Opers
Quantum Geometry
& Integrability

Peter Koroteev

Talk at Sheffield University 3/06/2023

My Background

- I work in Representation Theory and Algebraic Geometry with applications to Mathematical Physics, in particular, to Integrable Systems
- A theoretical physicist by training, I have now almost completely switched to pure math. Still, I try to write one or two papers per year in hep-th
- The term `*Physical Mathematics*' (in a nutshell — using string theory/QFT intuition to prove math theorems) is perhaps the most precise two-word description of my research

Current Research

- **Integrable Systems from Algebraic Geometry**
  Enumerative counts for Nakajima quiver varieties, Opers, Geometric Langlands Correspondence.

- **Geometric Representation Theory**
  Quantization by Branes. Algebras from deformation quantization of some nice families of hyperKähler spaces.
  [Gukov, PK, Nawata, Pei, Saberi] Monograph SpringerBriefs

- **Physics and Mathematics of $\mathcal{N}=2$ gauge theories and their stringy origins**
  The BPS/CFT correspondence

Early Career Work

- Cosmology
- Nonperturbative aspects of Supersymmetric Quantum Field Theories
- Condensed Matter applications
- Resurgence in QFT

Papers on AG&Integrability

[arXiv:23xx.xxxxx]
The qDE/IM Correspondence
E. Frenkel, P. Koroteev, A. M. Zeitlin

[arXiv:2208.08031]
The Zoo of Opers and Dualities
P. Koroteev, A. M. Zeitlin

[arXiv:2108.04184] **Crelle Journal**
q-Opers, QQ-systems, and Bethe Ansatz II: Generalized Minors
P. Koroteev, A. M. Zeitlin

[arXiv:2105.00588]
3d Mirror Symmetry for Instanton Moduli Spaces
P. Koroteev, A. M. Zeitlin

[arXiv:2007.11786] **J. Inst. Math. Jussieu**
Toroidal q-Opers
P. Koroteev, A. M. Zeitlin

[arXiv:2002.07344] **JEMS**
q-Opers, QQ-Systems, and Bethe Ansatz
E. Frenkel, P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1805.00986] **Commun.Math.Phys. 381 (2021) 175**
A-type Quiver Varieties and ADHM Moduli Spaces
P. Koroteev

[arXiv:1811.09937] **Commun.Math.Phys. 381 (2021) 641**
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality
P. Koroteev, D. S. Sage, A. M. Zeitlin

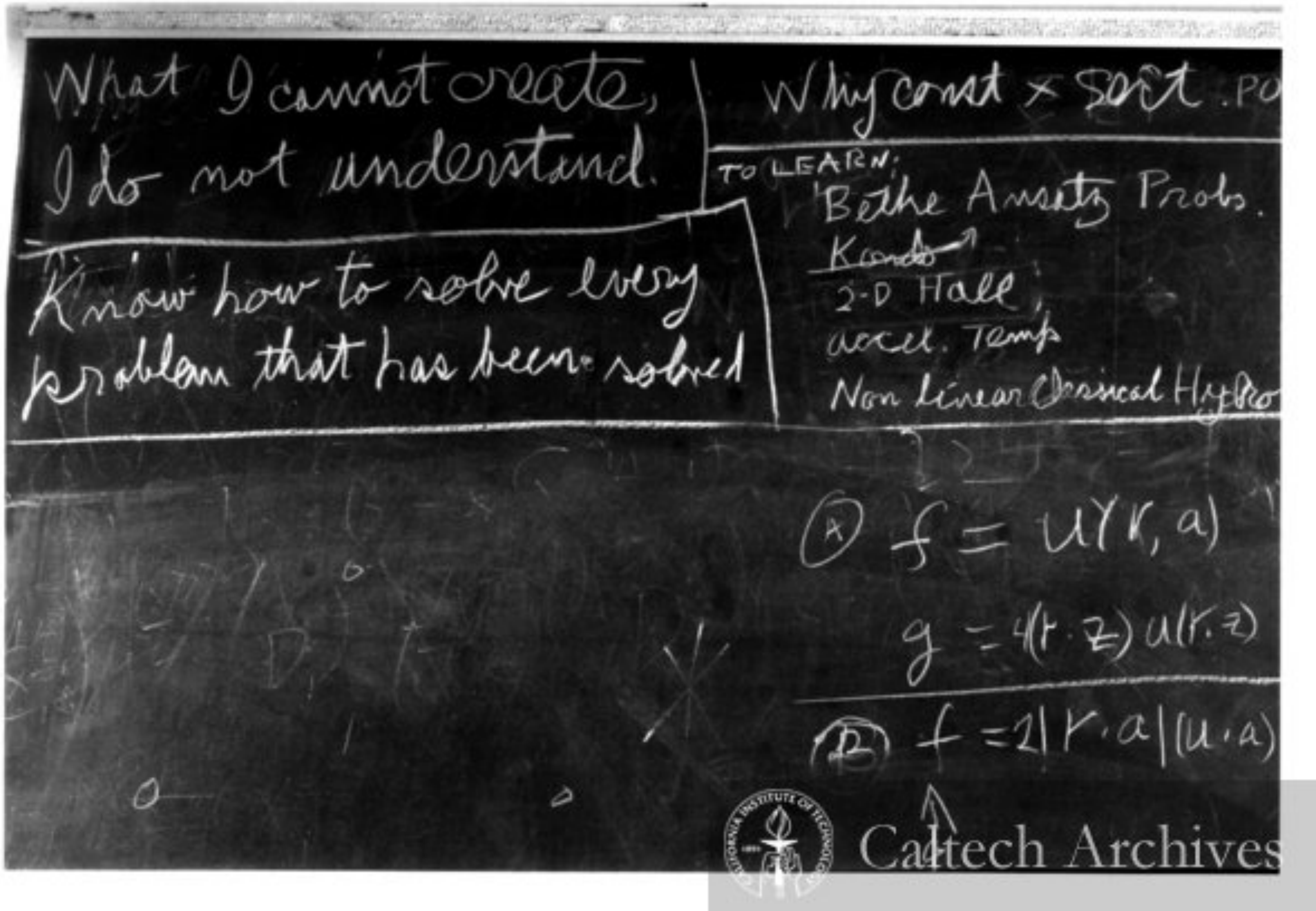
[arXiv:1802.04463] **Math.Res.Lett. 28 (2021) 435**
qKZ/tRS Duality via Quantum K-Theoretic Counts
P. Koroteev, A. M. Zeitlin

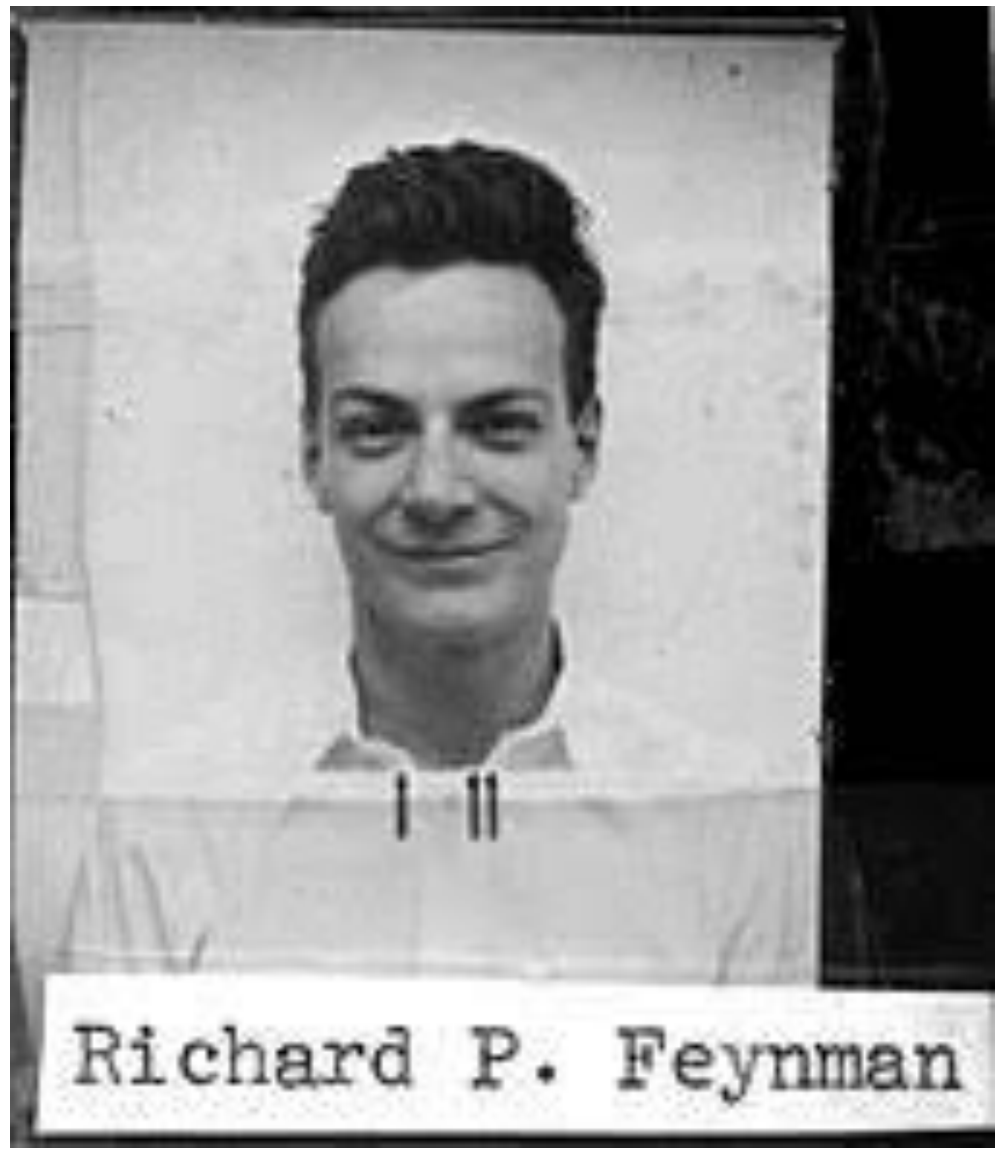
[arXiv:1705.10419] **Selecta Math. 27 (2021) 87**
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

Classical Integrability

- Classical integrable systems of n d.o.f. have n integrals of motion that are in involution with each other $\{H_i, H_j\}_{\text{PB}} = 0$.
- Examples include many-body systems like Calogero, Ruijsenaars, DELL, etc $H_2 = \sum \frac{p_i^2}{2m} + \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$, and continuous (1+1) dimensional models like KdV, Intermediate Long Wave, etc.
- The former can be defined algebraically. The latter admit soliton solutions and are connected to the former. Both were shown to be connected to the Seiberg-Witten solution of $\mathcal{N} = 2$ theories and to geometry

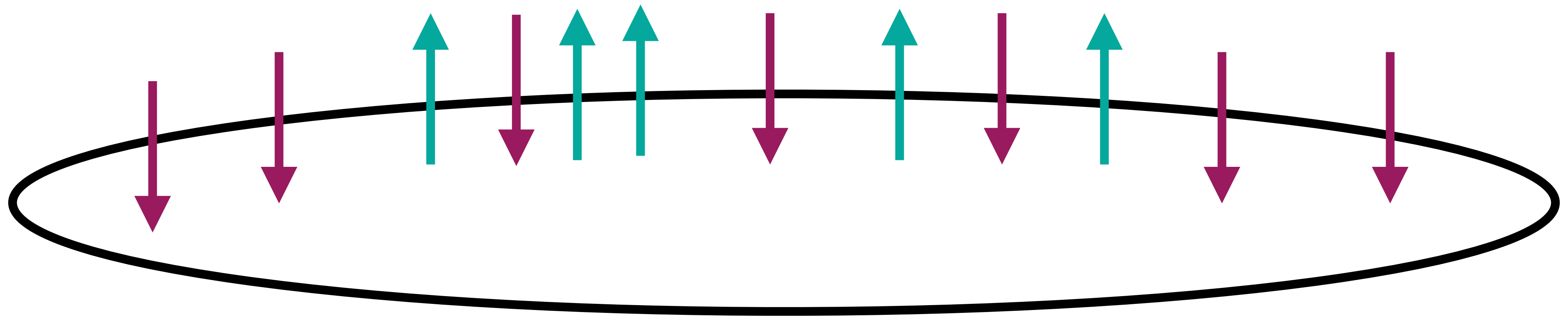
[arXiv:1510.00972] Lett.Math.Phys. **108** (2018) 45
[arXiv:1601.08238] J.Math.Phys. **57** (2016) 112302





I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better.

Quantum XXZ Spin Chain



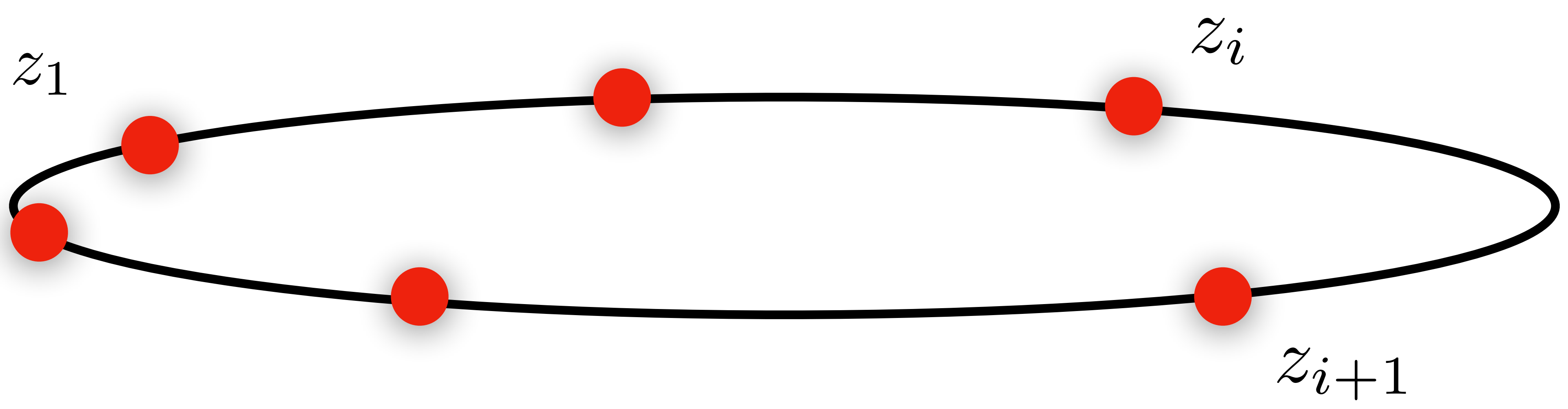
Bethe Ansatz Equations arise during diagonalization of spin chain Hamiltonian in sectors with k excitations: $\exp \frac{\partial Y}{\partial \sigma_i} = 1$

Planck's constant $\hbar$

twist eigenvalues z_i

equivariant parameters (anisotropies) a_i

Classical Many-Body System



Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \dots, n$$

Coupling constant $\hbar$

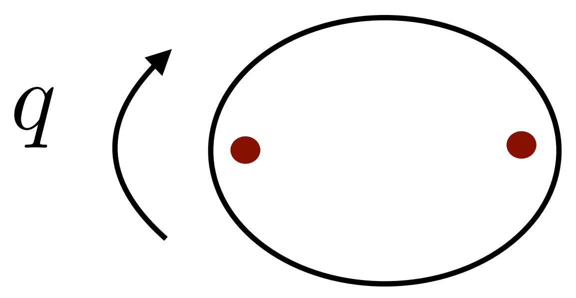
coordinates z_i

energy (eigenvalues of Hamiltonians) $e_i(a_i)$

q-Opers

Riemann sphere with multiplication

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$u \mapsto qu$$



Section $s(u)$

Connection $A(u) : E \to E^q$

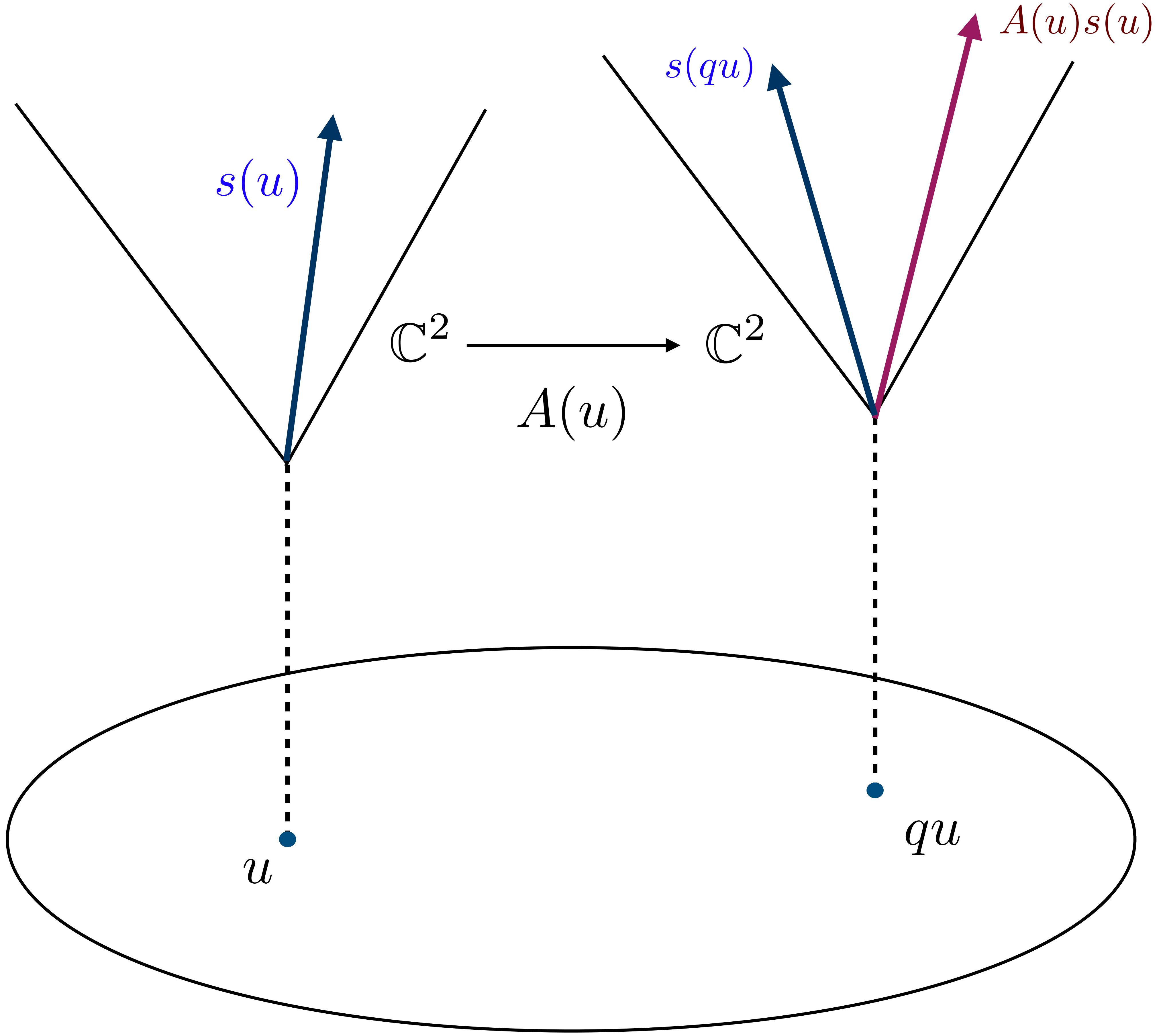
q-gauge transformation

$$A(u) \mapsto g(qu)A(u)g(u)^{-1}$$

(SL(2),q)-oper condition

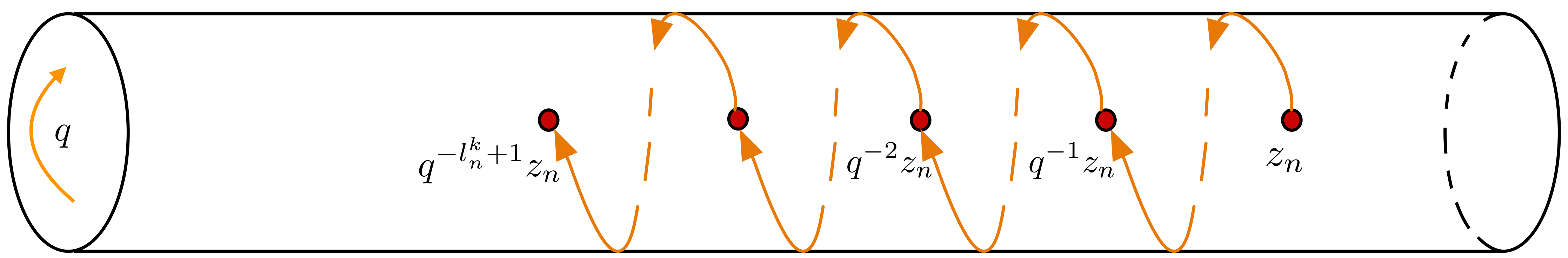
$$s(qu) \wedge A(u)s(u) \neq 0$$

Vector bundle E of rank 2



Singularities and Twists

Allow singularities $s(qu) \wedge A(u)s(u) = \Lambda(u)$ $\Lambda(u) = \prod_{l,j_l}(u - q^{j_l} a_l)$



Add Twists $Z = g(qu)A(u)g(u)^{-1}$

Section $s(u) = \begin{pmatrix} Q_+(u) \\ Q_-(u) \end{pmatrix}$ Twist element $Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition with $A(u) = Z$ – SL(2) QQ-system $\zeta^{-1} Q_+(u) Q_-(qu) - \zeta Q_+(qu) Q_-(u) = \Lambda(u)$

Difference Equation $D_q(s) = As$.

Scalar difference operator $\left(D_q^2 - T(qu) D_q - \frac{\Lambda(qu)}{\Lambda(u)} \right) s_1 = 0$

Trig Ruijsenaars-Schneider Hamiltonians

(SL(2),q)-oper condition

$$\det \begin{pmatrix} Q_+(u) & \zeta Q_+(qu) \\ Q_-(u) & \zeta^{-1} Q_-(qu) \end{pmatrix} = \Lambda(u)$$

Let $\quad Q_+(u) = u - p_+ \qquad Q_-(u) = u - p_-$

$$u^2 - u\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \frac{q\zeta - q\zeta^{-1}}{\zeta^{-1} - \zeta} p_-\right] + p_+ p_- = (u - a_+)(u - a_-)$$

$T_1 \qquad T_2$

qOper condition yields
tRS Hamiltonians!

$$\det(u - T) = (u - a_+)(u - a_-)$$

tRS Model with 2 Particles

Relativistic Hamiltonians

$$T_1 = \frac{\zeta_1 - q\zeta_2}{\zeta_1 - \zeta_2} p_1 + \frac{\zeta_2 - q\zeta_1}{\zeta_2 - \zeta_1} p_2$$

$$T_2 = p_1 p_2$$

Symplectic form

$$\Omega = \sum \frac{dp_i}{p_i} \wedge \frac{d\zeta_i}{\zeta_i}$$

Integrals of motion

$$T_i = E_i$$

Coordinates ζ_i, momenta p_i coupling constant q, energies E_i

Nonrelativistic limit

$$p_i = \exp \frac{P_i}{c} \qquad \zeta_i = \exp \frac{X_i}{c} \qquad T_{\text{Calogero}} = \lim_{c\to\infty} T_{\text{tRS}} - n\, mc^2$$

Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

$$qMT - TM = u \otimes v^T$$

The group $GL(N;\mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

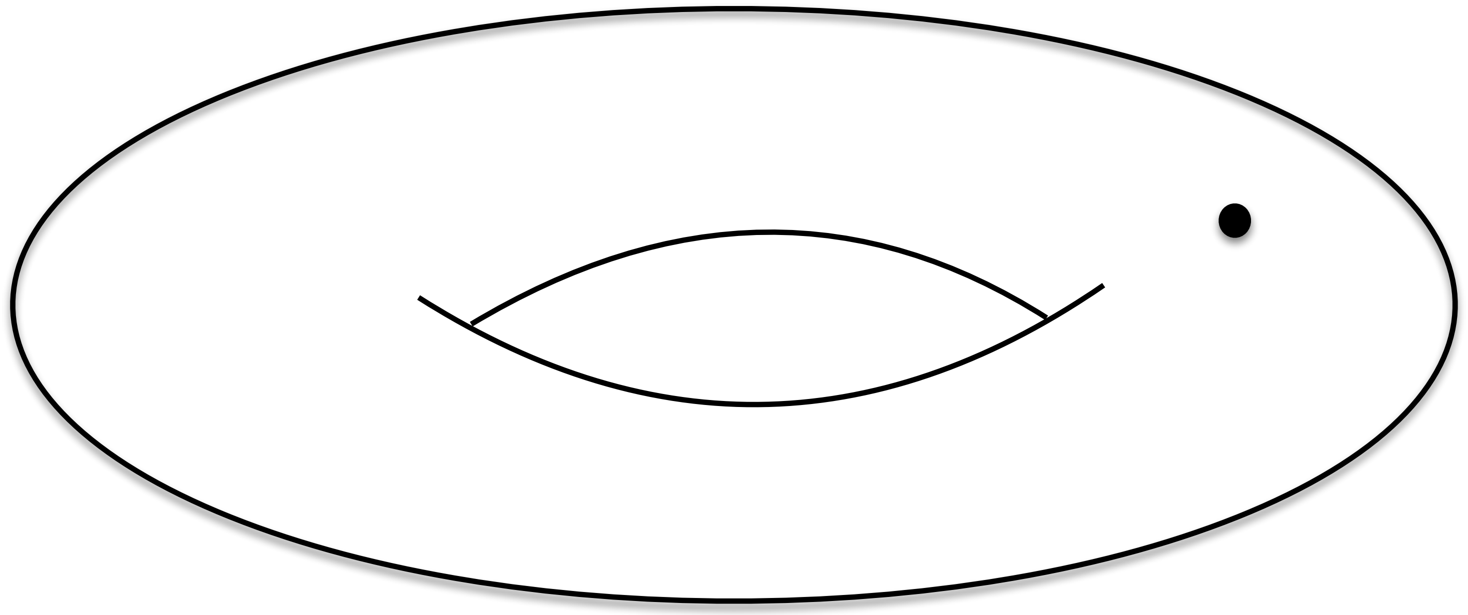
$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space** $\mathscr{M}$

Flat connections on punctured torus

tRS Integrable Hamiltonians are ~TrT^k

T-Lax matrix



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \text{diag}(q, \ldots, q, q^{n-1})$$

[my DAHA paper with Gukov, Nawata, Pei, Saberi
[arXiv:2206.03565] **SpringerBriefs** in press]

XXZ Bethe Equations

Consider the QQ-system equation $$\zeta^{-1}Q_+(u)Q_-(qu) - \zeta Q_+(qu)Q_-(u) = \Lambda(u)$$

Q_+ vanishes at Bethe roots $$Q_+(u) = \prod_{j=k}^{m}(u - s_j)$$

Framing $$\Lambda(u) = \prod_{l,j_l}(u - q^{j_l}a_l)$$

Evaluate QQ at $u = s_i$ $$-\zeta Q_+(qs_i)Q_-(s_i) = \Lambda(s_i)$$

Then at $u = q^{-1}s_i$ $$\zeta^{-1}Q_+(q^{-1}s_i)Q_-(s_i) = \Lambda(q^{-1}s_i)$$

Dividing one by another yields Bethe equations $$\prod_{l=1}^{n}\frac{s_i - q^{r_l}a_l}{s_i - a_l} = \zeta^2 q^k \prod_{j=1}^{k}\frac{qs_i - s_j}{s_i - qs_j}$$

Notice that we did not use Q_- at all

These equations appear as relations in quantum equivariant K-theory of $T^*Gr_{k,n}$ where q scales the cotangent direction

The Ubiquitous QQ-System

Bethe Ansatz equations for XXX, XXZ models — eigenvalues of Baxter operators

[Mukhin, Varchenko] ....

Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Nekrasov-Shatashvili] [Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\hat{g})$

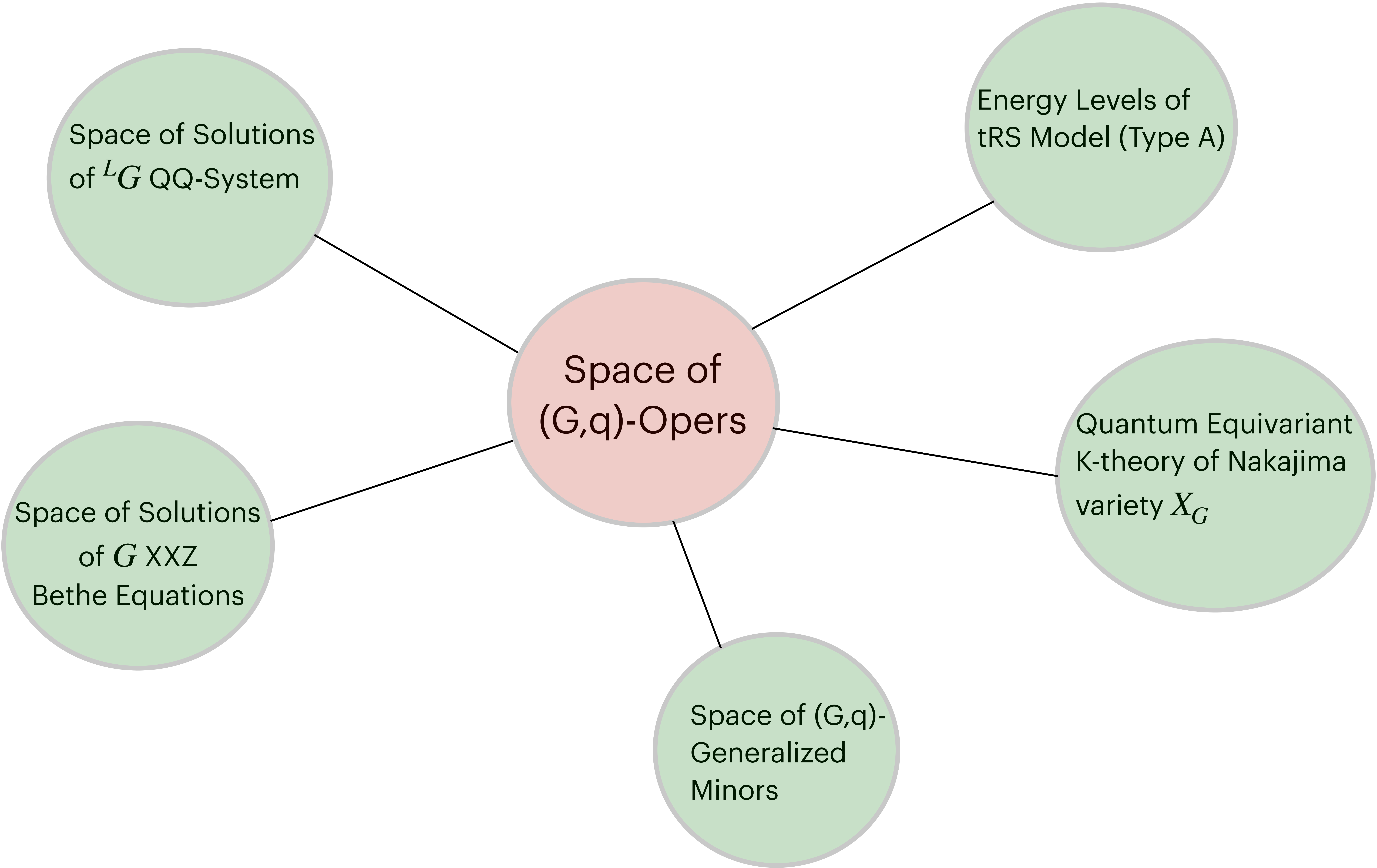
[Frenkel, Hernandez] ....

Spectral determinants in the QDE/IM Correspondence

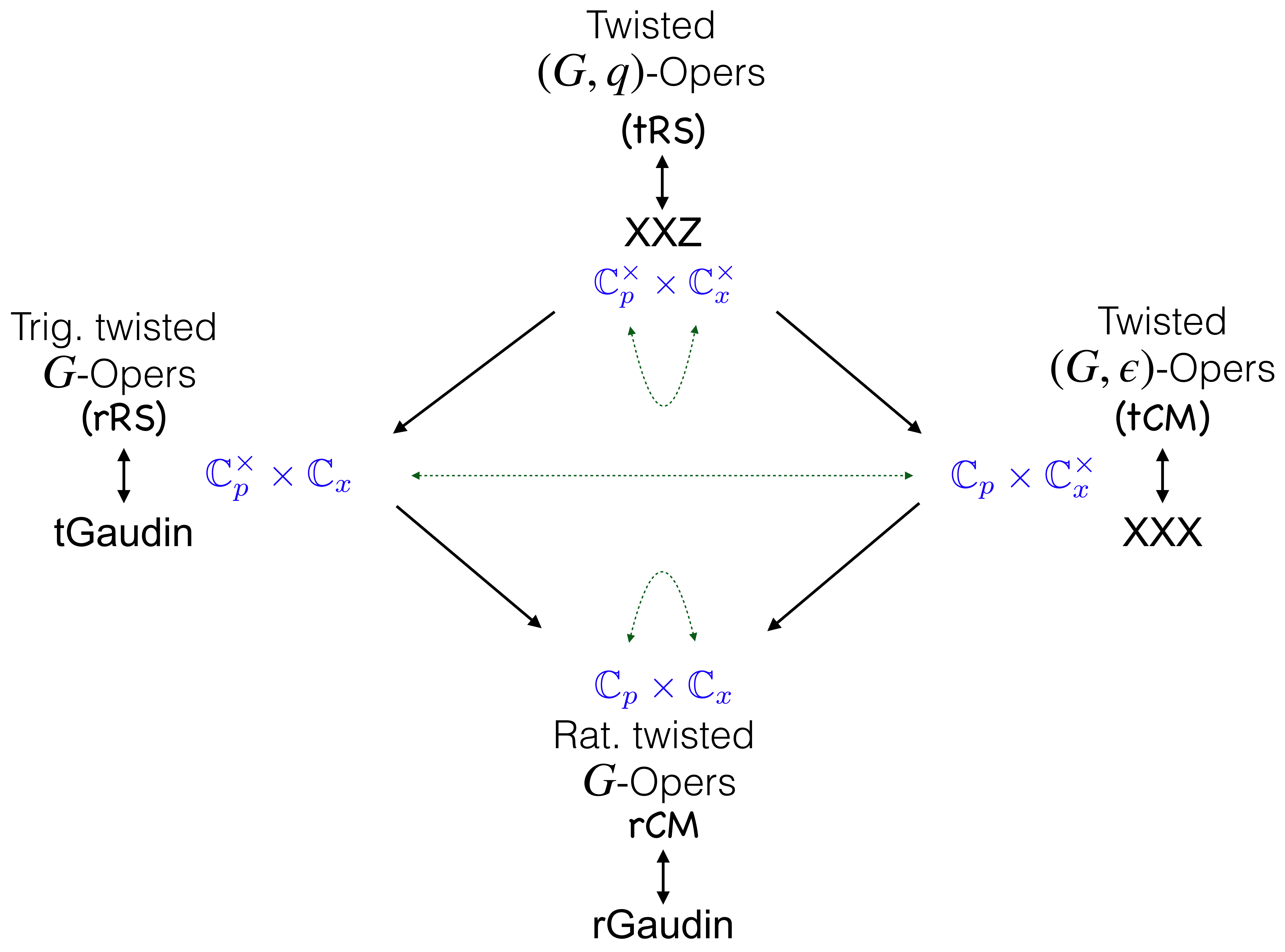
[Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

Relations between generalized minors in cluster algebra calculations

[Fomin Zelevinski][PK Zeitlin]


Space of Solutions
of ^{L}G QQ-System
Energy Levels of
tRS Model (Type A)
Space of
(G,q)-Opers
Quantum Equivariant
K-theory of Nakajima
variety X_G
Space of Solutions
of G XXZ
Bethe Equations
Space of (G,q)-
Generalized
Minors

Network of Dualities



(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathfrak{F}_G, A, \mathfrak{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathfrak{F}_{B_-}$ is a reduction of $\mathfrak{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathfrak{F}_G \longrightarrow \mathfrak{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally $$A(u) = n'(u)\prod_i (\phi_i(u)_i^{\check{\alpha}} s_i)\, n(u)$$

$$\phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N_-(u) = [B_-(u), B_-(u)]$$

q-Opers and q-Langlands

[Frenkel, PK, Zeitlin, Sage, 2021, to appear in **JEMS**]

Miura (G,q)-oper with singularities $A(u) = \prod_i g_i(u)^{\check{\alpha}_i} e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}$

Theorem: There is a 1-to-1 correspondence between the set of nondegenerate Z-twisted (G,q)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

$$\widetilde{\xi}_i\, Q^i_-(u)Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u)Q^i_+(u) = \Lambda_i(u) \prod_{j>i} \left[Q^j_+(\hbar u)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(u)\right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

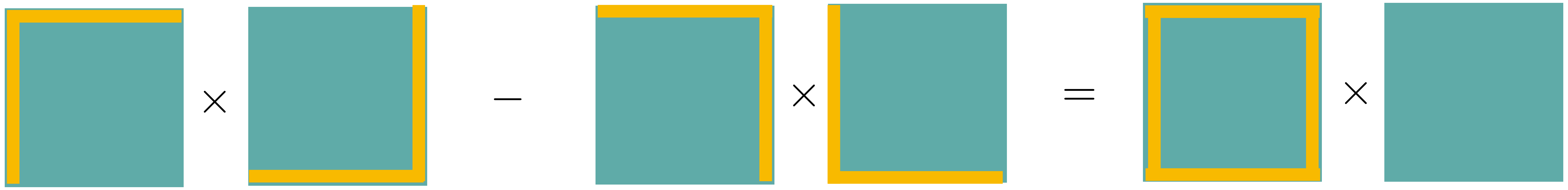
$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Cluster Algerbras

[PK, Zeitlin, 2022, J.Reine Angew.Math. **2023** 271]

The QQ-system $\xi_{i+1}\, Q^i_-(u) Q^i_+(u+\epsilon) - \xi_i Q^i_-(u+\epsilon) Q^i_+(u) = \Lambda_i(u) Q^{i+1}_+(u+\epsilon) Q^{i+1}_+(u)$

For $G = SL(n)$ obtain Lewis Carroll (Desnanot-Jacobi-Trudi) identity $M^1_1 M^2_i - M^1_i M^2_1 = M^{12}_{1i} M$



For general G obtain relation on generalized minors $\Delta^{\omega_i}(v(u)) = Q^i_+(u)$ [Fomin Zelevinsky]

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i} \Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i} \Delta_{u\cdot\omega_i, vw_i\cdot\omega_i} = \prod_{j\neq i} \Delta^{-a_{ji}}_{u\cdot\omega_j, v\cdot\omega_j},$$

$u, v \in W_G$

q-Langlands Correspondence

[Aganagic Frenkel Okounkov]

Two types of solutions of the qKZ equation:

Analytic in chamber of equivariant parameters $\{a_i\}$— conformal blocks of $U_\hbar(\widehat{g})$

Analytic in chamber of quantum parameters (twists) $\{\zeta_i\}$— conformal blocks for deformed W-algebra $W_{q,\hbar}({}^L\widehat{g})$

The q-Langlands correspondence

$U_\hbar(\widehat{g})$ ⟷ $W_{q,\hbar}({}^L\widehat{g})$

$\downarrow$ (left) ; $\downarrow$ $q \to 1$ (right)

XXZ Bethe ; $(G, \hbar)$-oper

Equivalence of categories

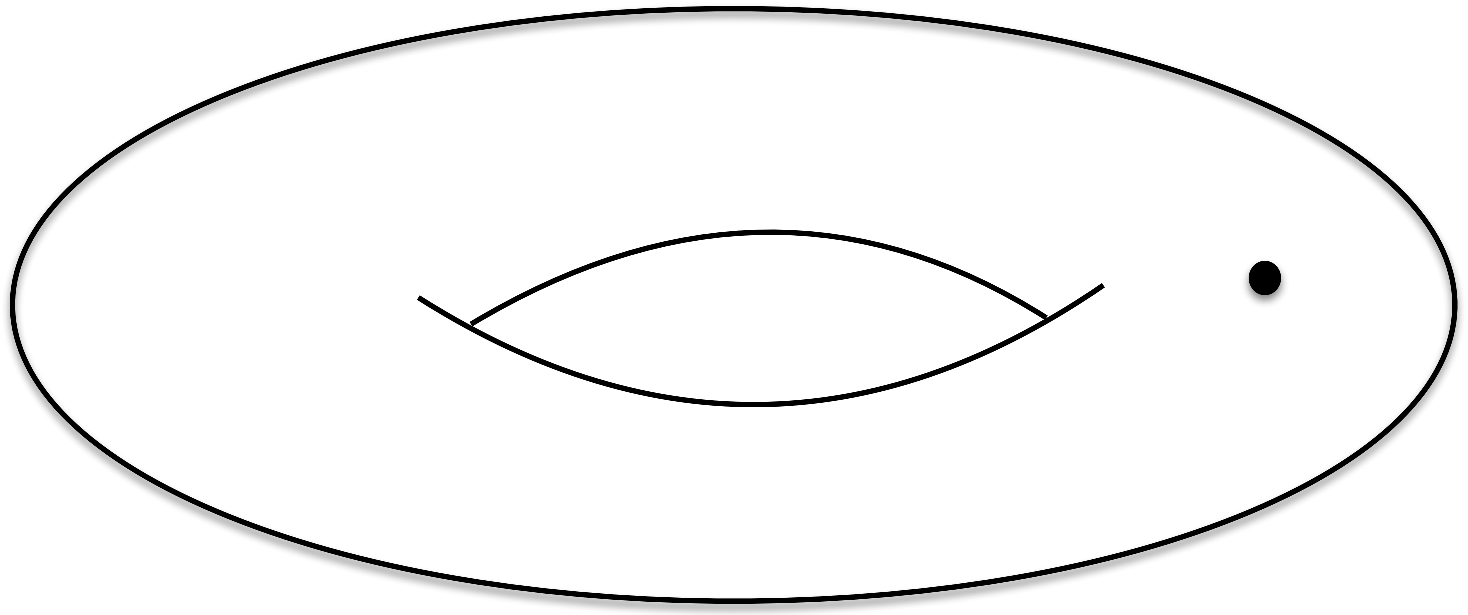
$\mathrm{Dmod}_\kappa(Bun_G)$ ⟷ $\mathrm{Dmod}_{-\frac{1}{m\kappa}}(Bun_{{}^LG})$

Book | May 2023

Branes and DAHA Representations

• **Authors:** Du Pei , Ingmar Saberi , Peter Koroteev , Satoshi Nawata , Sergei Gukov

Geometric representation theory of double affine Hecke algebra (DAHA) in terms of Hitchin moduli space of once-punctured torus



Deformation Quantization

Spherical $\mathfrak{sl}_2$ DAHA

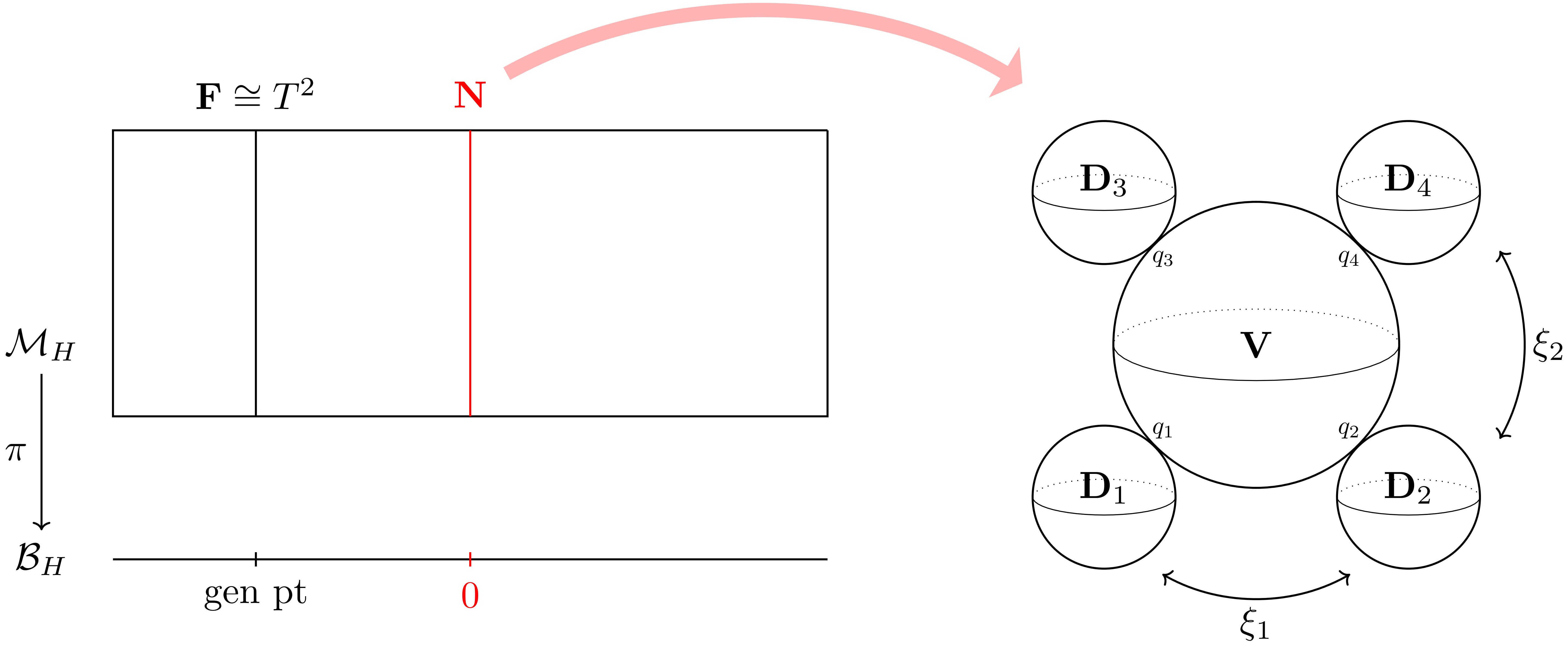
(line ops in $\mathcal{N} = 2^*$ theory)

$$qxy - yx = (q - q^{-1})z + \text{cyclic}$$

$$\rho : \pi_1(C_p) \to \mathrm{SL}(2, \mathbb{C})$$

$$x = \mathrm{Tr}(\rho(\mathfrak{m})),\ y = \mathrm{Tr}(\rho(\mathfrak{l})),\ \text{and } z = \mathrm{Tr}\big(\rho(\mathfrak{m}\mathfrak{l}^{-1})\big)$$

Wilson   't Hooft   Dyonic



Categorification

$$\mathbf{Hom}(\mathcal{B}_{cc}, -) : D^b\mathbf{ABrane}(\mathcal{X}) \longrightarrow D^b\mathbf{Rep}(\mathcal{O}^q(\mathcal{X}))$$