Math 221 : Linear algebra and geometry

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Lectures M, T, W, Th
add. office hrs   TA office hours every Tu, Fr

HW assigned every week
Midterm on week 3
Final

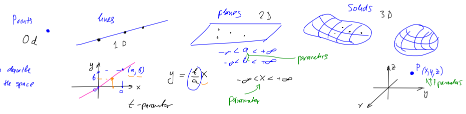
HW: 20%   A: 90%+
Midterm: 30%   B: 80% ish
Final: 50%   C <

Introduction: dimensions, dots, lines, planes.

Matrices: Linear equations, diagonalization, etc

Algebra: Groups, Rings, Fields, Vector space
Eigenvalue equations, eigenvectors, eigenvalues

① Introduction   Points 0d   lines 1 D   planes 2 D   Solids 3 D



Dimension ⇕ # parameters which describe every point of the space

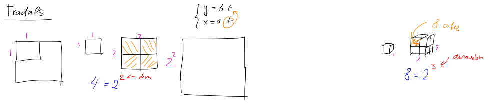
$-\infty < a < +\infty$, $-\infty < b < +\infty$ — parameters

$y = \frac{b}{a}x$ $-\infty < x < +\infty$ — parameter

t - parameter
$\begin{cases} y = bt \\ x = at \end{cases}$

$P(x, y, z)$ 3 parameters

Fractals



$4 = 2^2$ ← dim   8 cubes $8 = 2^3$ ← dimension

Systems of equations describing various sets of points

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ f_2(x_1, \dots, x_n) = 0 \\ \vdots \\ f_m(x_1, \dots, x_n) = 0 \end{cases}$$

$f_1 \dots f_n$ - any functions

Algebraic geometry $f_1 \dots f_n$ - polynomial in $x_1 \dots x_n$
$f_1 = x_1^7 + x_1x_2x_3 + x_3^7$

Linear algebra $f_1 \dots f_n$ - linear in $x_1 \dots x_n$

$f(x_1 \dots x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n + b$
where $a_1, a_2, \dots, a_n, b$ - real or complex
$\mathbb{R}$ $\mathbb{C}$

$f_1(x_1 \dots x_n) = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_1 = 0$
$f_2(x_1 \dots x_n) = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_2 = 0$
$\dots$
$f_m(x_1 \dots x_n) = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + b_m = 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

Q What is the geometric meaning of the set of solutions of linear equations?

Ex $\begin{cases} 5x_1 + 6x_2 = 1 \\ x_1 - x_2 = 3 \end{cases}$ 1 solution

$x_1 = x_2 + 3$
$5x_2 + 15 + 6x_2 = 1$
$11x_2 = -14 \Rightarrow x_2 = -\frac{14}{11}$
$x_1 = \frac{19}{11}$

Ex $2\times \begin{cases} 2x_1 + 4x_2 = 5\times 2 \\ 4x_1 + 8x_2 = 10 \end{cases}$ ∞ - many solutions

Ex $\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \end{cases}$ $\emptyset$ - empty set, no solutions
pair of parallel lines

Matrix notation

$\times(-5)$ $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 \quad - 5x_3 = 10 \end{cases}$

Left hand side
$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ 2x_2 - 8x_3 \\ 5x_1 - 5x_3 \end{pmatrix}$$

$A\vec{x} = \vec{b}$ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$

Right hand side $\begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$

Augmented matrix 3×4
$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right)$$
A $\vec{b}$

Q: Solve the system

① Multiply first equation by (-5) and add it to the third equation

$x_1 - 2x_2 + x_3 = 0$
$2x_2 - 8x_3 = 8$
$\times(-5)$ $0 + 10x_2 - 10x_3 = 10$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{pmatrix}$$

② Multiply second equation by (-5) and add to the third equation

$x_1 - 2x_2 + x_3 = 0$
$\frac{1}{2}$ $2x_2 - 8x_3 = 8$
$\frac{1}{30}$ $0 + 30x_3 = -30$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{pmatrix}$$

③ Divide the second equation by 2, divide 3rd equation by 30

$x_1 - 2x_2 + x_3 = 0$
$x_2 - 4x_3 = 4$
$4\times$ $x_3 = -1$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ & 1 & -4 & 4 \\ & & 1 & -1 \end{pmatrix}$$

④ Multiply third equation by 4 and add it to the second equation

$x_1 - 2x_2 + x_3 = 0$
$2\times$ $x_2 + 0 = 0$
$(-1)\times$ $x_3 = -1$

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$ ← is of Echelon form

⑤ Multiply third equation by (-1) and add it to the first equation
second equation by 2 and add it to the first equation

$x_1 + 0 + 0 = 1$
$x_2 + 0 = 0$
$x_3 = -1$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$\{x_1 = 1, x_2 = 0, x_3 = -1\}$

Matrices of echelon form

$$\begin{array}{c} 1 \rightarrow \\ 2 \rightarrow \\ \\ \end{array} \begin{pmatrix} 5 & 7 & 0 & 8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ ← echelon form

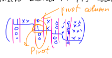
$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 5 & 7 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$ - Not of Echelon form

Echelon form
1) All nonzero rows are above any rows with all zeros
2) Each leading entry of a row is in a column to the right of leading entry of the row above it
3) All elements in a column below a leading entry are zeros

Reduced Echelon form
4) The leading entry in each nonzero row is 1
5) Each leading 1 is the only nonzero element in its column

Echelon form

Reduced Echelon form



Pivot positions   Row elimination algorithm

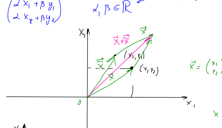
A pivot position of matrix A is a location in A that corresponds to a leading 1 in the reduced Echelon form of A

Vectors in $\mathbb{R}^2$ A vector in $\mathbb{R}^2$ is an ordered pair (x_1, x_2) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$
components of $\vec{x}$

Operations on vectors. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

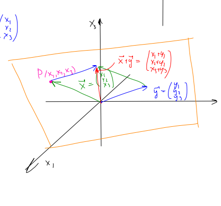
$\alpha\vec{x} + \beta\vec{y} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}$ $\alpha, \beta \in \mathbb{R}$ ← real numbers
linear property

$10\begin{pmatrix} 0 \\ 5 \end{pmatrix} + 7\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -7 \\ 50 \end{pmatrix}$



$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\vec{x} + \vec{x} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$

Vectors in $\mathbb{R}^3$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$



Vectors in $\mathbb{R}^n$ $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ times}}$ $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\vec{x} \longleftrightarrow (x_1, x_2, x_3, \dots, x_n)$

Properties of operations with vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$

commutativity
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
- $\vec{u} + \vec{0} = \vec{u}$
- $-\vec{u} + \vec{u} = \vec{0}$
- $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$
- $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
- $1 \cdot \vec{u} = \vec{u}$
- $\alpha(\beta\vec{u}) = \alpha\beta\vec{u}$

group
$+ \quad \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
$\vec{u}, \vec{v} \mapsto \vec{u} + \vec{v}$

$\mathbb{R}^n$ is a module over $\mathbb{R}$

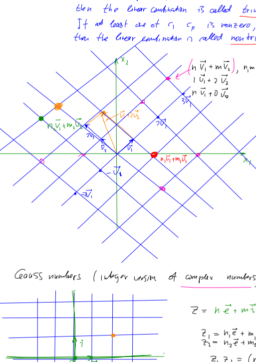
Linear combinations Def Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$ - vectors in $\mathbb{R}^n$
$c_1, c_2, \dots, c_p \in \mathbb{R}$ - scalars
$\mathbb{Z}$

Then the linear combination of $\vec{v}_1, \vec{v}_2, \dots \vec{v}_p$ with coefficients $c_1, c_2, \dots c_p$ (weights) is $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \sum_{i=1}^{p} c_i v_i$

Ex $\vec{v}_1 + 5\vec{v}_2 - 100\vec{v}_3$

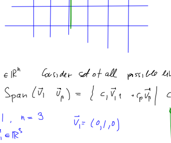
If all $c_1 \dots c_p$ are equal to zero then the linear combination is called trivial
If at least one of $c_1 \dots c_p$ is nonzero, than the linear combination is called nontrivial

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$n_1\vec{v}_1 + m_1\vec{v}_2 + n_2\vec{v}_1 + m_2\vec{v}_2$
$= (n_1 + n_2)\vec{v}_1 + (m_1 + m_2)\vec{v}_2$

Gauss numbers (integer version of complex numbers)



$z = n\vec{e} + m\vec{i}$ $i \cdot i = -e$
$i \cdot e = i$
$\vec{e}\,\vec{e} = \vec{e}$

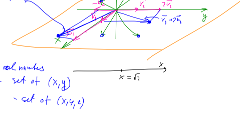
$z_1 = n_1\vec{e} + m_1\vec{i}$
$z_2 = n_2\vec{e} + m_2\vec{i}$
$z_1 z_2 = (n_1\vec{e} + m_1\vec{i})(n_2\vec{e} + m_2\vec{i})$
$= \underbrace{(n_1n_2 - m_1m_2)}_{\text{real part}}\vec{e} + \underbrace{(n_1m_2 + n_2m_1)}_{\text{imaginary part}}\vec{i}$

Def $\vec{v}_1 \dots \vec{v}_p \in \mathbb{R}^n$ Consider set of all possible linear combinations of these vectors
$\text{Span}(\vec{v}_1 \dots \vec{v}_p) = \{c_1\vec{v}_1 + \dots + c_p\vec{v}_p \mid c_i \in \mathbb{R}\}$

Ex $p = 1$, $n = 3$ $\vec{v}_1 = (0, 1, 0)$ $\text{Span}(\vec{v}_1) = \{c_1\vec{v} \mid c_1 \in \mathbb{R}\}$ - line $= \mathbb{R}$
$\vec{v}_1 \in \mathbb{R}^3$
$2\vec{v}_1 = (0, 2, 0)$ $c_1\vec{v}_1 = (0, c_1, 0)$

$p = 2$, $n = 3$ $\vec{v}_1 = (1, 0, 0)$
$\vec{v}_2 = (0, 1, 0)$
$\text{Span}(\vec{v}_1, \vec{v}_2) = \{c_1\vec{v} + c_2\vec{v}_2 \mid c_1, c_2 \in \mathbb{R}\} = \mathbb{R}^2 \subset \mathbb{R}^3$
$(x, y, 0)$ (x, y, z) subset   xy plane

$\vec{v}_1 + 2\vec{v}_2$
$3\vec{v}_1 - \vec{v}_2$



$\text{Span}(\vec{a}, \vec{b})$ - yz-plane
$\vec{a} = (0, 1, 0)$
$\vec{b} = (0, 0, 1)$ $c_1\vec{a} + c_2\vec{b}$

line - $\mathbb{R}$ - set of real numbers $x = \sqrt{7}$
plane - $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ - set of (x, y)
3d space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ - set of (x, y, z)

$\text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^3$
$\vec{v}_1 = (1, 0, 0)$
$\vec{v}_2 = (0, 1, 0)$
$\vec{v}_3 = (0, 0, 1)$

$\text{Span}(\vec{v}_1, \vec{v}_2)$ - line

Homogeneous system of equations
$A\vec{x} = \vec{0}$ $A_{n\times m}$ - matrix
$\vec{a}_1x_1 + \vec{a}_2x_2 + \dots + \vec{a}_mx_m = 0$
If $x_1 = \dots = x_n = 0$

$\vec{v} = \left(\frac{4}{3}, 0, 1\right)$

A Nontrivial solution exists if there is at least one free variable

$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$ reduced echelon form augmented matrix $\begin{pmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 - \frac{4}{3}x_3 = 0 \\ x_2 = 0 \end{array}$ ← line in $\mathbb{R}^3$

Set of solutions of $A\vec{x} = 0$ is a line $\begin{cases} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \end{cases} = \text{Span}(\vec{v})$

Nonhomogeneous system
$A\vec{x} = \vec{b}$ $A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$ $A\vec{p} = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix} = \vec{b}$

$\vec{x} = \vec{p} + \vec{v}_h$ ← solution of $A\vec{x} = 0$ $\vec{p} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$
any solution of $Ax = b$

Checking $A\vec{x} = A(\vec{p} + \vec{v}_h) = A\vec{p} + A\vec{v}_h = \vec{0} = \vec{b}$

Linear Independence Def $v_1 \dots v_p \in \mathbb{R}^n$ is linearly independent if the only linear combination of these vectors which is equal to zero is trivial
$c_1\vec{v}_1 + \dots + c_p\vec{v}_p = 0 \Rightarrow c_1 = \dots = c_p = 0$

$v_1 \dots v_p \in \mathbb{R}^n$ is linearly dependent if for $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = 0$ at least one of $c_1, \dots c_p$ is nonzero
(or there is a nontrivial linear combination which is equal to zero)

Ex $\vec{v}_1 = (0, 1)$ $p = 2, n = 2$ in $\mathbb{R}^2$
$\vec{v}_2 = (1, 0)$
$c_1\vec{v}_1 + c_2\vec{v}_2 = 0$ linearly independent
$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array}$

Ex $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $p = 3$ $n = 2$
linearly dependent
$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{0}$
1   -2   1
$\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 - c_3 = 0 \end{cases}$

$A \cdot \vec{c} = \vec{0}$ $p = 10$, $n = 2$ linearly dependent

Ex Assume not parallel to each other $(\vec{v}_1 \dots \vec{v}_p)$ in $\mathbb{R}^n$
$p > n \Rightarrow$ vectors are linearly dependent
$p \le n \Rightarrow$ linearly independent

$A = \boxed{}$ $A = (\vec{a}_1, \vec{a}_2, \dots \vec{a}_m)$ If $\vec{a}_1 \dots \vec{a}_m$ are linearly independent then $A\vec{x} = 0$ has only trivial solution
$\vec{a}_1 \underset{0}{x_1} + \vec{a}_2 \underset{0}{x_2} + \dots + \vec{a}_m \underset{0}{x_m} = 0$

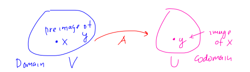
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 = x_3 = 0$

Linear Transformations

$A_{n\times m} =$ n $\boxed{}$ m $A_{n\times m} \cdot X_m$

$A \quad \mathbb{R}^3 \longrightarrow \mathbb{R}^2$
$\vec{x} \longmapsto \vec{y}$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}}_{A} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
$\vec{x} \in \mathbb{R}^3$ $\vec{y} \in \mathbb{R}^2$



Examples $n = m = 2$ $A_{n\times n}$ $\mathbb{R}^n \longrightarrow \mathbb{R}^n$
$\vec{x} \longmapsto \vec{y}$

- Identity transformation $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
$\forall\, \vec{x} \quad 1\vec{x} = \vec{x}$

- Shear transformation $S_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ $S\vec{x} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + \alpha x_2 \\ x_2 \end{pmatrix}$ $\alpha = 0.5$