# Branching Formula for $q$-Toda Function of Type $B$

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We present a proof of the explicit formula for the asymptotically free eigenfunctions of the $B_n$ $q$-Toda operator which was conjectured by Ayumu Hoshino and J.S. This formula can be regarded as a branching formula from the $B_n$ $q$-Toda eigenfunction restricted to the $A_{n-1}$ $q$-Toda eigenfunctions. The proof is given by a contigulation relation of the $A_{n-1}$ Toda eigenfunctions and a recursion relation of the branching coefficients.

## 1. $A_{n-1}$ CASE

First we briefly recall the asymptotically free eigenfunctions for $A_{n-1}$ $q$-Toda operator. Let $x = (x_1, \ldots, x_n)$ and $s = (s_1, \ldots, s_n)$ be a pair of $n$-tuples of indeterminates. Let $\delta = (n-1, n-2, \ldots, 0)$, and write $t^\delta u = (t^{n-1}u_1, t^{n-2}u_2, \ldots, u_n)$ etc. for short. Let

$$D^{A_{n-1}}(x|q,t) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i},$$

be the Macdonald operator of type $A_{n-1}$.

**Definition 1.1.** *Set*

$$\mathsf{D}^{A_{n-1}}(x|s|q,t) = x^{-\lambda} D^{A_{n-1}}(x|q,t) x^{\lambda} = \sum_{i=1}^{n} s_i \prod_{j<i} \frac{1 - tx_i/x_j}{1 - x_i/x_j} \prod_{k>i} \frac{1 - x_k/tx_i}{1 - x_k/x_i} T_{q,x_i},$$

*where* $x^\lambda = \prod_i x_i^{\lambda_i}$, *and* $s = q^\lambda t^\delta$ *namely* $s_i = q^{\lambda_i} t^{n-i}$.

**Definition 1.2.** *Set*

$$\mathsf{D}^{A_{n-1}\mathrm{Toda}}(x|s|q) = \sum_{i=1}^{n-1} s_i (1 - x_{i+1}/x_i) T_{q,x_i} + s_n T_{q,x_n}.$$

**Proposition 1.3.** *We have*

$$\mathsf{D}^{A_{n-1}\mathrm{Toda}}(x|s|q) = \lim_{t \to 0} \mathsf{D}^{A_{n-1}}(t^{-\delta} x|s|q,t).$$

**Definition 1.4.** *Let* $\mathsf{M}^{(n)}$ *be the set of strictly upper triangular matrices with nonnegative integer entries:* $\mathsf{M}^{(n)} = \{\theta = (\theta_{i,j})_{1\le i,j\le n} | \theta_{i,j} \in \mathbb{Z}_{\ge 0}, \theta_{i,j} = 0 \text{ if } i \ge j\}$. *Set*

$$
\begin{aligned}
&c_n(\theta; s_1, \cdots, s_n; q, t)\\
=&\prod_{k=2}^{n} \prod_{1\le i\le j\le k-1} \frac{(q^{\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j+1,a})} t s_{j+1}/s_i; q)_{\theta_{i,k}}}{(q^{\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j+1,a})} q s_{j+1}/s_i; q)_{\theta_{i,k}}} \frac{(q^{-\theta_{j,k}+\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j,a})} q s_j/t s_i; q)_{\theta_{i,k}}}{(q^{-\theta_{j,k}+\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j,a})} s_j/s_i; q)_{\theta_{i,k}}}.
\end{aligned}
$$

*Here we have used the notation*

$$
(a; q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a).
$$

**Definition 1.5.** *Define* $f^{A_{n-1}}(x|s|q,t) \in \mathbb{Q}(s,q,t)[[x_2/x_1, \ldots, x_n/x_{n-1}]]$ *by*

$$
f^{A_{n-1}}(x|s|q,t) = \sum_{\theta\in \mathsf{M}^{(n)}} c_n(\theta; s; q, t) \prod_{1\le i<j\le n} (x_j/x_i)^{\theta_{i,j}}.
$$

**Proposition 1.6.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$*, and set* $s = t^\delta q^\lambda$ *(*$s_i = t^{n-i} q^{\lambda_i}$*). Then we have*

$$
\mathsf{D}^{A_{n-1}}(x|s|q,t)\ f^{A_{n-1}}(x|s|q,t) = \sum_{i=1}^{n} s_i\ f^{A_{n-1}}(x|s|q,t).
$$

**Definition 1.7.** *Set*

$$c_n^{\mathrm{Toda}}(\theta; s_1, \cdots, s_n; q) = \lim_{t\to 0} c_n(\theta; s_1, \cdots, s_n; q, t) \prod_{1\le i<j\le n} t^{(j-i)\theta_{i,j}}$$

$$= \prod_{k=2}^{n} \prod_{1\le i\le j\le k-1} \frac{1}{(q^{\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j+1,a})} q s_{j+1}/s_i; q)_{\theta_{i,k}}} \frac{q^{\theta_{i,k}}}{(q^{\theta_{j,k}-\theta_{i,k}-\sum_{a=k+1}^{n}(\theta_{i,a}-\theta_{j,a})} q s_i/s_j; q)_{\theta_{i,k}}},$$

*and*

$$f^{A_{n-1}\mathrm{Toda}}(x|s|q) = \sum_{\theta\in\mathsf{M}^{(n)}} c_n^{\mathrm{Toda}}(\theta; s; q) \prod_{1\le i<j\le n} (x_j/x_i)^{\theta_{i,j}}.$$

**Proposition 1.8.** *We have*

$$\mathsf{D}^{A_{n-1}\mathrm{Toda}}(x|s|q)\ f^{A_{n-1}\mathrm{Toda}}(x|s|q) = \sum_{i=1}^{n} s_i\ f^{A_{n-1}\mathrm{Toda}}(x|s|q).$$

## 2. $B_n$ CASE

**Definition 2.1.** *Set*

$$\begin{aligned} D^{B_n}(x|q,t) =& \sum_{i=1}^{n} \frac{(1-tx_i)}{t^{n-1/2}(1-x_i)} \prod_{j\neq i} \frac{(1-tx_ix_j)(1-tx_i/x_j)}{(1-x_ix_j)(1-x_i/x_j)} T_{q,x_i}^{+1} \\ &+ \sum_{i=1}^{n} \frac{(1-t/x_i)}{t^{n-1/2}(1-1/x_i)} \prod_{j\neq i} \frac{(1-tx_j/x_i)(1-t/x_ix_j)}{(1-x_j/x_i)(1-1/x_ix_j)} T_{q,x_i}^{-1}. \end{aligned}$$

This is a special case of the Koornwinder operator $\mathcal{D}_x(a,b,c,d|q,t)$ to the parameter $(a,b,c,d)=(t,-1,q^{1/2},-q^{1/2})$ as

$$\mathcal{D}_x(t,-1,q^{1/2},-q^{1/2}|q,t) = D^{B_n}(x|q,t) - \frac{t^n-t^{-n}}{t^{1/2}-t^{-1/2}}.$$

**Definition 2.2.** *Set*

$$\begin{aligned} &\mathsf{D}^{B_n}(x|s|q,t) \\ =& \sum_{i=1}^{n} s_i \frac{1-1/tx_i}{1-1/x_i} \prod_{j<i} \frac{(1-1/tx_ix_j)(1-tx_i/x_j)}{(1-1/x_ix_j)(1-x_i/x_j)} \prod_{k>i} \frac{(1-1/tx_ix_k)(1-x_k/tx_i)}{(1-1/x_ix_k)(1-x_k/x_i)} T_{q,x_i}^{+1} \\ &+ \sum_{i=1}^{n} s_i^{-1} \frac{1-t/x_i}{1-1/x_i} \prod_{j<i} \frac{(1-x_i/tx_j)(1-t/x_ix_j)}{(1-x_i/x_j)(1-1/x_ix_j)} \prod_{k>i} \frac{(1-tx_k/x_i)(1-t/x_ix_k)}{(1-x_k/x_i)(1-1/x_ix_k)} T_{q,x_i}^{-1}, \end{aligned}$$

*namely*

$$\mathsf{D}^{B_n}(x|s|q,t) = x^{-\lambda} D^{B_n}(x|q,t) x^{\lambda},$$

*where* $s=q^{\lambda}t^{\delta+1/2}$, $(s_i = q^{\lambda_i}t^{n-i+1/2} \quad (1\le i\le n))$.

**Definition 2.3.** *Set*

$$\begin{aligned} \mathsf{D}^{B_n \mathrm{Toda}}(x|s|q) =& \sum_{i=1}^{n-1} s_i(1-x_{i+1}/x_i)T_{q,x_i} + s_n(1-1/x_n)T_{q,x_n} + \\ &+ s_1^{-1}T_{q,x_1}^{-1} + \sum_{i=2}^{n} s_i^{-1}(1-x_i/x_{i-1})T_{q,x_i}^{-1}. \end{aligned}$$

**Proposition 2.4.** *We have*

$$\mathsf{D}^{B_n \mathrm{Toda}}(x|s|q) = \lim_{t\to 0} \mathsf{D}^{B_n}(t^{-\delta-1}x|s|q,t),$$

*where* $t^{-\delta-1}x$ *means* $(t^{-n}x_1, t^{-n+1}x_2, \ldots, t^{-1}x_n)$.

**Definition 2.5.** *The asymptotically free eigenfunction* $f^{B_n\mathrm{Toda}}(s|x|q)$ *of type* $B_n$ *q-Toda system is defined as*

$$f^{B_n\mathrm{Toda}}(s|x|q) = \sum_{i_1,\ldots,i_n\geq 0} c^{B_n}_{i_1,\ldots,i_n}(s_1,\ldots,s_n,q)(x_2/x_1)^{i_1}\cdots(x_n/x_{n-1})^{i_{n-1}}\cdots(1/x_n)^{i_n},$$

$$\mathsf{D}^{B_n\mathrm{Toda}}(x|s|q)f^{B_n\mathrm{Toda}}(s|x|q) = (s_1+\cdots+s_n+1/s_1+\cdots+1/s_n)f^{B_n\mathrm{Toda}}(s|x|q).$$

**Theorem 2.6.** *Let* $\theta = (\theta_1,\ldots,\theta_n)$*. The* $f^{B_n\mathrm{Toda}}(x|s|q)$ *is expanded in terms of* $f^{A_{n-1}\mathrm{Toda}}(x|s|q)$ *as the following brunching formula*

$$\begin{aligned} &f^{B_n\mathrm{Toda}}(x_1,\ldots,x_n|s_1,\ldots,s_n|q) \\ &= \sum_{\theta_1,\ldots,\theta_n\geq 0} e^{B_n/A_{n-1}}_{\theta}(s|q)\cdot\prod_{i=1}^{n} x_i^{-\theta_i}\cdot f^{A_{n-1}\mathrm{Toda}}(x_1,\ldots,x_n|q^{-\theta_1}s_1,\ldots,q^{-\theta_n}s_n|q), \end{aligned}$$

*where we have*

$$e^{B_n/A_{n-1}}_{\theta}(s|q) = \prod_{k=1}^{n}\frac{q^{(n-k+1)\theta_k}}{(q)_{\theta_k}(q/s_k^2)_{\theta_k}}\cdot\prod_{1\leq i<j\leq n}\frac{1}{(qs_j/s_i)_{\theta_i}(q^{\theta_j-\theta_i}qs_i/s_j)_{\theta_i}}\frac{(q/s_is_j)_{\theta_i+\theta_j}}{(q/s_is_j)_{\theta_i}(q/s_is_j)_{\theta_j}}.$$

**Examples:**

$$e_{\theta}^{B_n/A_{n-1}}(s|q) = \prod_{k=1}^{n} \frac{q^{(n-k+1)\theta_k}}{(q)_{\theta_k}(q/s_k^2)_{\theta_k}} \cdot \prod_{1\le i<j\le n} \frac{1}{(qs_j/s_i)_{\theta_i}(q^{\theta_j-\theta_i}qs_i/s_j)_{\theta_i}} \frac{(q/s_is_j)_{\theta_i+\theta_j}}{(q/s_is_j)_{\theta_i}(q/s_is_j)_{\theta_j}}.$$

We have

$$\begin{aligned}
&e_{(\theta_1,\theta_2)}^{B_2/A_1}(s_1,s_2|q)\\
=&\frac{q^{2\theta_1}}{(q)_{\theta_1}(q/s_1^2)_{\theta_1}}\frac{q^{\theta_2}}{(q)_{\theta_2}(q/s_2^2)_{\theta_2}}\frac{1}{(qs_2/s_1)_{\theta_1}(q^{\theta_2-\theta_1}qs_1/s_2)_{\theta_1}}\frac{(q/s_1s_2)_{\theta_1+\theta_2}}{(q/s_1s_2)_{\theta_1}(q/s_1s_2)_{\theta_2}},
\end{aligned}$$

$$\begin{aligned}
&e_{(\theta_1,\theta_2,\theta_3)}^{B_3/A_2}(s_1,s_2,s_3|q)\\
=&\frac{q^{3\theta_1}}{(q)_{\theta_1}(q/s_1^2)_{\theta_1}}\frac{q^{2\theta_2}}{(q)_{\theta_2}(q/s_2^2)_{\theta_2}}\frac{q^{\theta_3}}{(q)_{\theta_3}(q/s_3^2)_{\theta_3}}\\
&\times\frac{1}{(qs_2/s_1)_{\theta_1}(q^{\theta_2-\theta_1}qs_1/s_2)_{\theta_1}}\frac{1}{(qs_3/s_1)_{\theta_1}(q^{\theta_3-\theta_1}qs_1/s_3)_{\theta_1}}\frac{1}{(qs_3/s_2)_{\theta_2}(q^{\theta_3-\theta_2}qs_2/s_3)_{\theta_2}}\\
&\times\frac{(q/s_1s_2)_{\theta_1+\theta_2}}{(q/s_1s_2)_{\theta_1}(q/s_1s_2)_{\theta_2}}\frac{(q/s_1s_3)_{\theta_1+\theta_3}}{(q/s_1s_3)_{\theta_1}(q/s_1s_3)_{\theta_3}}\frac{(q/s_2s_3)_{\theta_2+\theta_3}}{(q/s_2s_3)_{\theta_2}(q/s_2s_3)_{\theta_3}}.
\end{aligned}$$

## 3. CONTIGUITY RELATION

**Proposition 3.1.** *The $q$-Toda functions of type $A$ satisfy the contiguity relation*

$$f^{A_{n-1}\mathrm{Toda}}(x_1,\dots,x_{n-1},qx_n|s|q)$$

$$=\sum_{k=1}^{n}(-1)^{n-k}\frac{q^{n-k}\prod_{i=k+1}^{n-1}s_i/s_k}{\prod_{i=k+1}^{n}(1-s_i/s_k)(1-qs_i/s_k)}(x_n/x_k)f^{A_{n-1}\mathrm{Toda}}(x_1,\dots,x_n|q^{-\varepsilon_k}\cdot s|q).$$

*Here, we have used the notation*

$$q^{\pm\varepsilon_i}\cdot s=(s_1,\dots s_{i-1},q^{\pm 1}s_i,s_{i+1},\dots,s_n).$$

This is equivalent to the identity

$$\prod_{i=1}^{n-1}a_i=\sum_{k=1}^{n}(s_k/s_n)\frac{\prod_{i=1}^{n-1}(1-a_is_k/s_i)}{\prod_{\substack{1\le i\le n\\ i\neq k}}(1-s_k/s_i)}.$$

## 4. RECURSION RELATIONS FOR $e_\theta^{B_n/A_{n-1}}(s|q)$

**Proposition 4.1.** *We have*

$$\sum_{i=1}^{n}\Big((1-q^{-\theta_i})s_i+(1-q^{\theta_i})s_i^{-1}\Big)\,e_\theta^{B_n/A_{n-1}}(s|q)$$

$$=\sum_{k=1}^{n} s_n\frac{(-1)^{n-k+1}q^{-\theta_n+\delta_{k,n}}q^{n-k}\prod_{i=k+1}^{n-1}(q^{-\theta_i+\theta_k-1}s_i/s_k)}{\prod_{i=k+1}^{n}(1-q^{-\theta_i+\theta_k-1}s_i/s_k)(1-qq^{-\theta_i+\theta_k-1}s_i/s_k)}\,e^{B_n/A_{n-1}}_{(\theta_1\ldots,\theta_k-1,\ldots,\theta_n)}(s|q).$$

This is equivalent to the identity

$$\sum_{i=1}^{n}\big((1-Q_i)s_i+(1-Q_i^{-1})s_i^{-1}\big)=\sum_{k=1}^{n}s_k\frac{\prod_{i=1}^{n}(1-Q_is_i/s_k)(1-Q_i^{-1}/s_is_k)}{\prod_{i\neq k}(1-s_i/s_k)(1-1/s_is_k)}.$$

## 5. PROOF

We want to show

$$f^{B_n \mathrm{Toda}}(x_1,\dots,x_n|s_1,\dots,s_n|q)$$

$$= \sum_{\theta_1,\dots,\theta_n \geq 0} e_\theta^{B_n/A_{n-1}}(s|q) \cdot \prod_{i=1}^n x_i^{-\theta_i} \cdot f^{A_{n-1}\mathrm{Toda}}(x_1,\dots,x_n|q^{-\theta_1}s_1,\dots,q^{-\theta_n}s_n|q).$$

Recall that we have

$$\mathsf{D}^{B_n\mathrm{Toda}}(x|s|q) = \sum_{i=1}^{n-1} s_i(1-x_{i+1}/x_i)T_{q,x_i} + s_n(1-1/x_n)T_{q,x_n} + \\ + s_1^{-1}T_{q,x_1}^{-1} + \sum_{i=2}^{n} s_i^{-1}(1-x_i/x_{i-1})T_{q,x_i}^{-1}.$$

The action of $\mathsf{D}^{B_n\mathrm{Toda}}(x|s|q)$ on the right hand side gives

$$\mathsf{D}^{B_n\mathrm{Toda}}(x|s|q)\cdot(\mathrm{RHS})$$

$$= \sum_{\theta\in\mathbb{Z}^n_{\geq 0}} e_\theta^{B_n/A_{n-1}}(s|q) \prod_{i=1}^n x_i^{-\theta_i} \cdot \Big\{ \mathsf{D}^{A_{n-1}\mathrm{Toda}}(x|s|q) - q^{-\theta_n}s_n/x_n T_{q,x_n}$$

$$+ \mathsf{D}^{A_{n-1}\mathrm{Toda}}((x_{n-i+1}^{-1})_{i=1}^n|(q^{\theta_{n-i+1}}s_{n-i+1}^{-1})_{i=1}^n|q) \Big\} f^{A_{n-1}\mathrm{Toda}}(x|(q^{-\theta_i}s_i)_{i=1}^n|q)$$

$$= \sum_{\theta\in\mathbb{Z}^n_{\geq 0}} e_\theta^{B_n/A_{n-1}}(s|q) \prod_{i=1}^n x_i^{-\theta_i} \cdot \Big\{ \sum_{i=1}^n q^{-\theta_i}s_i + \sum_{i=1}^n q^{\theta_i}s_i^{-1} - q^{-\theta_n}s_n/x_n T_{q,x_n} \Big\} \\ \times f^{A_{n-1}\mathrm{Toda}}(x|(q^{-\theta_i}s_i)|q).$$

Note that we have used the symmetry

$$f^{A_{n-1}\mathrm{Toda}}(x|s|q) = f^{A_{n-1}\mathrm{Toda}}((x_{n-i+1}^{-1})_{1\leq i\leq n}|(s_{n-i+1}^{-1})_{1\leq i\leq n}|q).$$

Using the contiguity relation, we have

$\mathsf{D}^{B_n\mathrm{Toda}}(x|s|q)\cdot(\mathrm{RHS})$

$$= \sum_{\theta\in\mathbb{Z}^n_{\geq 0}} e_\theta^{B_n/A_{n-1}}(s|q)\prod_{i=1}^n x_i^{-\theta_i}\cdot\left\{\sum_{i=1}^n(q^{-\theta_i}s_i+q^{\theta_i}s_i^{-1})f^{A_{n-1}\mathrm{Toda}}(x|(q^{-\theta_l}s_l)|q)\right.$$

$$-\,q^{-\theta_n}\sum_{k=1}^n(-1)^{n-k}(s_n/x_k)\frac{q^{n-k}\prod_{i=k+1}^{n-1}(q^{-\theta_i+\theta_k}s_i/s_k)}{\prod_{i=k+1}^n(1-q^{-\theta_i+\theta_k}s_i/s_k)(1-qq^{-\theta_i+\theta_k}s_i/s_k)}$$

$$\left.\times\, f^{A_{N-1}\mathrm{Toda}}(x|q^{-\varepsilon_k}\cdot(q^{-\theta_l}s_l)_{1\leq l\leq N}|q).\right\}$$

Now we can apply the recursion relations for $e_\theta^{B_n/A_{n-1}}(s|q)$ having the result

$$\mathsf{D}^{B_n\mathrm{Toda}}(x|s|q)\cdot(\mathrm{RHS})=\sum_{i=1}^n(s_i+s_i^{-1})\cdot(\mathrm{RHS}\ )$$

This completes the proof.

**Questions:**

- What do we have for other cases, i.e. for $C_n$ or $D_n$ $q$-Toda?
- How can we study the asymptotically free eigenfunctions of the $B_n, C_n, D_n$ Macdonald systems, or Koornwinder systems with arbitrary $a, b, c, d$ parameters?
- In $A_n$ (affine $A_n$) case, we have good understandings from the point of view of the geometry of the Laumon (affine Laumon) spaces. Can it be possible to find some good scheme of resolutions of singularities concerning the Drinfeld zastava spaces of types $B_n, C_n, D_n$.
- Is it possible to identify $f^{B_n \mathrm{Toda}}(x_1, \ldots, x_n | s_1, \ldots, s_n | q)$ with some kind of Nekrasov partition functions?
- If yes, can we find some (toroidal?) quantum group setting in which we have a certain amount of intertwining operators (topological vertex operators?) creating automatically (as matrix elements) such partition functions?
- In the affine $A_n$ case, it is expected that we have a non-stationary analogue of the Ruijsenaars elliptic system, where the $\mathbb{Z}_n$ orbifold Nekrasov partition functions play the role of the asymptotically free eigenfunctions. Can we expect a sort of non-stationary analogues of the van Diejen, Hikami-Komori elliptic $BC$ systems?
- Are there some DELL versions of the $B_n, C_n, D_n$ $q$-Toda/Macdonald systems?