Hello and welcome to class!

Last time

We discussed matrix arithmetic.

Today

We introduce the notion of linear subspace, and study some linear subspaces naturally associated to a matrix or linear transformation.

Linear subspaces

Definition

A subset $V \subset \mathbb{R}^n$ is a linear subspace if it contains $\mathbf{0}$ and all linear combinations of its elements.

That is, if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are any collection of vectors in V, and $c_1, c_2, \ldots, c_n$ are scalars, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \qquad \text{is in } V$$

Linear subspaces

An equivalent formulation:

For every $\mathbf{v}, \mathbf{w}$ in V and scalars c, d, the vector $c\mathbf{v} + d\mathbf{w}$ is in V.

After all, the expression $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ can be computed by adding only two vectors at a time.

Trivial examples of linear subspaces

The subset $\{\mathbf{0}\} \subset \mathbb{R}^n$ is a linear subspace:

Any linear combination of $\mathbf{0}$'s is again $\mathbf{0}$, hence in the subset.

Trivial examples of linear subspaces

The subset $\mathbb{R}^n \subset \mathbb{R}^n$ is a linear subspace:

It contains $\mathbf{0}$, and any linear combination of vectors in $\mathbb{R}^n$ is some vector in $\mathbb{R}^n$.

Linear subspaces of $\mathbb{R}$

The only linear subspaces of $\mathbb{R}$ are $\{\mathbf{0}\}$ and $\mathbb{R}$.

If $V \subset \mathbb{R}$ is a linear subspace, and v is a nonzero vector in V, then any w in $\mathbb{R}$ can be written as

$$w = \left(\frac{w}{v}\right) \cdot v$$

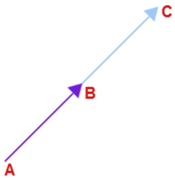
hence is a multiple of v.

Since V is linear and contains v, it must contain w.

Linear subspaces of $\mathbb{R}^2$

Any line through the origin in $\mathbb{R}^2$ is a linear subspace.

It contains zero, and — recall the geometric rule for adding vectors — any sum of vectors on the line remains on the line.



Try it yourself!

Is it a linear subspace of $\mathbb{R}^2$?

The origin. yes

The x-axis. yes

The y-axis. yes

The union of the x-axis and the y-axis. no

Try it yourself!

Is it a linear subspace of $\mathbb{R}^2$?

The hyperbola $xy = 1$ no

The graph of $y = 5x$ yes

The graph of $y = 5x + 3$ no

The graph of $y = sin(x)$ no

Try it yourself!

Is it a linear subspace of $\mathbb{R}^2$?

The circle $x^2 + y^2 = 1$ no

The circle $(x-1)^2 + (y-1)^2 = 1$ no

All of $\mathbb{R}^2$ except the origin no

All of $\mathbb{R}^2$ yes

Linear subspaces of $\mathbb{R}^2$

Say V is a linear subspace of $\mathbb{R}^2$.

If V contains a nonzero vector, then it must contain the linear span of that vector, i.e. the line through it.

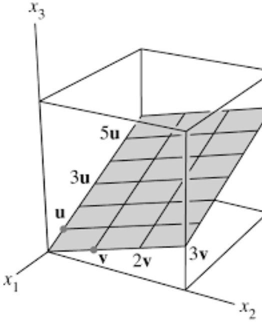
If V contains any two vectors not on the same line through the origin, then since these vector span, V must be all of $\mathbb{R}^2$.

So V must be $\{\mathbf{0}\}$, a line through the origin, or $\mathbb{R}^2$.

Linear subspaces of $\mathbb{R}^3$

Any plane through the origin in $\mathbb{R}^3$ is a linear subspace.

It contains zero, and — recall the geometric rule for adding vectors — any sum of vectors in the plane remains in the plane.



Linear subspaces of $\mathbb{R}^3$

Any linear subspace in $\mathbb{R}^3$ is a $\{\mathbf{0}\}$, a line or plane through the origin, or all of $\mathbb{R}^3$.

Indeed, any nonzero vector spans a line; two non-colinear vectors span a plane, and three non-coplanar vectors span all of $\mathbb{R}^3$.

Linear spans

The linear span of any collection of vectors is a linear subspace.

Indeed, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the vectors,

and $\mathbf{w} = \sum a_i \mathbf{v}_i$ and $\mathbf{x} = \sum b_i \mathbf{v}_i$ are linear combinations of the $\mathbf{v}_i$, then any linear combination of $\mathbf{w}$ and $\mathbf{x}$,

$$c\mathbf{w} + d\mathbf{x} = c\left(\sum a_i \mathbf{v}_i\right) + d\left(\sum b_i \mathbf{x}_i\right) = \sum (ca_i + db_i)\mathbf{v}_i$$

is again a linear combination of the $\mathbf{v}_i$.

Ranges

The range of a linear transformation is a linear subspace.

If $\mathbf{v}$ and $\mathbf{v}'$ are in the range of a transformation T, that means there are some $\mathbf{w}$ and $\mathbf{w}'$ with $T(\mathbf{w}) = \mathbf{v}$ and $T(\mathbf{w}') = \mathbf{v}'$.

So, any linear combination

$$\begin{aligned} c\mathbf{v} + d\mathbf{v}' &= cT(\mathbf{w}) + dT(\mathbf{w}') \\ &= T(c\mathbf{w} + d\mathbf{w}') \end{aligned}$$

is in the range of T.

Column space

If A is the matrix of the linear transformation T, the range of T is the span of the columns of A. We call this the column space of A.

Indeed, these columns are $T(\mathbf{e}_1), \cdots, T(\mathbf{e}_n)$, so for any $\mathbf{v}$ in the range, i.e. $\mathbf{v} = T(\mathbf{w})$, we can write

$$\mathbf{w} = \sum c_i \mathbf{e}_i$$

hence

$$\mathbf{v} = T(\mathbf{w}) = T\left(\sum c_i \mathbf{e}_i\right) = \sum c_i T(\mathbf{e}_i)$$

Column space

Example

In a reduced echelon matrix,

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the column space is the set of vectors with zeroes in non-pivot rows. It's spanned by the pivot columns. The same is true for an echelon matrix.

Null space

The set of vectors sent to zero by a linear transformation is a linear subspace.

Indeed, if $T(\mathbf{v}) = 0$ and $T(\mathbf{w}) = 0$, then

$$T(c\mathbf{v} + d\mathbf{w}) = cT(\mathbf{v}) + dT\mathbf{w} = 0 + 0 = 0$$

This is called the null space of the linear transformation, or of the corresponding matrix.

Bases

Definition

A basis of a linear subspace V is a collection of linearly independent vectors which span V.

Example

The empty set is a basis of the vector space $\{\mathbf{0}\}$

Example

The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ give a basis for $\mathbb{R}^2$

Bases

Example

The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ give a basis for $\mathbb{R}^n$.

Example

In a reduced echelon matrix, we already saw the pivot columns span the column space. Since they're distinct $\mathbf{e}_i$, they are linearly independent, hence give a basis.

Bases

Theorem

Any linear subspace V of $\mathbb{R}^n$ has a finite basis.

Proof (and procedure for finding one):

Begin with the empty set of vectors, then enter the following loop:

- If the current set of vectors spans V, stop.
- Otherwise, take any vector in V but not in the span of the current set, and add it to the set.

Bases

To see that this procedure gives a basis, note first that the set of vectors being built is always linearly independent — at each step, we add a vector not in the linear span of the others.

From this, we conclude that the procedure must terminate: every linearly independent subset of $\mathbb{R}^n$ has at most n elements.

The procedure only terminates once a spanning set for V is found.

Thus every linear subspace of $\mathbb{R}^n$ has a basis.

Finding a basis

If we begin with a finite spanning set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ for V, then the following variant gives an actual algorithm.

To find a basis:

Begin with the empty set of vectors, then enter the following loop:

- If the current set of vectors spans V, stop.
- Otherwise, take the lowest indexed vector among the $\mathbf{v}_i$ not in the span of the current set, and add it to the set.

Finding a basis

To see if $\mathbf{v}_{n+1}$ is in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, one forms the augmented matrix

$$\left[\begin{array}{ccc|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n & \mathbf{v}_{n+1} \end{array}\right]$$

and solves the corresponding system by row reduction.

Instead of doing this each time we want to check linear dependence among some of the $\mathbf{v}_i$, we can do it all at once.

Finding a basis for the column space

A linear dependence between the columns of

$$M = \left[\begin{array}{ccc} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array}\right]$$

is by definition an expression of the form $\sum x_i \mathbf{v}_i = 0$,

which in turn is an equation $M\mathbf{x} = 0$.

Finding a basis for the column space

A sequence of row operations is performed by multiplying M on the left by a sequence of elementary matrices,

$$M \rightsquigarrow E_n E_{n-1} \cdots E_1 M$$

Or writing $R = E_n E_{n-1} \cdots E_1$, simply $M \rightsquigarrow RM$.

Since R is invertible, $RM\mathbf{x} = 0$ if and only if $M\mathbf{x} = 0$, and therefore row operations preserve linear dependencies between the columns.

Finding a basis for the column space

In a reduced echelon matrix, a basis of the column space is given by the by the columns with pivots.

Thus in any matrix, a basis for the column space is given by the columns which will have the pivots after row reduction.

Make sure to use the columns of the original matrix!

Finding a basis for the column space

Example

Find a basis for the space spanned by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$$

The first step is to put them as the columns of a matrix.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 2 & 4 & -1 & 1 & 1 \\ 3 & 6 & -1 & 2 & 4 \\ 1 & 2 & 0 & 1 & 4 \end{bmatrix}$$

Finding a basis for the column space

Now row reduce it

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 2 & 4 & -1 & 1 & 1 \\ 3 & 6 & -1 & 2 & 4 \\ 1 & 2 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 & 7 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding a basis for the column space

The final matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has pivots in the first, third, and fifth columns.

Thus the first, third, and fifth columns of the original matrix

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 2 & 4 & -1 & 1 & 1 \\ 3 & 6 & -1 & 2 & 4 \\ 1 & 2 & 0 & 1 & 4 \end{bmatrix}$$

form a basis for its column space.

Finding a basis for the null space

We can also look for a basis for the null space of a matrix A, or in other words, a basis for the space of solutions to $A\mathbf{x} = 0$.

Actually we already know how to do this: row reduce the augmented matrix $[A \mid 0]$, and then read off the answer.

Finding a basis for the null space

Find a basis for the null space of

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 2 & 4 & -1 & 1 & 1 \\ 3 & 6 & -1 & 2 & 4 \\ 1 & 2 & 0 & 1 & 4 \end{bmatrix}$$

Finding a basis for the null space

First row reduce it

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & -1 & 0 \\ 2 & 4 & -1 & 1 & 1 & 0 \\ 3 & 6 & -1 & 2 & 4 & 0 \\ 1 & 2 & 0 & 1 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 2 & 2 & 7 & 0 \\ 0 & 0 & 1 & 1 & 5 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Finding a basis for the null space

Row reduction doesn't change the space of solutions, so we read them off the reduced echelon matrix:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We introduce free parameters for the non-pivot columns — s and t for columns 2 and 4 — and then read off the answer

$$\begin{aligned} x_1 &= -2s - t \\ x_2 &= s \\ x_3 &= -t \\ x_4 &= t \\ x_5 &= 0 \end{aligned}$$

Finding a basis for the null space

Or in other words,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Or in other words, a basis for the null space is given by

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$