# Hello and welcome to class!

## Last time

We had been discussing orthogonality, and in particular how to compute orthogonal projections in term of an orthonormal bases.

## This time

We will discuss how to produce an orthonormal basis.

# 1d case

Given a basis of a 1-d vector space $V$

I.e., a nonzero vector $\mathbf{v} \in V$

You can get an orthonormal basis

$$\frac{\mathbf{v}}{||\mathbf{v}||}$$

# 1d case: example

Given the basis $(1,2,3,4)$ of the 1-d vector space $\mathrm{Span}((1,2,3,4))$

You can get an orthonormal basis

$$\frac{(1,2,3,4)}{||(1,2,3,4)||} = \frac{1}{\sqrt{30}}(1,2,3,4)$$

# 2d case

Given a basis of a 2-d vector space $V$

I.e., two nonzero vectors $\mathbf{v}, \mathbf{w} \in V$, neither a multiple of the other

First we want to modify $\mathbf{w}$ so it becomes orthogonal to $\mathbf{v}$, by projection: $\mathbf{w}' = \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$

Now normalize: $\mathbf{v}' := \frac{\mathbf{v}}{||\mathbf{v}||}$, and $\mathbf{w}'' = \frac{\mathbf{w}'}{|\mathbf{w}'|}$.

# 2d example

Given the basis $(1,2,3),(2,3,1)$ of the 2d space they span

First project $(2,3,1)$ to the orthogonal complement of $(1,2,3)$.

$$(2,3,1) - \frac{(2,3,1)\cdot(1,2,3)}{(1,2,3)\cdot(1,2,3)}(1,2,3) = (2,3,1) - \frac{11}{14}(1,2,3)$$

$$= \frac{1}{14}(17,20,-19)$$

And then normalize.

$$\frac{1}{\sqrt{14}}(2,3,4) \qquad \frac{1}{\sqrt{17^2+20^2+19^2}}(17,20,-19)$$

# Try it yourself

Find an orthonormal basis of the span of $(1, 1, 0)$ and $(1, 0, 1)$.

# Gram-Schmidt

Consider a vector space $V \subset \mathbb{R}^n$ with basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Project $\mathbf{v}_2$ to the subspace orthogonal to $\mathbf{v}_1$

Project $\mathbf{v}_3$ to the subspace orthogonal to $\mathbf{v}_1, \mathbf{v}_2$

Project $\mathbf{v}_3$ to the subspace orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Etc.

Rescale everything to ensure they are unit vectors.

# Gram-Schmidt

That is, from $\mathbf{v}_1, \ldots, \mathbf{v}_k$, form:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 \\
\mathbf{u}_2 &= \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 \\
\mathbf{u}_3 &= \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 \\
\mathbf{u}_4 &= \mathbf{v}_4 - \left(\frac{\mathbf{v}_4 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \left(\frac{\mathbf{v}_4 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 - \left(\frac{\mathbf{v}_4 \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\right)\mathbf{u}_3 \\
&\vdots
\end{aligned}$$

And then rescale all the $\mathbf{u}_i$.

# Example

Apply the Gram-Schmidt process to

$$(1,1,1,0),(1,0,1,1),(1,1,0,1)$$

We take $\mathbf{v}_1 = (1,1,1,0), \mathbf{v}_2 = (1,0,1,1), \mathbf{v}_3 = (1,1,0,1)$.

Then $\mathbf{u}_1 = \mathbf{v}_1 = (1,1,1,0)$,

$$\mathbf{u}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 = (1,0,1,1) - \frac{2}{3}(1,1,1,0) = \frac{1}{3}(1,-2,1,3)$$

# Example

$$\mathbf{u}_1 = (1,1,1,0) \qquad \mathbf{u}_2 = \frac{1}{3}(1,-2,1,3) \qquad \mathbf{v}_3 = (1,1,0,1)$$

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 \\
&= (1,1,0,1) - \frac{2}{3}(1,1,1,0) - \frac{2}{15}(1,-2,1,3) \\
&= \frac{1}{15}((15,15,0,15) - (10,10,10,0) - (2,-4,2,6)) \\
&= \frac{1}{15}(3,9,-12,9)
\end{aligned}$$

Finally, rescaling:

$$\frac{1}{\sqrt{3}}(1,1,1,0) \qquad \frac{1}{\sqrt{15}}(1,-2,1,3) \qquad \frac{(3,9,-12,-9)}{\sqrt{3^2+9^2+12^2+9^2}}$$

# Try it yourself

Apply the Gram-Schmidt process to

$$(1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0)$$

# Orthogonal matrices

For a square matrix $A$, the following are equivalent:

- $A$ has orthonormal rows
- $AA^T = I$
- $A^T = A^{-1}$
- $A^T A = I$
- $A$ has orthonormal columns

Such matrices are called orthogonal. They also preserve dot products:

$$(Av) \cdot (Aw) = (Av)^T (Aw) = v^T A^T A w = v^T w = v \cdot w$$

# QR decomposition

Observe that the steps we took in the Gram-Schmidt process each involve adding multiples of earlier vectors to later ones.

$$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \textit{lower} \\ \textit{triangular} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_k \end{bmatrix}$$

The inverse of a lower triangular matrix is lower triangular, so:

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_k \end{bmatrix} = \begin{bmatrix} \textit{lower} \\ \textit{triangular} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_k \end{bmatrix}$$

# QR decomposition

The vectors $\mathbf{v}_i$ could have been anything, so:

$$\begin{bmatrix} \textit{any} \\ \textit{matrix} \end{bmatrix} = \begin{bmatrix} \textit{lower} \\ \textit{triangular} \end{bmatrix} \begin{bmatrix} \textit{orthogonal} \\ \textit{rows} \end{bmatrix}$$

Taking transposes:

$$\begin{bmatrix} \textit{any} \\ \textit{matrix} \end{bmatrix} = \begin{bmatrix} \textit{orthogonal} \\ \textit{columns} \end{bmatrix} \begin{bmatrix} \textit{upper} \\ \textit{triangular} \end{bmatrix}$$

This is called the QR decomposition.