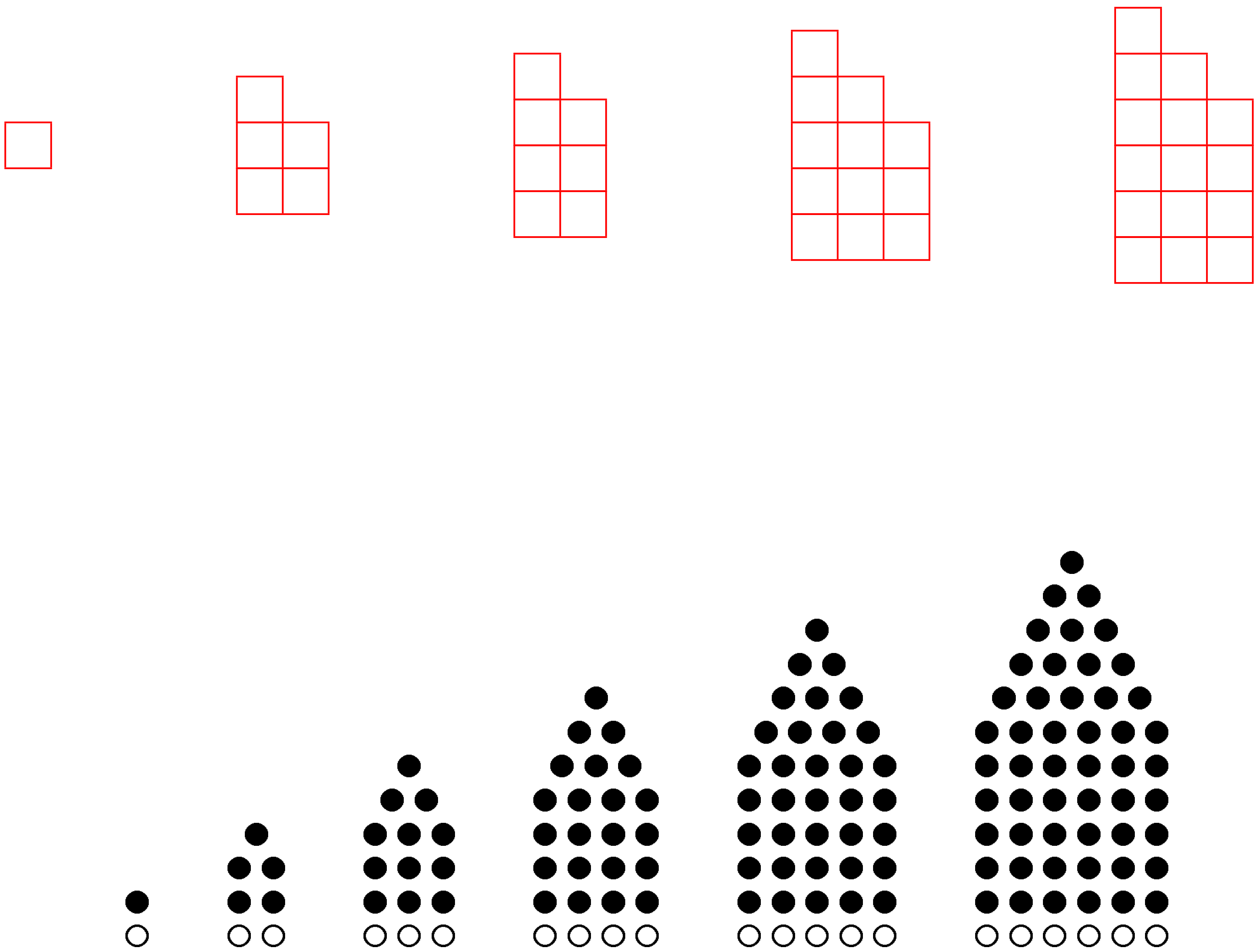
n=10



n=12



Pentagonal Numbers



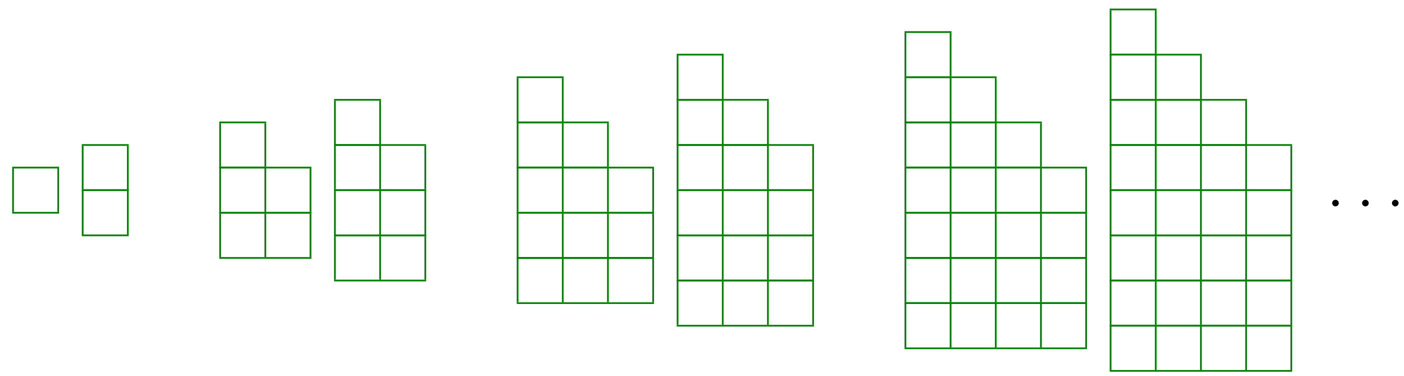
$1,2 \quad 5,7 \quad 12,15 \quad 22,26 \quad 35,40 \quad 51,57$

Pentagonal Numbers

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
T_n	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
S_n	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64
P_n	100	77	57	40	26	15	7	2	0	1	5	12	22	35	51	70	92

1,2, 5,7, 12,15, 22,26, 35,40, 51,57, 70,77, 92,100,

Pentagonal Numbers

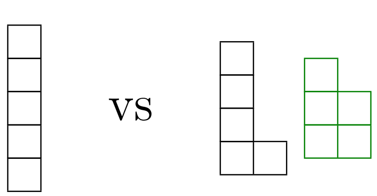


Show that $P_n = \frac{n(3n-1)}{2}$

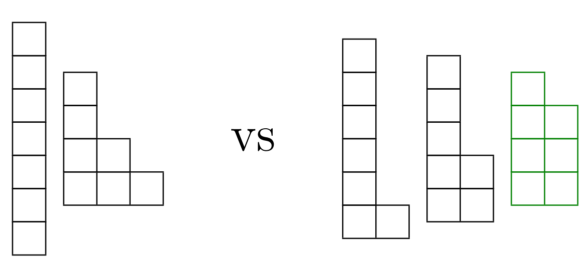
$(P_1, P_{-1}) \quad (P_2, P_{-2}) \quad (P_3, P_{-3}) \quad (P_4, P_{-4}) \quad \ldots$

Theorem: If n is not a pentagonal number, then the number of even distinct partitions of n, call it $q_e(n)$ equals the number of odd distinct partitions of n, call it $q_o(n)$. So $q_e(n) = q_o(n)$ and so the total number of distinct partitions of n, call it $q(n)$ is $q(n) = 2q_o(n)$ which is even.

If n is a pentagonal number, say $n = P_j$, then $q_e(n) = q_o(n) + (-1)^j$ and so $q(n) = 2q_o(n) + (-1)^j$ which is odd.



n=5



n=7

Leonhard Euler multiplied

$$(1-z)(1-z^2) = 1\times(1-z^2) - z\times(1-z^2) = 1\times1 - 1\times z^2 - z\times1 - z\times(-z^2) = 1-z-z^2+z^3$$

$$(1-z)(1-z^2)(1-z^3) = (1-z-z^2+z^3)(1-z^3) = (1-z-z^2+z^3-z^3)1+(1-z-z^2+z^3)(-z^3)$$
$$= 1 - z - z^2 + z^3 - z^3 - z^3 + z^4 + z^5 - z^6 = 1 - z - z^2 + z^4 + z^5 - z^6$$

$$\phi(z) = \prod_{k=1}^{\infty}(1-z^k) = (1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\ldots$$

Euler found that…

$$
\begin{array}{rlllllllllll}
\varphi_1(x) = 1 - x & & & & & & & & & & \\
\varphi_2(x) = 1 - x & -x^2 & +x^3 & & & & & & & & \\
\varphi_3(x) = 1 - x & -x^2 & & +x^4 & +x^5 & -x^6 & & & & & \\
\varphi_4(x) = 1 - x & -x^2 & & & +2x^5 & & & -x^8 & -x^9 & +x^{10} & \\
\varphi_5(x) = 1 - x & -x^2 & & & +x^5 & +x^6 & +x^7 & -x^8 & -x^9 & -x^{10} & \ldots \\
\varphi_6(x) = 1 - x & -x^2 & & & +x^5 & & +2x^7 & & -x^9 & -x^{10} & \ldots \\
\varphi_7(x) = 1 - x & -x^2 & & & +x^5 & & +x^7 & +x^8 & & -x^{10} & \ldots \\
\varphi_8(x) = 1 - x & -x^2 & & & +x^5 & & +x^7 & & +x^9 & & \ldots \\
\varphi_9(x) = 1 - x & -x^2 & & & +x^5 & & +x^7 & & & +x^{10} & \ldots \\
\varphi_{10}(x) = 1 - x & -x^2 & & & +x^5 & & +x^7 & & & & \ldots
\end{array}
$$

What Euler wrote about his identity. "In considering the partitions of numbers, I examined, a long time ago, the expression

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)\ldots,$$

in which the product is assumed to be infinite. In order to see what kind of series will result, I multiplied actually a great number of factors and found

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\ldots$$

The exponents of x are the same which enter into the above formula; [1] also the signs $+$ and $-$ arise twice in succession. It suffices to undertake this multiplication and to continue it as far as it is deemed proper to become convinced of the truth of these series. Yet I have no other evidence for this, except a long induction which I have carried out so far that I cannot in any way doubt the law governing the formation of these terms and their exponents. I have long searched in vain for a rigorous demonstration of the equation between the series and the above infinite product $(1-x)(1-x^2)(1-x^3)\ldots$, and I proposed the same question to some of my friends with whose ability in these matters I am familiar, but all have agreed with me on the truth of this transformation of the product into a series, without being able to unearth any clue of a demonstration."

Infinite Product

$$\phi(z) = \prod_{k=1}^{\infty}(1 - z^k) = (1 - z)(1 - z^2)(1 - z^3)(1 - z^4) \cdot \ldots$$

$$\phi(z) = 1 - z^{1} - z^{2} + z^{5} + z^{7} - z^{12} - z^{15} + z^{22} + z^{26} - \ldots$$

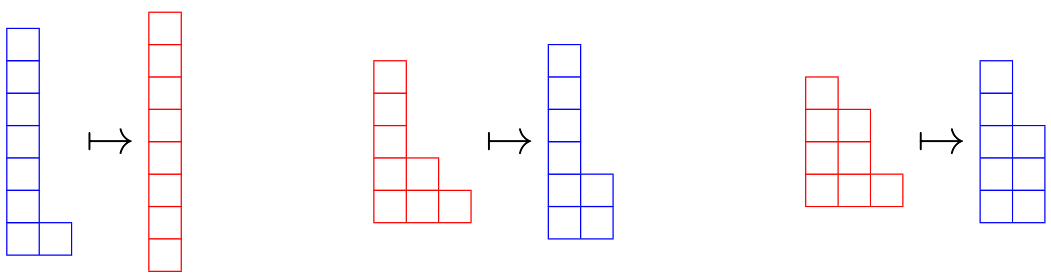
Pentagonal numbers!!

$$\phi(z) = 1 + \sum_{n=1}^{\infty}(-1)^n \left(z^{\frac{3n^2-n}{2}} + z^{\frac{3n^2+n}{2}}\right) = \sum_{n=-\infty}^{\infty}(-1)^n z^{\frac{3n^2+n}{2}}$$

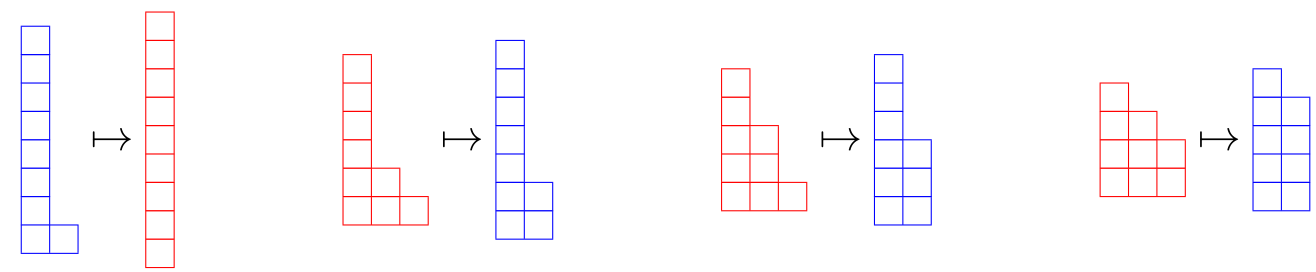
Claim: coefficient for z^n equals: #distinct even partitions of n -#distinct odd partitions of n

Theorem If n is not a pentagonal number, then $q_e(n) = q_o(n)$ and so $q(n) = 2q_o(n)$ which is even. If n is a pentagonal number, say $n = P_j$, then $q_e(n) = q_o(n) + (-1)^j$ and so $q(n) = 2q_o(n) + (-1)^j$ which is odd.

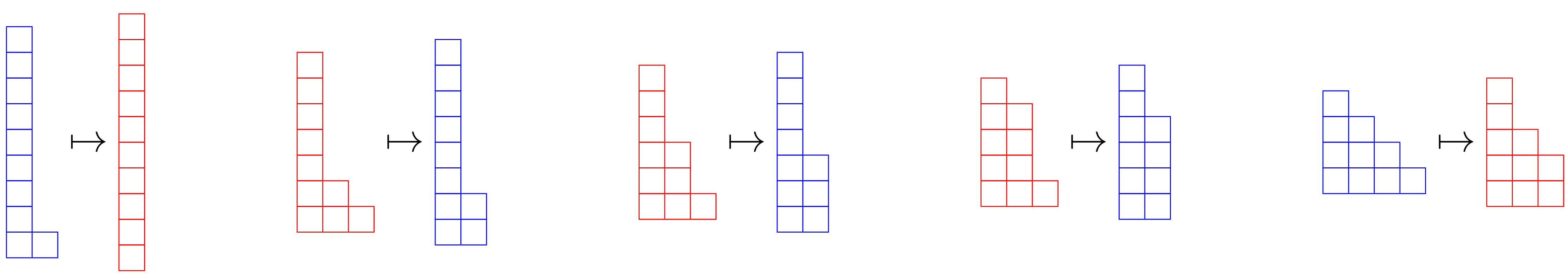
n=8



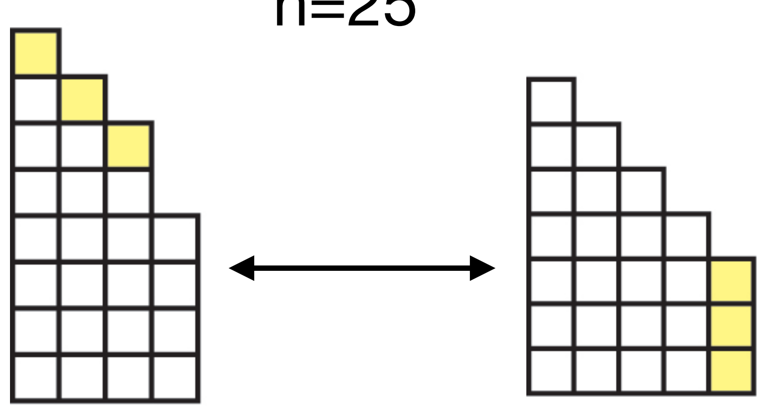
n=9



n=10



n=25



So the rule is

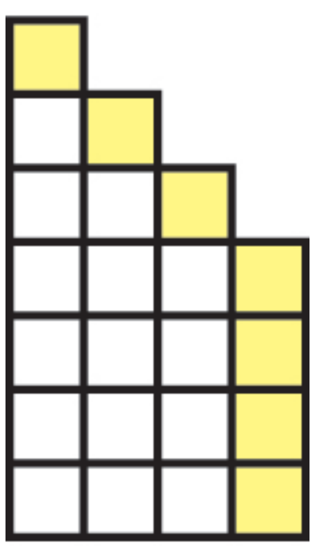
If the number of blocks in the rightmost column is smaller or equal than the number of blocks in the topmost diagonal then move them from that column to the top diagonal

Otherwise move the top diagonal to the rightmost column

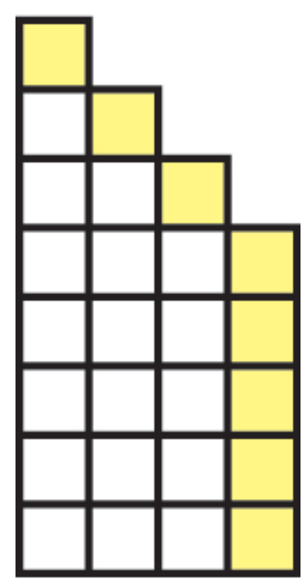
Problem with pentagonal numbers

$$q_e(n) = q_o(n) + (-1)^j.$$

$P_4 = 22$



$P_{-4} = 26$



Generating Function

$$\mathbf{p}(z) = \frac{1}{\phi(z)} = \prod_{k=1}^{\infty} \frac{1}{1-z^k} = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\ldots}$$

Recall that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \ldots$$

so

$$\frac{1}{1-z^k} = 1 + z^k + z^{2k} + z^{3k} + z^{4k} + \ldots$$

Generating function

$$\mathbf{p}(z) = \frac{1}{\phi(z)} = \prod_{k=1}^{\infty} \frac{1}{1-z^k} = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)\cdot\ldots}$$

$$\mathbf{p}(z) = (1 + z + z^2 + z^3 + \ldots)(1 + z^2 + z^4 + z^6 + \ldots)(1 + z^3 + z^6 + z^9 + \ldots)\cdot\ldots$$

$$\mathbf{p}(z) = \prod_{k=1}^{\infty}(1 + z^k + z^{2k} + z^{3k} + \ldots)$$

Generating Function

$$\mathbf{p}(z) = (1 + z + z^2 + z^3 + \dots)(1 + z^2 + z^4 + z^6 + \dots)(1 + z^3 + z^6 + z^9 + \dots) \cdot \dots$$

Now we need to collect terms in front of each power of z. Each term z^n in the resulting product will look like

$$z^{k_1} \cdot z^{2k_2} \cdot z^{3k_3} \cdot \dots \cdot z^{mk_m} = z^{k_1 + 2k_2 + 3k_3 + \dots + mk_m}$$

We want to count the number of such products with $k_1 + 2k_2 + 3k_3 + \dots + mk_m = n$

$$n = k_1 + 2k_2 + \dots + mk_m = \underbrace{1 + \dots + 1}_{k_1} + \underbrace{2 + \dots + 2}_{k_2} + \dots + \underbrace{m + \dots + m}_{k_m}$$

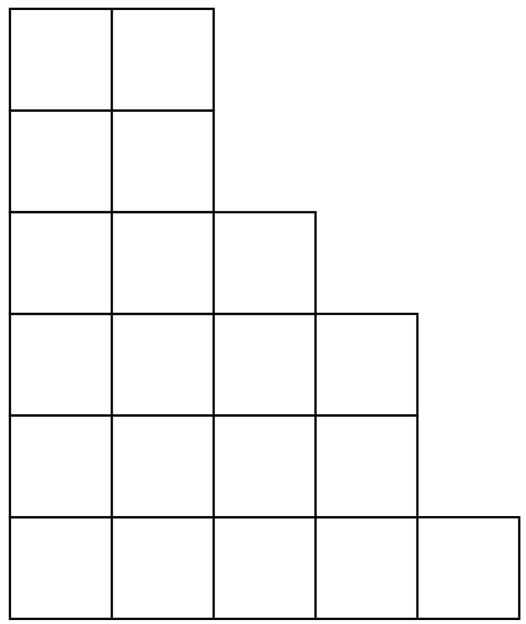
which is the number of partitions of n $\{\underbrace{m, \dots, m}_{k_m}, \underbrace{m-1, \dots, m-1}_{k_{m-1}}, \dots, \underbrace{2, \dots, 2}_{k_2}, \underbrace{1, \dots, 1}_{k_1}\}$

$$\mathbf{p}(z) = 1 + p(1)z + p(2)z^2 + p(3)z^3 + p(4)z^4 +$$

Example

consider partition $\{6, 6, 4, 3, 1\}$ of 20

$m = 6$ and $k_6 = 2, k_5 = 0, k_4 = 1, k_3 = 1, k_2 = 0, k_1 = 1.$



$$\mathbf{p}(z) = 1 + p(1)z + p(2)z^2 + p(3)z^3 + p(4)z^4 +$$

$$\mathbf{p}(z) = 1+z+2z^2+3z^3+5z^4+7z^5+11z^6+15z^7+22z^8+30z^9+42z^{10}+56z^{11}+77z^{12}+101z^{13}+\ldots$$

Recurrent Formula

Now use the Euler identity:

$$\varphi(x)\mathbf{p}(x) = 1.$$

$$(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} \dots)(1 + \mathbf{p}(1)x + \mathbf{p}(2)x^2 + \mathbf{p}(3)x^3 + \dots) = 1,$$

$$\begin{array}{ll}
\mathbf{p}(1) & -1 = 0 \\
\mathbf{p}(2) & -\mathbf{p}(1) - 1 = 0 \\
\mathbf{p}(3) & -\mathbf{p}(2) - \mathbf{p}(1) = 0 \\
\mathbf{p}(4) & -\mathbf{p}(3) - \mathbf{p}(2) = 0 \\
\mathbf{p}(5) & -\mathbf{p}(4) - \mathbf{p}(3) + 1 = 0 \\
\mathbf{p}(6) & -\mathbf{p}(5) - \mathbf{p}(4) + \mathbf{p}(1) = 0 \\
\mathbf{p}(7) & -\mathbf{p}(6) - \mathbf{p}(5) + \mathbf{p}(2) + 1 = 0 \\
\mathbf{p}(8) & -\mathbf{p}(7) - \mathbf{p}(6) + \mathbf{p}(3) + \mathbf{p}(1) = 0
\end{array}$$

"Read Euler, read Euler, he is the master of us all."

P. Laplace

{1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627}

$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-p(n-22)-p(n-26)+\dots$$

$$\begin{aligned} 5 &= 3 + 2\,, \\ 11 &= 7 + 5 - 1\,, \\ 15 &= 11 + 7 - 2 - 1\,, \\ 56 &= 42 + 30 - 11 - 5\,, \\ 77 &= 56 + 42 - 15 - 7 + 1\,, \\ 101 &= 77 + 56 - 22 - 11 + 1\,. \end{aligned}$$

Euler's ruller

	→
	+
	+
	−
	−
	+
	+

→	**1**
+	$1 = \mathbf{p}(0)$

→	**2**
+	$1 = \mathbf{p}(1)$
+	$1 = \mathbf{p}(0)$

→	**3**
+	$2 = \mathbf{p}(2)$
+	$1 = \mathbf{p}(1)$
	$1 = \mathbf{p}(0)$

→	**5**
+	$3 = \mathbf{p}(3)$
+	$2 = \mathbf{p}(2)$
	$1 = \mathbf{p}(1)$
	$1 = \mathbf{p}(0)$

→	**7**
+	$5 = \mathbf{p}(4)$
+	$3 = \mathbf{p}(3)$
	$2 = \mathbf{p}(2)$
	$1 = \mathbf{p}(1)$
−	$1 = \mathbf{p}(0)$

→	**11**
+	$7 = \mathbf{p}(5)$
+	$5 = \mathbf{p}(4)$
	$3 = \mathbf{p}(3)$
	$2 = \mathbf{p}(2)$
−	$1 = \mathbf{p}(1)$
	$1 = \mathbf{p}(0)$

→	**15**
+	$11 = \mathbf{p}(6)$
+	$7 = \mathbf{p}(5)$
	$5 = \mathbf{p}(4)$
	$3 = \mathbf{p}(3)$
−	$2 = \mathbf{p}(2)$
	$1 = \mathbf{p}(1)$
−	$1 = \mathbf{p}(0)$

→	**22**
+	$15 = \mathbf{p}(7)$
+	$11 = \mathbf{p}(6)$
	$7 = \mathbf{p}(5)$
	$5 = \mathbf{p}(4)$
−	$3 = \mathbf{p}(3)$
	$2 = \mathbf{p}(2)$
−	$1 = \mathbf{p}(1)$
	$1 = \mathbf{p}(0)$

Partitions

n	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	2	3	5	7	11	15	22	30	42	56	77
# odd	1	1	2	2	3	4	5	6	8	10	12	15
# distinct	1	1	2	2	3	4	5	6	8	10	12	15
# symmetric	1	0	1	1	1	1	1	2	2	2	2	3
# odd&distinct	1	0	1	1	1	1	1	2	2	2	2	3

Homework

- Write generating functions for distinct and odd partitions.
  Prove that distinct = odd using generating functions

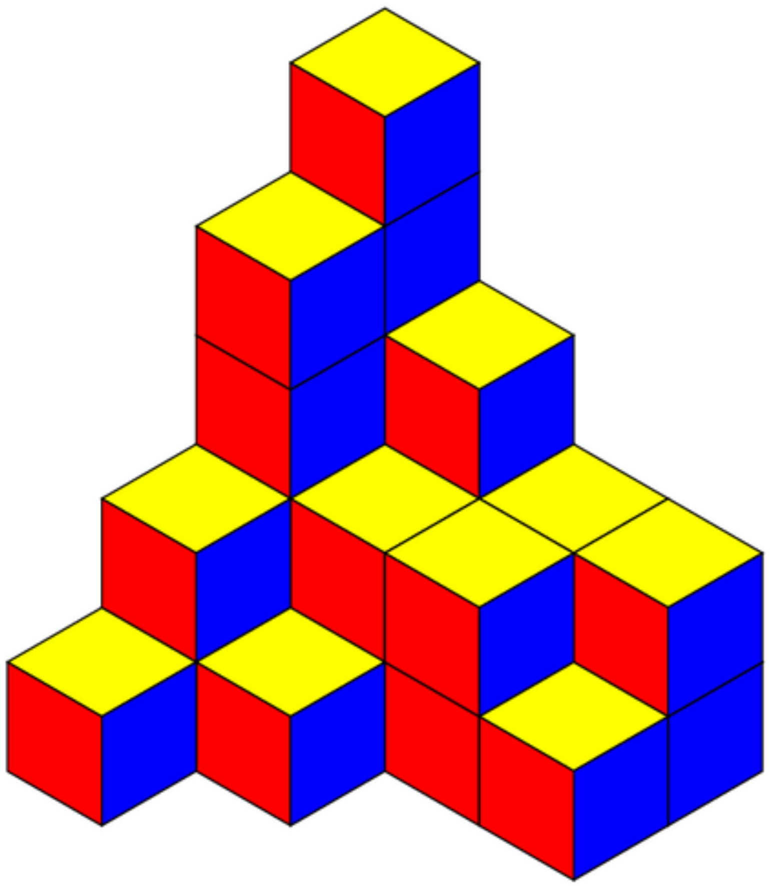
Homework

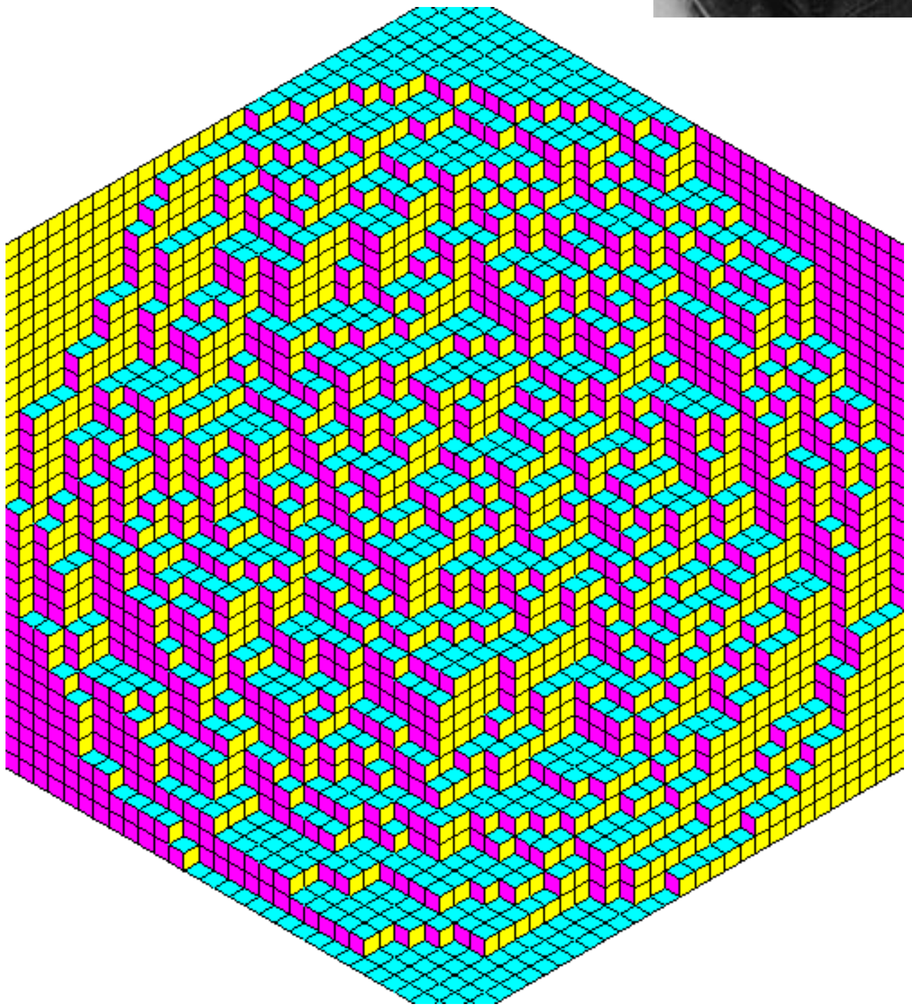
- Prove that $$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \text{primes}} \frac{p^s}{p^s - 1}$$

Homework

- Prove that $\mathbf{p}(n) < F_n$ where F_n is the n-th Fibonacci number

Plane Partitions





$$\sum_{n=0}^{\infty} \mathrm{PL}(n) x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + \cdots$$

Appendix

p(9)

{{**9**}, {8, 1}, {7, 2}, {7, 1, 1}, {6, 3}, {6, 2, 1}, {6, 1, 1, 1}, {5, 4}, {5, 3, 1}, {5, 2, 2},
{5, 2, 1, 1}, {5, 1, 1, 1, 1}, {4, 4, 1}, {4, 3, 2}, {4, 3, 1, 1}, {4, 2, 2, 1}, {4, 2, 1, 1, 1},
{4, 1, 1, 1, 1, 1}, {3, 3, 3}, {3, 3, 2, 1}, {3, 3, 1, 1, 1}, {3, 2, 2, 2}, {3, 2, 2, 1, 1},
{3, 2, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 1}, {2, 2, 2, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1},
{2, 1, 1, 1,1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(10)

{{**10**}, {9, 1}, {8, 2}, {8, 1, 1}, {7, 3}, {7, 2, 1}, {7, 1, 1, 1}, {6,4}, {6, 3, 1}, {6, 2, 2}, {6, 2, 1, 1}, {6, 1, 1, 1, 1}, {5, 5}, {5, 4, 1}, {5, 3, 2}, {5, 3, 1, 1}, {5, 2, 2, 1}, {5, 2, 1, 1, 1}, {5, 1, 1, 1, 1,1}, {4, 4, 2}, {4, 4, 1, 1}, {4, 3, 3}, {4, 3, 2, 1}, {4, 3, 1, 1, 1}, {4, 2, 2, 2}, {4, 2, 2, 1, 1}, {4, 2, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 1}, {3, 3, 2, 2}, {3, 3, 2, 1, 1}, {3, 3, 1, 1, 1, 1}, {3, 2, 2, 2, 1}, {3, 2, 2, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2}, {2, 2, 2, 2, 1, 1}, {2, 2, 2, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}

p(11)

{{**11**}, {10, 1}, {9, 2}, {9, 1, 1}, {8, 3}, {8, 2, 1}, {8, 1, 1, 1}, {7, 4}, {7, 3, 1}, {7, 2, 2}, {7, 2, 1, 1}, {7, 1, 1, 1, 1}, {6, 5}, {6, 4, 1}, {6, 3, 2}, {6, 3, 1, 1}, {6, 2, 2, 1}, {6, 2, 1, 1, 1}, {6, 1, 1, 1, 1, 1}, {5, 5, 1}, {5, 4, 2}, {5, 4, 1, 1}, {5, 3, 3}, {5, 3, 2, 1}, {5, 3, 1, 1, 1}, {5, 2, 2, 2}, {5, 2, 2, 1, 1}, {5, 2, 1, 1, 1, 1}, {5, 1, 1, 1, 1, 1, 1}, {4, 4, 3}, {4, 4, 2, 1}, {4, 4, 1, 1, 1}, {4, 3, 3, 1}, {4, 3, 2, 2}, {4, 3, 2, 1, 1}, {4, 3, 1, 1, 1, 1}, {4, 2, 2, 2, 1}, {4, 2, 2, 1, 1, 1}, {4, 2, 1, 1, 1, 1, 1}, {4, 1, 1, 1, 1, 1, 1, 1}, {3, 3, 3, 2}, {3, 3, 3, 1, 1}, {3, 3, 2, 2, 1}, {3, 3, 2, 1, 1, 1}, {3, 3, 1, 1, 1, 1, 1}, {3, 2, 2, 2, 2}, {3, 2, 2, 2, 1, 1}, {3, 2, 2, 1, 1, 1, 1}, {3, 2, 1, 1, 1, 1, 1, 1}, {3, 1, 1, 1, 1, 1, 1, 1, 1}, {2, 2, 2, 2, 2, 1}, {2,2, 2, 2, 1, 1, 1}, {2, 2, 2, 1, 1, 1, 1, 1}, {2, 2, 1, 1, 1, 1, 1, 1, 1}, {2, 1, 1, 1, 1, 1, 1, 1, 1, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}}