Hello and welcome to class!

Last time

We discussed complex eigenvalues and eigenvectors, and then began the discussion of length, orthogonality, and the dot product.

This time

Orthogonal projection, orthogonal sets, orthonormal bases.

Midterm 2

Covers chapters 4-6.5 (depending on where we get next time), but you may need the material from the previous chapters to solve the problems.

It will be, in format, much like midterm 1.

Also like the practice midterm, available at https://math.berkeley.edu/~vivek/54slides/prmid2.pdf

Dot product and transpose

The dot product can be represented as multiplication by the transpose. If we write our vectors as columns, then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

Example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = [1\ 2\ 3] \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Orthogonal complements

Consider a linear subspace $V \subset \mathbb{R}^n$.

We write $V^\perp$ for the set of all vectors which are orthogonal to every vector in V.

It called the orthogonal complement of V.

Orthogonal complements

Equivalently, $V^\perp$ is the set of all vectors which have dot product zero with every element of V.

Note that $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$, which is zero if and only if $\mathbf{v}$ is zero.

So

$$V \cap V^\perp = \{0\}$$

Orthogonal complements

The orthogonal complement is a vector subspace.

I.e., if $\mathbf{x}, \mathbf{y} \in V^{\perp}$ are any vectors orthogonal to all vectors in V, then any linear combination of these vectors is also orthogonal to all vectors in V.

Proof:

If $\mathbf{x} \cdot \mathbf{v} = 0$ and $\mathbf{y} \cdot \mathbf{v} = 0$, then

$$(a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{v} = a\mathbf{x} \cdot \mathbf{v} + b\mathbf{y} \cdot \mathbf{v} = 0$$

Orthogonal complements

The orthogonal complement of V can be characterized as vectors orthogonal to all elements of some basis for V.

Indeed, suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is such a basis, and $\mathbf{x} \cdot \mathbf{v}_i = 0$ for all i. Then for any $\mathbf{v} \in V$, we can expand $\mathbf{v} = \sum c_i \mathbf{v}_i$, and observe

$$\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot \sum c_i \mathbf{v}_i = \sum c_i \mathbf{x} \cdot \mathbf{v}_i = 0$$

Thus $\mathbf{x}$ is orthogonal to any element of V.

Orthogonal complements

This allows us to compute (bases for) orthogonal complements.

Given a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ for V, to find a basis for $V^{\perp}$:

- Write the matrix $[\mathbf{v}_1, \ldots, \mathbf{v}_k]^T$, whose rows are the $\mathbf{v}_i$.
- Note that the kernel of this matrix is precisely the set of vectors whose dot product with all the $\mathbf{v}_i$ is zero.
- Determine (a basis for) the kernel of this matrix.

Example

Find a basis for the orthogonal complement in $\mathbb{R}^4$ of the subspace spanned by $(1, 0, 1, 0)$ and $(2, 3, 4, 5)$.

This is the same as finding a basis for the kernel of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

We row reduce:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2/3 & 5/3 \end{bmatrix}$$

The kernel is spanned by $(-1, -2/3, 1, 0)$ and $(0, -5/3, 0, 1)$.

Try it yourself!

Find a basis for the orthogonal complement in $\mathbb{R}^3$ of the subspace spanned by the vector $(1, 2, 3)$.

This is the same as finding the kernel of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

It's already row reduced.

The kernel is spanned by $(-2, 1, 0)$ and $(-3, 0, 1)$.

Orthogonal complements

(For notation, recall V was a subspace of $\mathbb{R}^n$.)

$V^\perp$ is the kernel of a matrix with $\dim V$ linearly independent rows and n columns, so $\dim V^\perp = n - \dim V$, i.e.

$$\dim V + \dim V^\perp = \dim \mathbb{R}^n$$

Recall from before, $V \cap V^\perp = \{0\}$.

Orthogonal complements

$V^\perp$ = "everything orthogonal to everything in V"

$(V^\perp)^\perp$ = "everything orthogonal to
everything orthogonal to everything in V"

Everything in V is orthogonal to $V^\perp$, so $V \subset (V^\perp)^\perp$. But from the previous slide, we have:

$\dim V + \dim V^\perp = n$ and similarly $\dim V^\perp + \dim(V^\perp)^\perp = n$

So we learn $\dim V = \dim(V^\perp)^\perp$, hence $V = (V^\perp)^\perp$.

Orthogonal complements

In words, $V = (V^{\perp})^{\perp}$ is telling you that:

Everything orthogonal to everything orthogonal to everything in V is already in V.

Orthogonal complements

Theorem

Let $V \subset \mathbb{R}^n$ *be a linear subspace. Any vector in* $\mathbb{R}^n$ *has a unique expression as sum of a vector in* V *and a vector in* $V^\perp$.

In fact, something more general is true:

Theorem

Given any two subspaces V, W *such that* $\dim V + \dim W = n$ *and* $V \cap W = \{0\}$, *any vector in* $\mathbb{R}^n$ *can be written uniquely as a sum of a vector in* V *and a vector in* W.

Example

Consider the subspace $V \subset \mathbb{R}^4$ spanned by $\mathbf{e}_1, \mathbf{e}_2$.

Its orthogonal complement $V^\perp$ is spanned by $\mathbf{e}_3, \mathbf{e}_4$.

Any vector (w, x, y, z) can be written as $(w, x, 0, 0) + (0, 0, y, z)$.

Proof

Take bases $\mathbf{v}_1, \ldots, \mathbf{v}_v$ of V and $\mathbf{w}_1, \ldots, \mathbf{w}_w$ of W.

Suppose $\sum a_i \mathbf{v}_i + \sum b_j \mathbf{w}_j = 0$. Then $\sum a_i \mathbf{v}_i = -\sum b_j \mathbf{w}_j$.

One side is in V and the other is in W, so both are in $V \cap W$, hence zero. $\{\mathbf{v}_i\}$ and $\{\mathbf{w}_j\}$ were bases, so a_i and b_j must be zero.

Proof

Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_v, \mathbf{w}_1, \ldots, \mathbf{w}_w\}$ is linearly independent.

This linearly independent set has n elements, so it's a basis for $\mathbb{R}^n$.

So any vector can be written as $\sum a_i \mathbf{v}_i + \sum b_j \mathbf{w}_j$. This a sum of a vector in V and a vector in W.

Proof

Such an expression is unique:

If $\mathbf{v}, \mathbf{v}' \in V$ and $\mathbf{w}, \mathbf{w}' \in W$ and $\mathbf{v} + \mathbf{w} = \mathbf{v}' + \mathbf{w}'$,

then $\mathbf{v} - \mathbf{v}' = \mathbf{w} - \mathbf{w}'$ is in V and in W hence is zero.

So $\mathbf{v} = \mathbf{v}'$ and $\mathbf{w} = \mathbf{w}'$.

Orthogonal projection

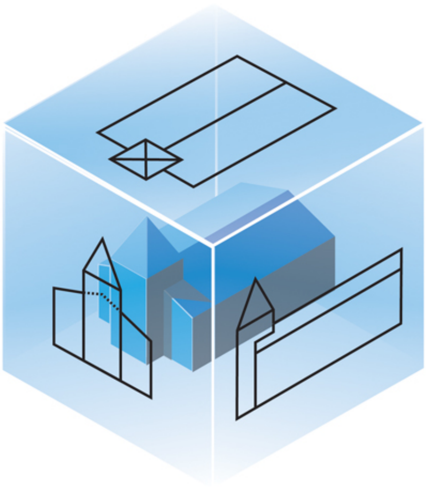
We have seen that if $V \subset \mathbb{R}^n$ is a vector subspace,

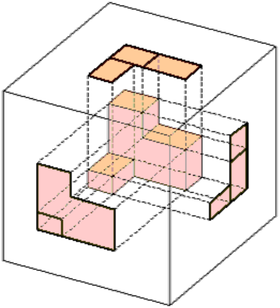
any vector $\mathbf{x}$ can be written uniquely as

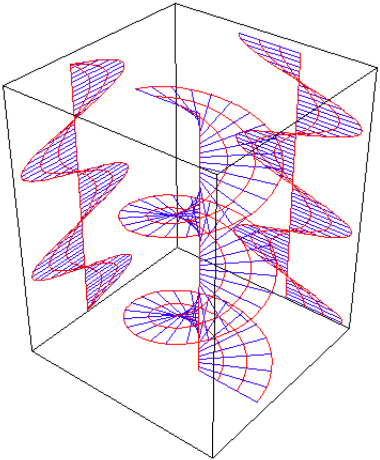
$$\mathbf{x} = \mathbf{v} + \mathbf{v}^\perp$$

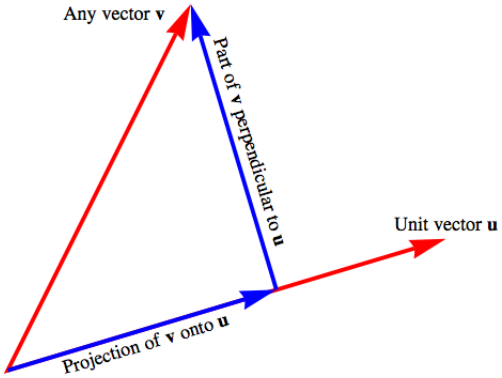
for some $\mathbf{v} \in V$ and $\mathbf{v}^\perp \in V^\perp$.

The vector $\mathbf{v}$ above is said to be the orthogonal projection of $\mathbf{x}$.









Any vector **v**
Part of **v** perpendicular to **u**
Unit vector **u**
Projection of **v** onto **u**

Computing orthogonal projections

Special case: V is one dimensional, hence the span of one vector $\mathbf{v}$.

Writing $\mathbf{x}$ as the sum of a vector in V and a vector in $V^\perp$ means: writing $\mathbf{x} = \lambda\mathbf{v} + \mathbf{w}$ where $\mathbf{v} \cdot \mathbf{w} = 0$. Taking dot products,

$$\mathbf{v} \cdot \mathbf{x} = \lambda \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} = \lambda \mathbf{v} \cdot \mathbf{v}$$

Or in other words,

$$\lambda = \frac{\mathbf{v} \cdot \mathbf{x}}{\mathbf{v} \cdot \mathbf{v}}$$

So the orthogonal projection of $\mathbf{x}$ to $\mathrm{Span}(\mathbf{v})$ is

$$\left(\frac{\mathbf{v} \cdot \mathbf{x}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}$$

Example

The orthogonal projection of $(1, 2, 3, 4)$

to the line spanned by $(1, 1, 1, 1)$ is

$$\left(\frac{(1,1,1,1)\cdot(1,2,3,4)}{(1,1,1,1)\cdot(1,1,1,1)}\right)\ (1,1,1,1) = \frac{10}{4}\cdot(1,1,1,1)$$

Computing orthogonal projections

More generally, suppose you have a basis $\mathbf{v}_1, \cdots, \mathbf{v}_k$ for V such that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$. Such a basis is called an orthogonal basis.

Then the orthogonal projection of $\mathbf{x}$ to V is the sum of the orthogonal projections to the $\mathbf{v}_i$. That is,

$$\mathrm{Proj}_V(\mathbf{x}) = \sum_i \left(\frac{\mathbf{v}_i \cdot \mathbf{x}}{\mathbf{v}_i \cdot \mathbf{v}_i} \right) \cdot \mathbf{v}_i$$

Proof: We compute the dot product $\mathbf{v}_j \cdot (\mathbf{x} - \mathrm{Proj}_V(\mathbf{x}))$:

$$\mathbf{v}_j \cdot \mathbf{x} - \mathbf{v}_j \cdot \sum_i \left(\frac{\mathbf{v}_i \cdot \mathbf{x}}{\mathbf{v}_i \cdot \mathbf{v}_i} \right) \mathbf{v}_i = \mathbf{v}_j \cdot \mathbf{x} - \mathbf{v}_j \cdot \mathbf{v}_j \left(\frac{\mathbf{v}_j \cdot \mathbf{x}}{\mathbf{v}_j \cdot \mathbf{v}_j} \right) = 0$$

Orthonormal bases

The formula

$$\mathrm{Proj}_V(\mathbf{x}) = \sum_i \left(\frac{\mathbf{v}_i \cdot \mathbf{x}}{\mathbf{v}_i \cdot \mathbf{v}_i} \right) \cdot \mathbf{v}_i$$

simplifies further when $\mathbf{v}_i \cdot \mathbf{v}_i = ||\mathbf{v}_i||^2 = 1$ for all i.

Such a basis is called an orthonormal basis.

Example

Any subset of the standard basis is an orthonormal basis for the linear space it spans.

Finding orthonormal bases:

If V is one dimensional: choose any nonzero vector v_1, and then divide it by its length. Now you have a unit vector,

$$\mathbf{v}_1' = \frac{1}{||\mathbf{v}_1||}\ \mathbf{v}_1$$

Finding orthonormal bases:

If V is two dimensional: choose any nonzero vector v_1, and then divide it by its length. Now you have a unit vector,

$$\mathbf{v}_1' = \frac{1}{||\mathbf{v}_1||}\ \mathbf{v}_1$$

Next, choose any vector $\mathbf{v}_2$ outside the span of $\mathbf{v}_1'$. Compute its orthogonal projection:

$$\mathbf{v}_2 = \lambda \mathbf{v}_1' + \mathbf{v}_2'$$

The vectors $\mathbf{v}_1', \mathbf{v}_2'$ form an orthogonal basis. Finally rescale

$$\mathbf{v}_2'' = \frac{1}{||\mathbf{v}_2'||}\ \mathbf{v}_2'$$

The vectors $\mathbf{v}_1', \mathbf{v}_2''$ form an orthonormal basis.