

A has r rows and c columns; $A : \mathbb{R}^c \rightarrow \mathbb{R}^r$

Columns below have equivalent conditions (except in parenthesis)

$A\mathbf{x} = 0$ implies $\mathbf{x} = 0$

pivot in every **column**

columns linearly independent

rows span all of \mathbb{R}^c

distinct points to distinct points

(can only happen if $c \leq r$)

$A\mathbf{x} = \mathbf{b}$ has solutions for any \mathbf{b}

pivot in every **row**

rows linearly independent

columns span all of \mathbb{R}^r

hits all of \mathbb{R}^r .

(can only happen if $r \leq c$)

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.

Span and linear independence

In particular, a collection of d vectors in \mathbb{R}^n

can only span if $d \geq n$,

and can only be linearly independent if $d \leq n$.

Example

Are the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ linearly independent?

Definitely not, there are too many. Let's put them in a matrix and row reduce anyway.

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

The row operations do not affect linear independence, and the final matrix has a zero row, so the original rows were not linearly independent. You can also see: there's not a pivot in every row, the columns don't span, etc. But note: the columns are linearly independent.

Example

Are the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 1 \\ 9 \end{bmatrix}$ linearly independent?

Put them in a matrix and row reduce!

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 5 & 1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 5 & 1 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The row operations do not affect linear independence, and the final matrix has a zero row, so **the original rows were not linearly independent**. You can also see: there's not a pivot in every row, the columns don't span, etc. But note: **the columns are not linearly independent**.

Try it yourself

Are the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 1 \\ 9 \\ 1 \end{bmatrix}$ linearly independent?

Put them in a matrix and row reduce!

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -1 & 1 & 1 \\ 5 & 1 & 9 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 5 & 1 & 9 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The final echelon matrix has no zero row, so the original rows are linearly independent.

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pivot in every **row**

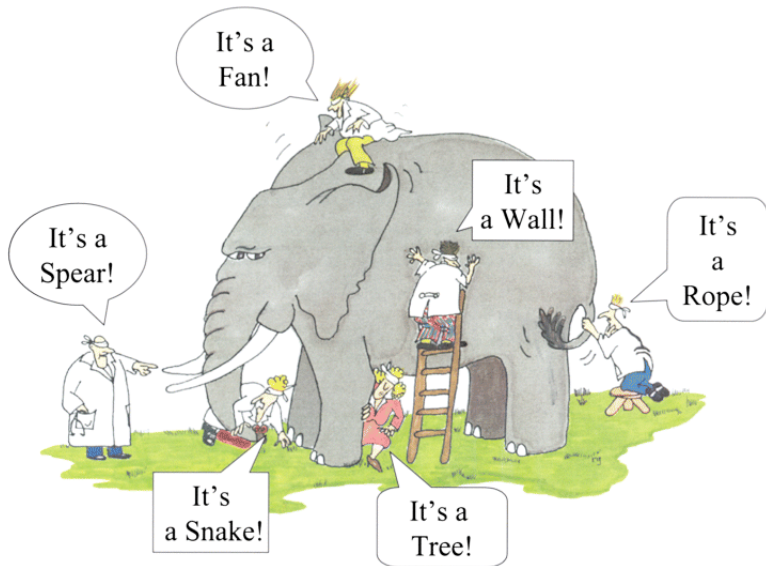
rows linearly independent

columns span all of \mathbb{R}^r

hits all of \mathbb{R}^r .

(can only happen if $r \leq c$)

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.



Scalar multiplication

Just like for vectors, multiplying a matrix by a scalar just means multiplying every element of the matrix by that scalar.

$$3 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

$$-2 \cdot \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 4 & -6 & -2 \end{bmatrix}$$

$$0 \cdot \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix addition

And you can add matrices **of the same size** by adding them termwise.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 5 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 2 & 2 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

they're not the same size

Matrix transpose

The transpose of the matrix is what you get by reflecting along a northwest-southeast diagonal. This makes the old first column into the new first row, etcetera.

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \\ -1 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 3 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = [1 \quad 2 \quad 3]$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

Matrix entries

We write $M_{i,j}$ or M_{ij} for the entry in the i' th row and j' th column of the matrix M .

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 5 & 8 \end{bmatrix}$$

$$A_{1,1} = 3, \quad A_{2,3} = 8$$

Matrix multiplication

Given two matrices, A, B , if A has as many columns as B has rows, then there is a matrix product AB .

The matrix product is determined by the formula

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$$

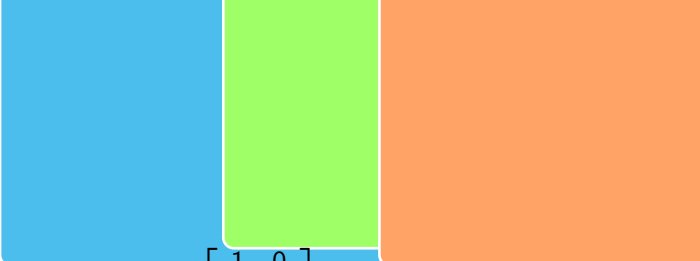
where n is the number of columns of A , or of rows of B .

AB has as many rows as A and as many columns as B .

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

Matrix multiplication

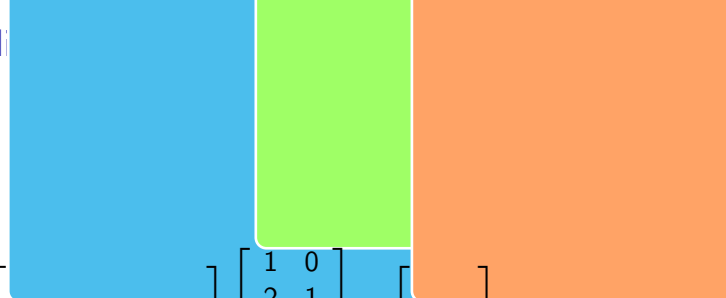


The diagram shows three colored rectangular blocks at the top: a blue block on the left, a green block in the middle, and an orange block on the right. Below these blocks is a matrix multiplication equation. The first matrix is 4x4, the second is 2x2, and the result is a 4x1 column vector. A blue horizontal line highlights the first row of the first matrix and the first column of the second matrix, which are used in the calculation shown below.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 \\ \\ \\ \end{bmatrix}$$

$$1 \times 1 + 2 \times 2 + 1 \times 1 + 2 \times 0 = 6$$

Matrix multipli

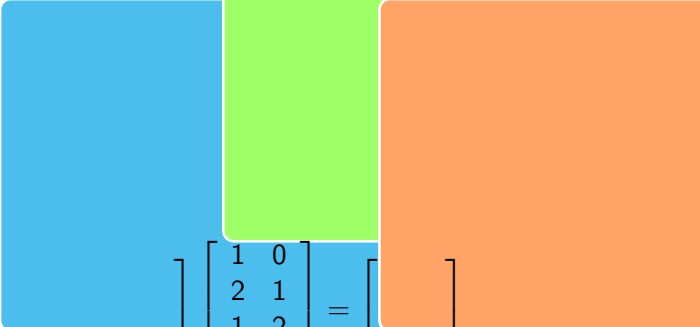


The diagram illustrates the calculation of the dot product of the first row of a 2x4 matrix and the first column of a 4x2 matrix. The 2x4 matrix is represented by a blue block, the 4x2 matrix by a green block, and the resulting 2x2 matrix by an orange block. The first row of the blue block and the first column of the green block are highlighted in a lighter blue, corresponding to the calculation shown below.

$$\begin{bmatrix} -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & \dots \\ \dots & \dots \end{bmatrix}$$

$$-1 \times 1 + 0 \times 2 + 1 \times 1 + 3 \times 0 = 0$$

Matrix multiplication

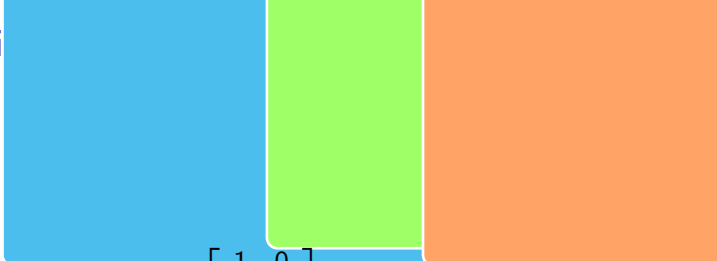


The diagram illustrates the dimensions of the matrices involved in the multiplication. A large blue rectangle represents the first matrix (4x4), a green rectangle represents the second matrix (2x2), and a large orange rectangle represents the resulting matrix (4x2). The blue rectangle is divided into four 2x2 quadrants. The green rectangle is divided into two 2x1 vertical strips. The orange rectangle is divided into four 2x1 vertical strips. The equation below shows the calculation of the first element of the result matrix, which is 8.

$$\begin{bmatrix} 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}$$

$$2 \times 1 + 3 \times 2 + 0 \times 1 + 1 \times 0 = 8$$

Matrix multipli

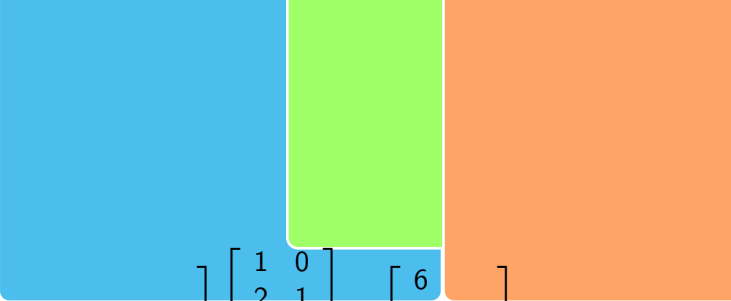


The diagram illustrates the dot product of two rows from a matrix multiplication. A large blue square represents the first row of the first matrix (1, 2, 1, 2). A green square represents the second column of the second matrix (1, 2, 3). An orange square represents the resulting element in the product matrix (10). A horizontal blue bar highlights the alignment of the blue and green squares.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 0 & 8 \end{bmatrix}$$

$$1 \times 0 + 2 \times 1 + 1 \times 2 + 2 \times 3 = 10$$

Matrix multip

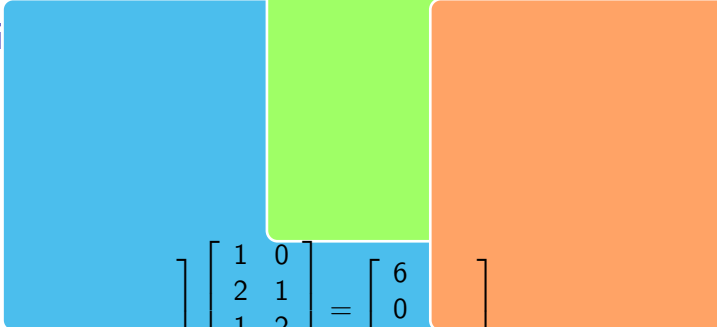


The diagram illustrates the dot product of two rows from a matrix multiplication. A large blue rectangle on the left represents the first row of the first matrix, with the values -1, 0, 1, and 3. A smaller blue rectangle on the right represents the second column of the second matrix, with the values 2 and 3. These two rectangles overlap, and the overlapping area is highlighted in a lighter blue. To the right of this overlap is a green rectangle representing the first row of the second matrix, with the values 1 and 0. To the right of the green rectangle is an orange rectangle representing the second column of the first matrix, with the values 6 and 11. The equation below shows the calculation of the dot product of the first row of the first matrix and the second column of the second matrix, resulting in 11.

$$\begin{bmatrix} -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 11 \\ 0 & 8 \end{bmatrix}$$

$$-1 \times 0 + 0 \times 1 + 1 \times 2 + 3 \times 3 = 11$$

Matrix multipli



The diagram shows three colored rectangles representing matrices. A large blue rectangle on the left represents a 1x4 matrix. A smaller light green rectangle is positioned above the blue one, representing a 2x2 matrix. A large orange rectangle is on the right, representing a 4x2 matrix. The blue rectangle is divided into four vertical sections, each containing a number: 2, 3, 0, and 1. The light green rectangle is divided into four quadrants, each containing a number: 1, 0, 2, and 3. The orange rectangle is divided into two vertical sections, each containing a number: 6 and 8. The equation below shows the multiplication of the blue matrix by the light green matrix, resulting in the orange matrix.

$$\begin{bmatrix} 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 8 & 6 \end{bmatrix}$$

$$2 \times 0 + 3 \times 1 + 0 \times 2 + 1 \times 3 = 6$$

Matrix multiplication

Another way to see it:

Write the second matrix as a “row of columns”

$$B = [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_n]$$

Then:

$$AB = A [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_n] = [A\mathbf{b}_1 \mid A\mathbf{b}_2 \mid \cdots \mid A\mathbf{b}_n]$$

The matrix-vector product is a special case.

Try it yourself

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = ?$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = ?$$

Matrix multiplication

The matrix product is **associative**, **distributes over matrix addition**, but **is not generally commutative**.

Indeed, the dimensions of A and B can be such that AB makes sense but BA does not; and we saw on the last slide that even if they both make sense, they need not be equal.

Try it yourself

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \quad ?$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \quad ?$$

The identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$$

The identity matrix

We write I_n for the matrix with 1's along the diagonal, and zeroes everywhere else.

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If $I_n \cdot M$ is defined, i.e., M has n rows, then

$$I_n \cdot M = M$$

If $M \cdot I_m$ is defined, i.e., M has m columns, then

$$M \cdot I_m = M$$

The identity matrix

We write I_n for the matrix with 1's along the diagonal, and zeroes everywhere else.

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the identity matrix I_n is the unique reduced echelon $n \times n$ matrix with a pivot in every row (or equivalently, every column).

Try it yourself

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = ?$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = ?$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = ?$$

Try it yourself

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 11 & 13 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix}$$

Elementary row operations

You can do an elementary row operation by multiplying on the left by the matrix which is obtained by performing that row operation on the identity matrix.

Try it yourself!

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = ?$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = ?$$

Try it yourself!

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix inversion

If A is a matrix, we say A is invertible if there is some other matrix B such that BA and AB are both identity matrices.

In this case we say that B is the inverse of A , and write it as A^{-1} .

The inverse is unique if it exists: if $BA = I$ and $AC = I$ then

$$B = BI = B(AC) = (BA)C = IC = C$$

Try it yourself!

Is the identity matrix invertible? **yes**

$$I_n \cdot I_n = I_n$$

Matrix inversion

If A is an invertible matrix, then the following equations are equivalent:

$$A\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

In particular,

$A\mathbf{x} = \mathbf{0}$ has only the zero solution $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$

For any \mathbf{b} , the equation $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$.

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$A\mathbf{x} = 0$ implies $\mathbf{x} = 0$

pivot in every **column**

columns linearly independent

rows span all of \mathbb{R}^c

one-to-one

(can only happen if $c \leq r$)

$A\mathbf{x} = \mathbf{b}$ has solutions for any \mathbf{b}

pivot in every **row**

rows linearly independent

columns span all of \mathbb{R}^r

onto

(can only happen if $r \leq c$)

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.

Matrix inversion

If A is an invertible matrix, then A is square, and all the conditions on the previous slide hold.

Conversely, for a square matrix, being invertible is equivalent to any one of these conditions.

Calculating the inverse

In particular, row-reducing an invertible matrix to reduced row-echelon form gives the identity matrix.

This leads to an algorithm for calculating the inverse.

Calculating the inverse

Row reduction is implemented by elementary matrices, so if M is invertible — hence can be row reduced to the identity — there exist some elementary matrices, E_1, \dots, E_k such that

$$E_k \cdots E_2 E_1 M = I$$

Multiplying by M^{-1} on both sides, (or recalling that the inverse was unique)

$$E_k \cdots E_2 E_1 = M^{-1}$$

Calculating the inverse

The equations

$$E_k \cdots E_2 E_1 \cdot M = I \qquad E_k \cdots E_2 E_1 \cdot I = M^{-1}$$

can be combined: putting the matrices M and I next to each other,

$$E_k \cdots E_2 E_1 \cdot [M \mid I] = [I \mid M^{-1}]$$

Calculating the inverse

Now remember what E_i do: **they are row operations**. Thus,

$$E_k \cdots E_2 E_1 \cdot [M \mid I] = [I \mid M^{-1}]$$

is simply asserting that $[I \mid M^{-1}]$ is obtained from $[M \mid I]$ by **row reduction**!

Calculating the inverse

To find the inverse of M ,

- ▶ Form the matrix $[M|I]$
- ▶ Row reduce it
- ▶ If the result has the form $[I|X]$ then $X = M^{-1}$
- ▶ If not, M was not invertible (not enough pivots).

Calculating the inverse

Find the inverse of $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

First put it next to the identity in a matrix.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right]$$

Calculating the inverse

Row reduce this matrix.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

Read off the inverse from the right of the matrix:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Try it yourself!

Find inverses for the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1} = ?$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}^{-1} = ?$$

Another way to think about calculating the inverse

The columns of the matrix A^{-1} are $A^{-1}\mathbf{e}_1, A^{-1}\mathbf{e}_2, \dots, A^{-1}\mathbf{e}_n$.

These are the solutions to the equations $A\mathbf{x} = \mathbf{e}_1, A\mathbf{x} = \mathbf{e}_2, \dots$

To find these solutions, we would row reduce the augmented matrixes $[A|\mathbf{e}_1], [A|\mathbf{e}_2], \dots$

Do them all at once by row reducing the matrix $[A|\mathbf{e}_1|\mathbf{e}_2|\dots|\mathbf{e}_n]$

$[\mathbf{e}_1|\mathbf{e}_2|\dots|\mathbf{e}_n]$ is just the identity matrix, so row reduce $[A|I]$.

The inverse of a 2×2 matrix

Consider an arbitrary 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{aligned} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - cb/a & -c/a & 1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -c/(ad - bc) & a/(ad - bc) \end{array} \right] \\ &\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & d/(ad - bc) & -b/(ad - bc) \\ 0 & 1 & -c/(ad - bc) & a/(ad - bc) \end{array} \right] \end{aligned}$$

The inverse of a 2×2 matrix

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse if (and in fact only if)

$ad - bc \neq 0$, and in this case its inverse is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called the **discriminant**.