Integrability Enumerative Counts & Opers

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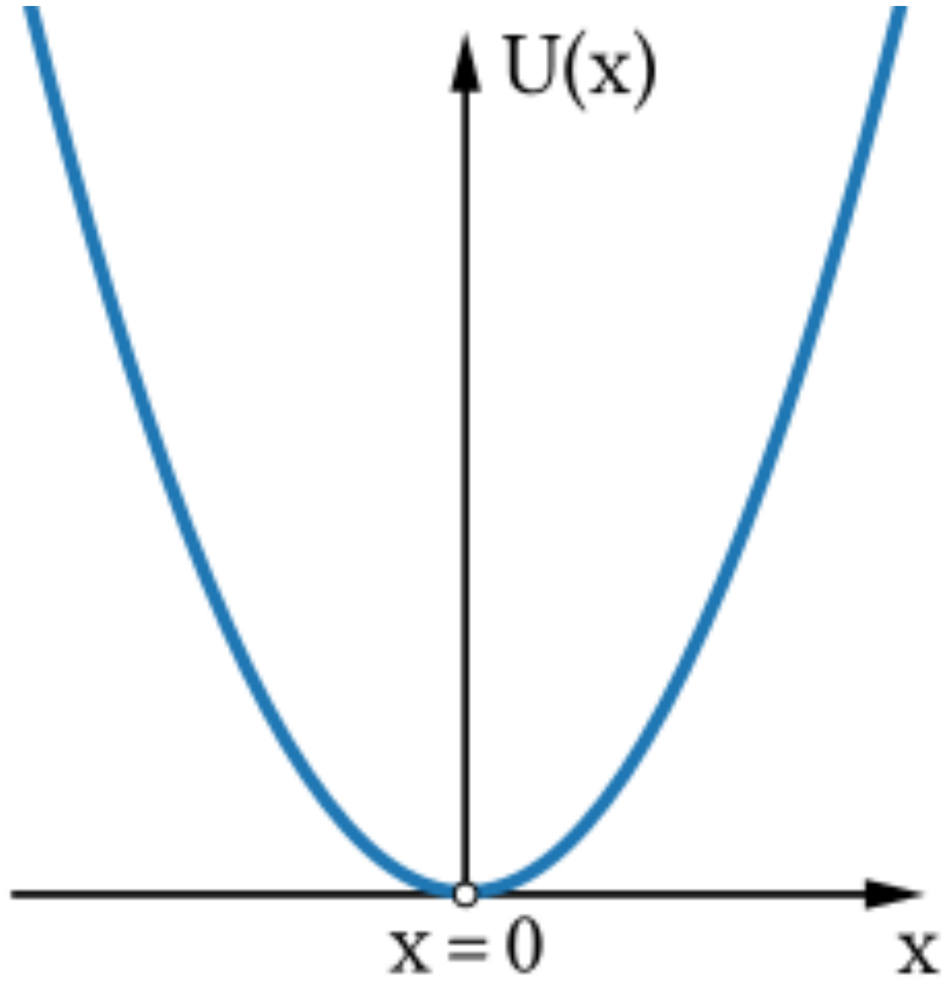
Talk at Ohio State University 1/23/2023

Symplectic Geometry

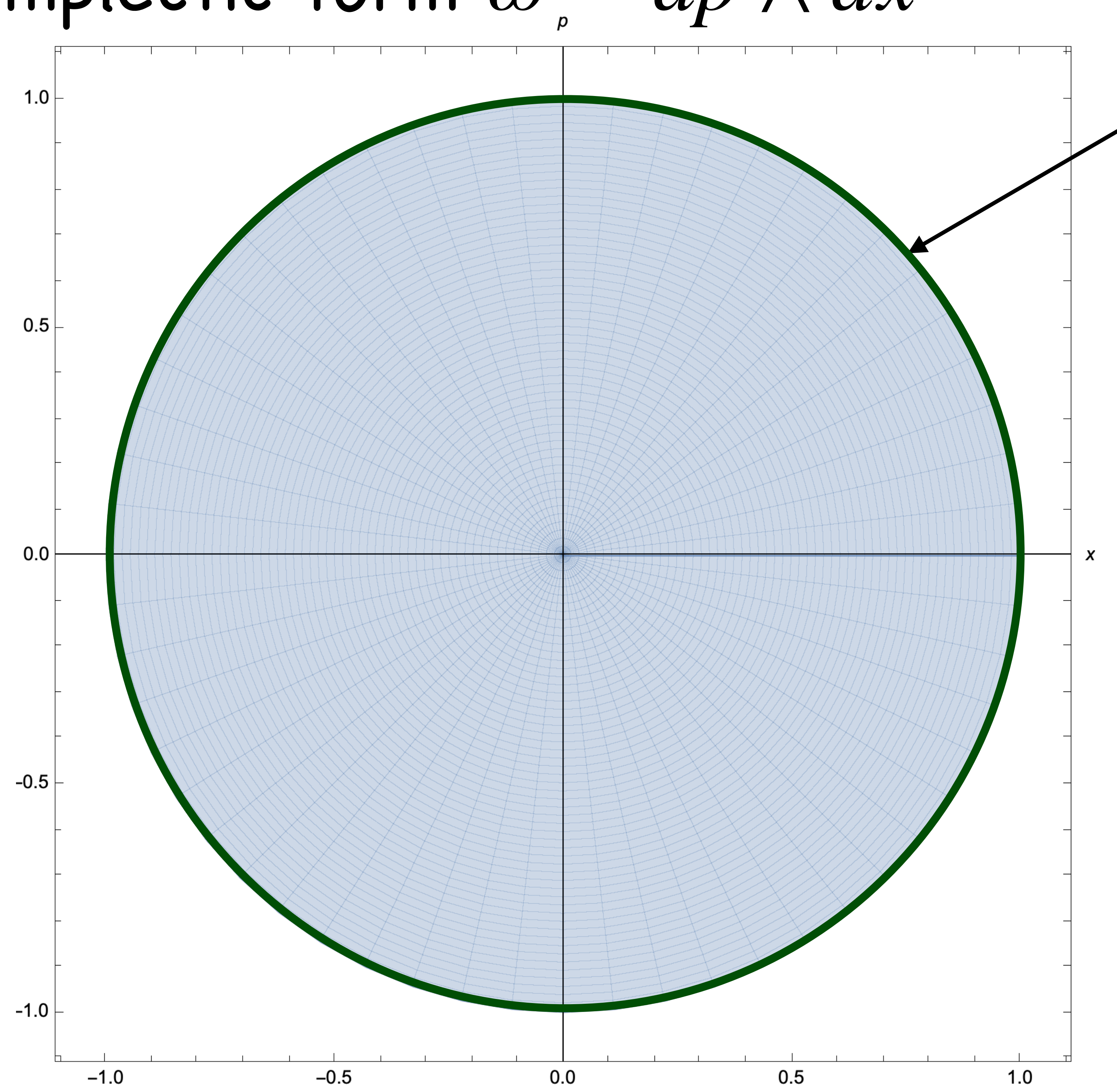
Harmonic oscillator $H = \frac{p^2}{2} + \frac{x^2}{2}$



Phase space – symplectic manifold $\mathscr{M}$

Symplectic form $\omega = dp \wedge dx$

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$



Lagrangian $\mathscr{L} \subset \mathscr{M}$ is a middle-dimensional submanifold and such that the restriction of the symplectic form on $\mathscr{L}$ vanishes $\omega|_{\mathcal{L}} = 0$

Classical Integrability

Equations of motion

$$\frac{df}{dt} = \{H_1, f\}$$

Integrability — family of n conserved quantities which Poisson commute with each other

$$\{H_i, H_j\} = 0 \quad i, j = 1, \ldots, n$$

Liouville-Arnold Theorem

Compact Lagrangians $\mathscr{L}$: $\{H_i = E_i\}$ are isomorphic to tori

Evolution in the neighborhood of $\mathscr{L}$ is linearized in action/angle variables $\{I_i, \varphi_i\}_{i=1}^{n}$

$$\frac{d\varphi_i}{dt} = \omega_i, \qquad \frac{dI_i}{dt} = 0$$

Action/angle variables are hard to find

History (1960–current)

Many-body integrable systems — Calogero, Toda, Ruijsenaars (more on this later)

Continuous integrable models in (1+1)-dimensions: Korteweg-de-Vries, Intermediate Long-Wave, etc.

$$u_t = 6uu_x - u_{xxx}$$

They admit soliton solutions. Sectors with N solitons are described by finite N-body integrable systems

[my work on (1+1) hydro with Scirappa]

[arXiv:1510.00972] Lett.Math.Phys. **108** (2018) 45

[arXiv:1601.08238] J.Math.Phys. **57** (2016) 112302

Inverse scattering method — Lax pair data → action/angle variables

Quantization

Coordinates and momenta become operators

$$p, x \mapsto \hat{p}, \hat{x}$$

Poisson brackets associated to ω become commutators

$$\{A, B\}_{P.B.} \mapsto [A, B]$$

Heisenberg algebra

$$[\hat{p}, \hat{x}] = -i\hbar$$

$$\hat{x} f(x) = x f(x)$$
$$\hat{p} f(x) = -i\hbar f'(x)$$

Lagrangian constraint

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$

Replaced by operator

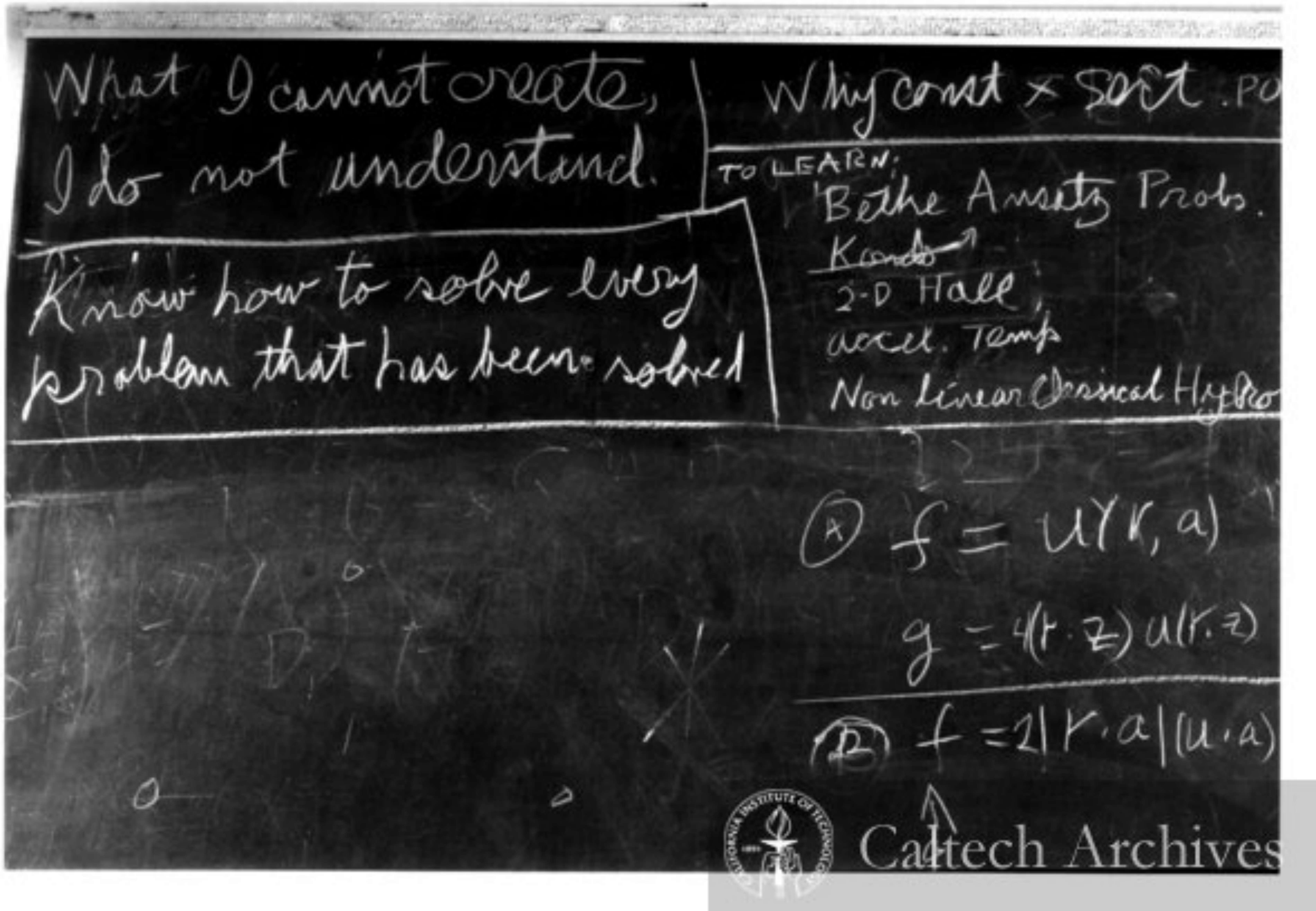
$$\left(\frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} - E\right) Z(x) = 0$$

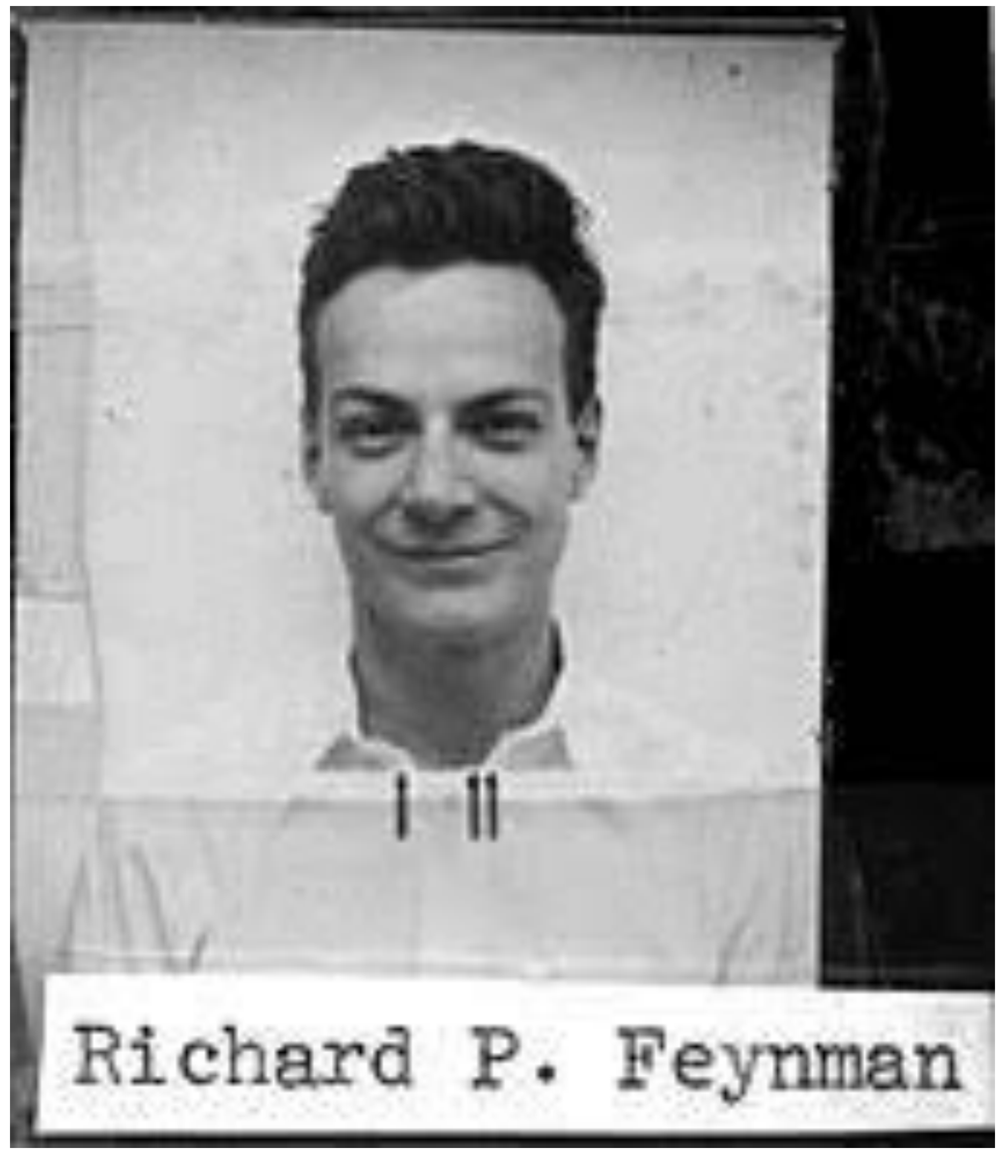
Integrability

$$[H_i, H_j] = 0$$

$$H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables — simultaneous diagonalization of H_i





I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better.

Motivation

Enumerative Algebraic Geometry

[Givental, Kim][Okounkov]
[Givental, Lee][Pushkar, Zeitlin, Smirnov]
[PK, Pushkar, Smirnov, Zeitlin]

Geometric (q-)Langlands Correspondence

[Frenkel] [Aganagic, Frenkel, Okounkov]
[Frenkel, PK, Sage, Zeitlin]

Quantum/Classical Integrable Systems

[PK, Gaiotto][PK, Zeitlin] [Matsuo, Cherednik]
[Bazhanov, Lukyanov, Zamolodchikov]
[Dorey, Tateo]
[Frenkel, PK, Zeitlin, in progress]

(G,q)-Opers

Literature

[arXiv:23xx.xxxxx]
The qDE/IM Correspondence
E. Frenkel, P. Koroteev, A. M. Zeitlin

[arXiv:2208.08031]
The Zoo of Opers and Dualities
P. Koroteev, A. M. Zeitlin

[arXiv:2108.04184] **Crelle Journal**
q-Opers, QQ-systems, and Bethe Ansatz II: Generalized Minors
P. Koroteev, A. M. Zeitlin

[arXiv:2105.00588]
3d Mirror Symmetry for Instanton Moduli Spaces
P. Koroteev, A. M. Zeitlin

[arXiv:2007.11786] **J. Inst. Math. Jussieu**
Toroidal q-Opers
P. Koroteev, A. M. Zeitlin

[arXiv:2002.07344] **JEMS**
q-Opers, QQ-Systems, and Bethe Ansatz
E. Frenkel, P. Koroteev, D. S. Sage, A. M. Zeitlin

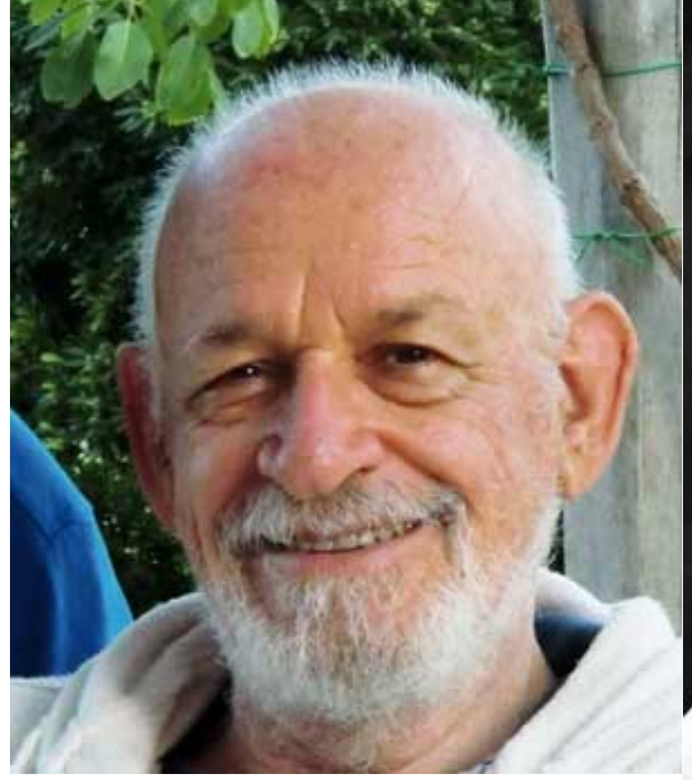
[arXiv:1805.00986] **Commun.Math.Phys. 381 (2021) 175**
A-type Quiver Varieties and ADHM Moduli Spaces
P. Koroteev

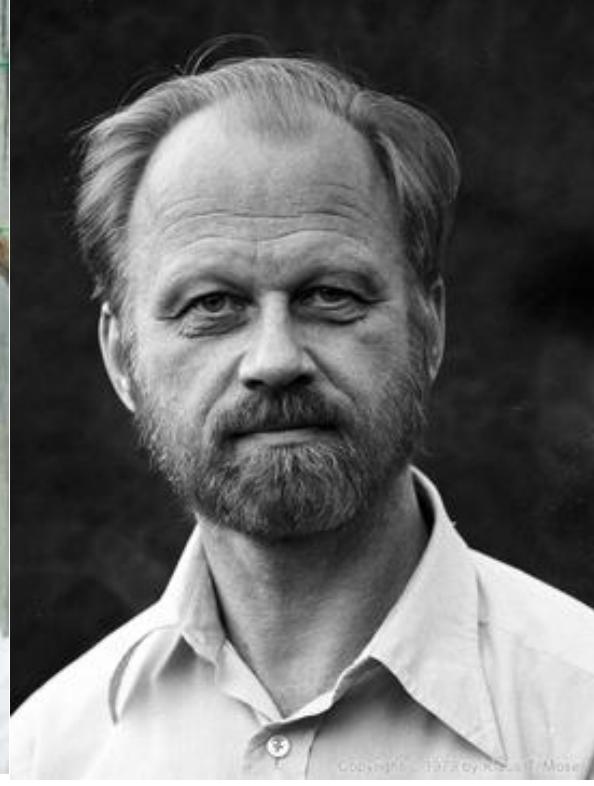
[arXiv:1811.09937] **Commun.Math.Phys. 381 (2021) 641**
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality
P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1802.04463] **Math.Res.Lett. 28 (2021) 435**
qKZ/tRS Duality via Quantum K-Theoretic Counts
P. Koroteev, A. M. Zeitlin

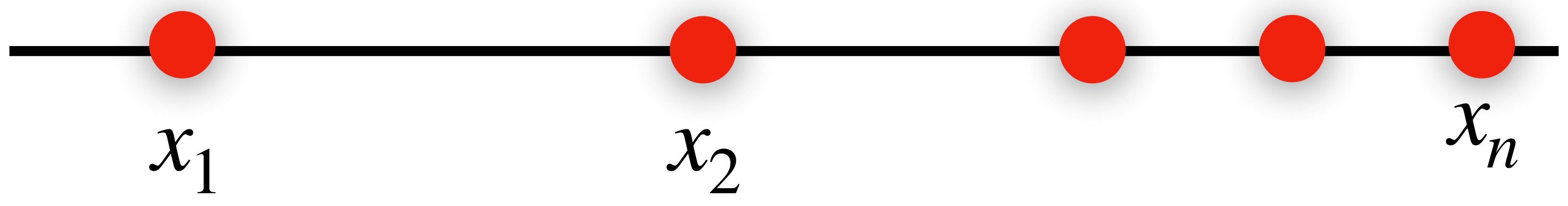
[arXiv:1705.10419] **Selecta Math. 27 (2021) 87**
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

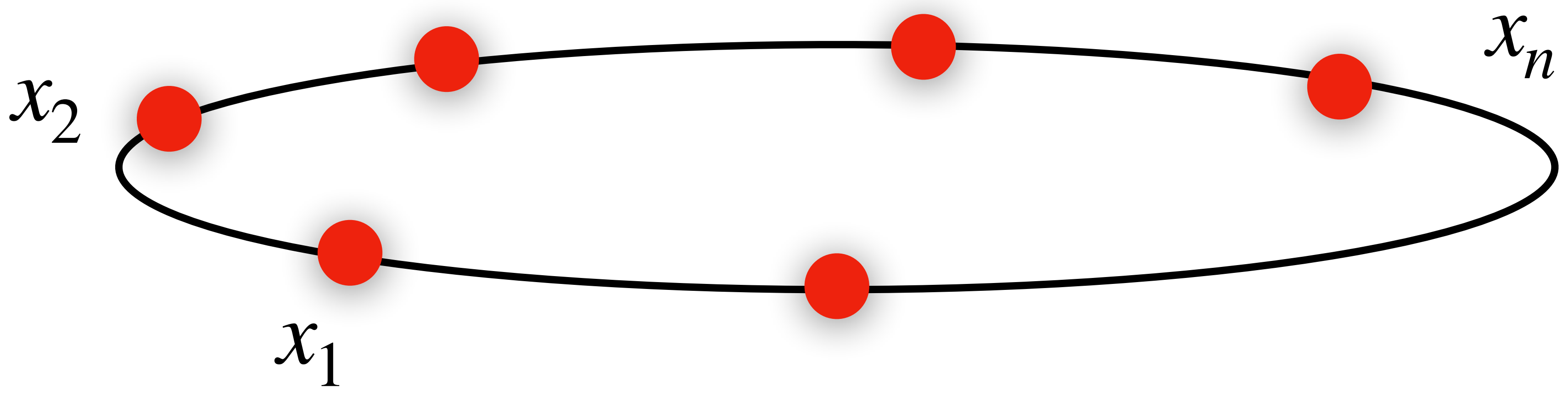
I. Many-Body Systems





Calogero in 1971 introduced a new integrable system. Moser in 1975 proved its integrability using Lax pair

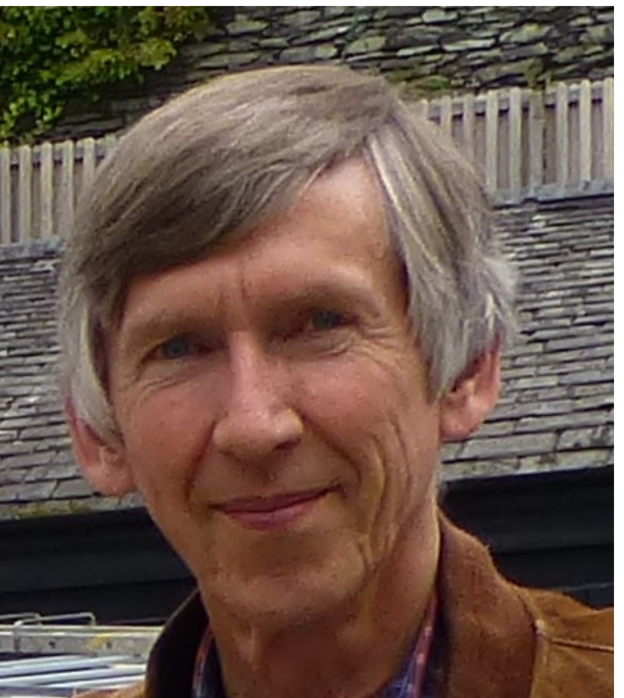




$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$

The **Calogero-Moser (CM)** system has several generalizations: rational CM $\rightarrow$ trigonometric CM $\rightarrow$ elliptic CM

$$V(x) \simeq \sum \frac{1}{(x_i - x_j)^2} \qquad V(x) \simeq \sum \frac{1}{\sinh(x_i - x_j)^2} \qquad V(x) \simeq \wp(x_j - x_i)$$



Another relativistic generalization called **Ruijsenaars-Schneider (RS)** family rRS $\rightarrow$ tRS $\rightarrow$ eRS

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

Example: tRS Model with 2 Particles

Hamiltonians

$$T_1 = \frac{\xi_1 - t\xi_2}{\xi_1 - \xi_2} p_1 + \frac{\xi_2 - t\xi_1}{\xi_2 - \xi_1} p_2$$

$$T_2 = p_1 p_2$$

Symplectic form

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{d\xi_i}{\xi_i}$$

Integrals of motion

$$T_i = E_i$$

Coordinates ξ_i, momenta p_i

coupling constant t, energies E_i

Quantization

$$p_i \xi_j = \xi_j p_i q^{\delta_{ij}} \qquad q \in \mathbb{C}^\times$$

tRS Momenta are shift operators

$$p_i f(\xi_i) = f(q\xi_i)$$

Eigenvalue Equations

$$T_i V = E_i V$$

Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

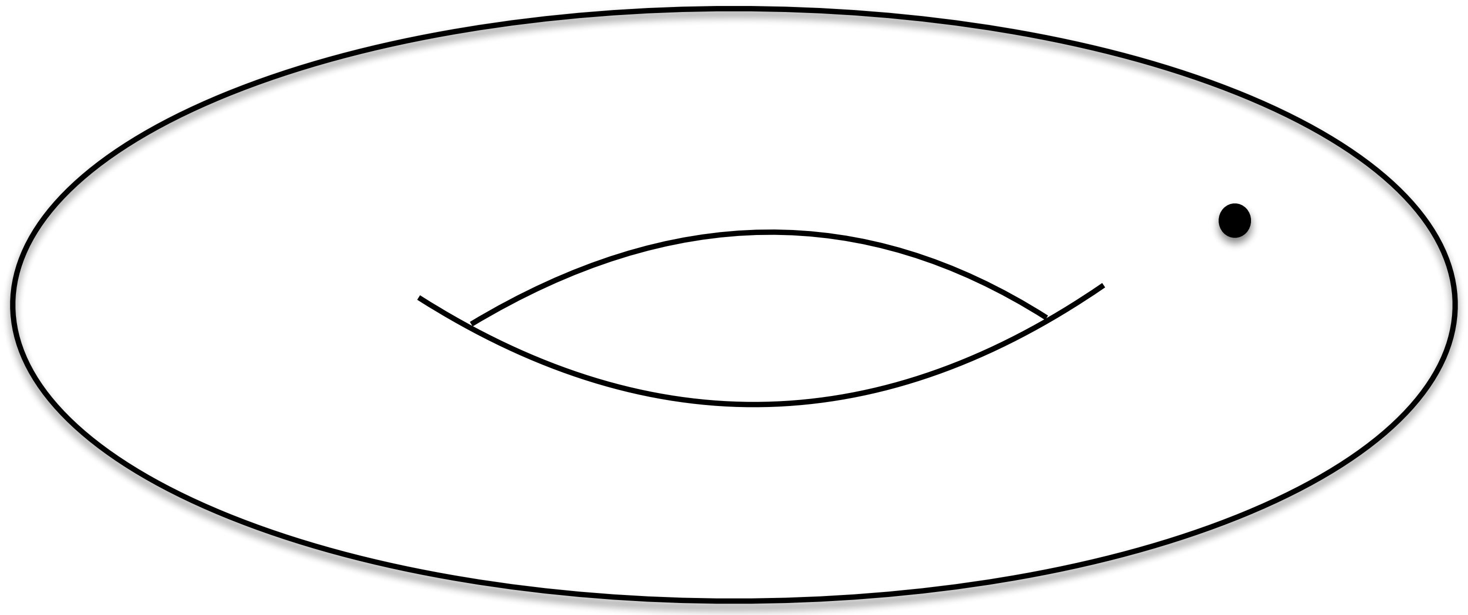
$$qMT - TM = u \otimes v^T$$

The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space** $\mathscr{M}$

Flat connections on punctured torus



$$\mathcal{M}_n = \{A, B, C\}/GL(n; \mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \text{diag}(q, \ldots, q, q^{n-1})$$

Integrable Hamiltonians are ~TrT^k

T-Lax matrix

[my DAHA paper with Gukov, Nawata, Pei, Saberi [arXiv:2206.03565] **SpringerBriefs** in press]

II. Quantum Integrability

Let $\mathfrak{g}$ Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}(t)$ loop algebra (Laurent poly valued in g)

Evaluation modules form a tensor category of $\hat{\mathfrak{g}}$

$$V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

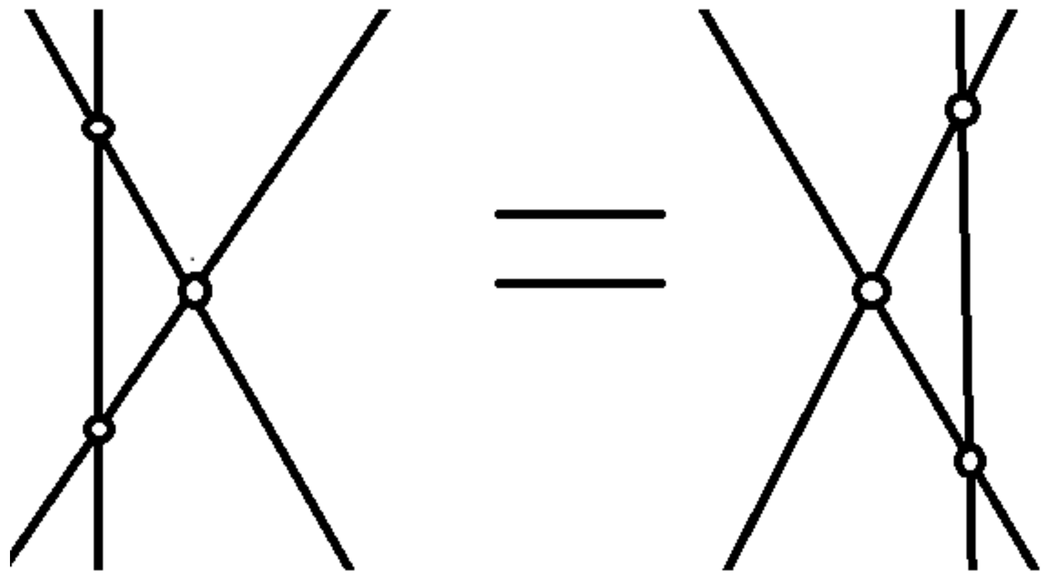
V_i are representations of $\mathfrak{g}$ a_i are special values of spectral parameter t

Quantum group is a noncommutative deformation $U_\hbar(\hat{\mathfrak{g}})$

with a nontrivial intertwiner — R-matrix

$$R_{V_1,V_2}(a_1/a_2) : V_1(a_1) \otimes V_2(a_2) \to V_2(a_2) \otimes V_1(a_1)$$

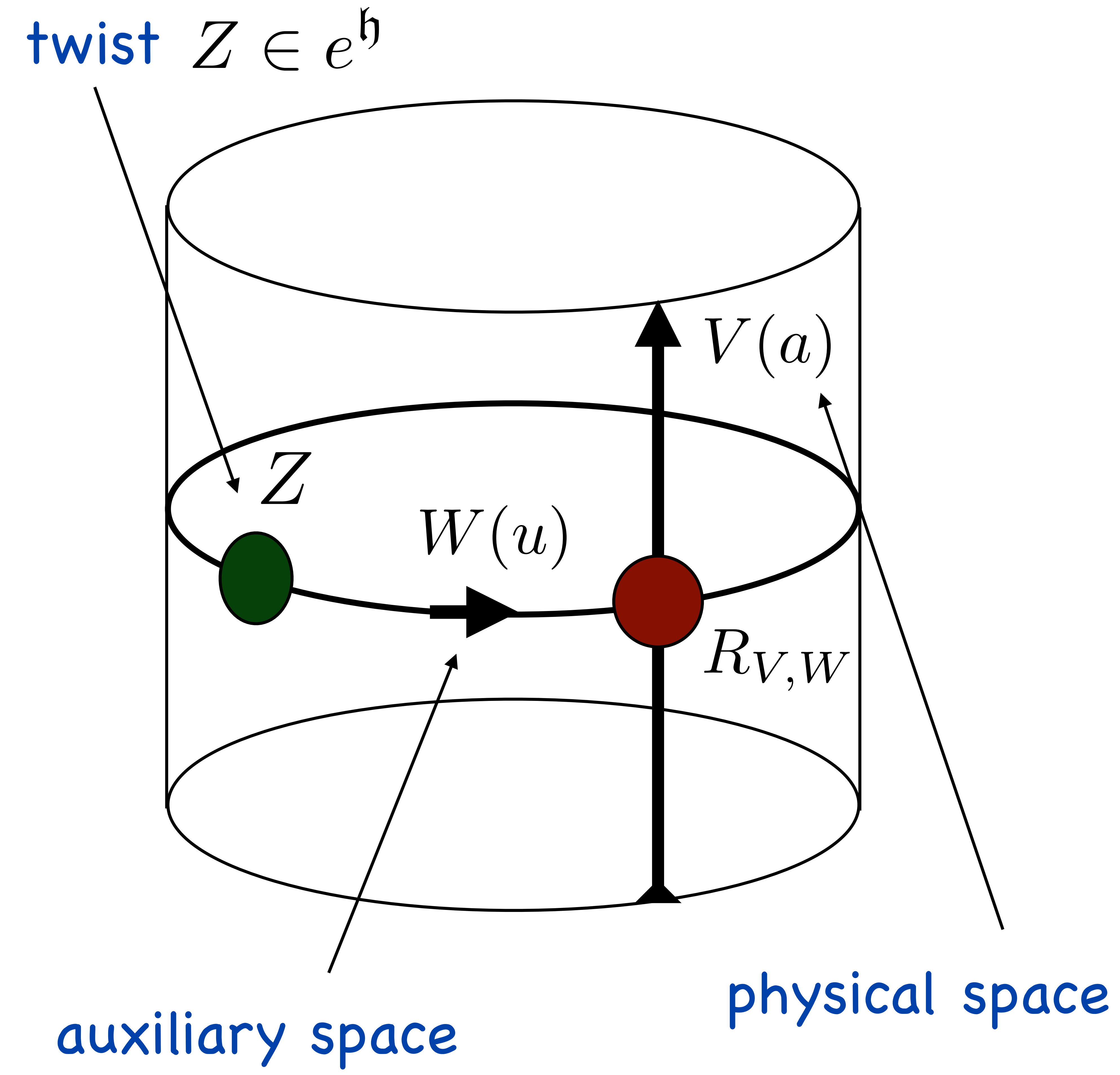
satisfying Yang-Baxter equation



Transfer Matrix

[Faddeev Reshetikhin Tachtajan]

The intertwiner represents an interaction vertex in integrable models. The quantum group is generated by matrix elements of R



Integrability comes from transfer matrices which generates Bethe algebra

$$T_W(u) = Tr_{W(u)}((Z \otimes 1) R_{V,W})$$

$$[T_W(u), T_W(u')] = 0$$

Transfer matrices are usually polynomials in u whose coefficients are the **integrals of motion**

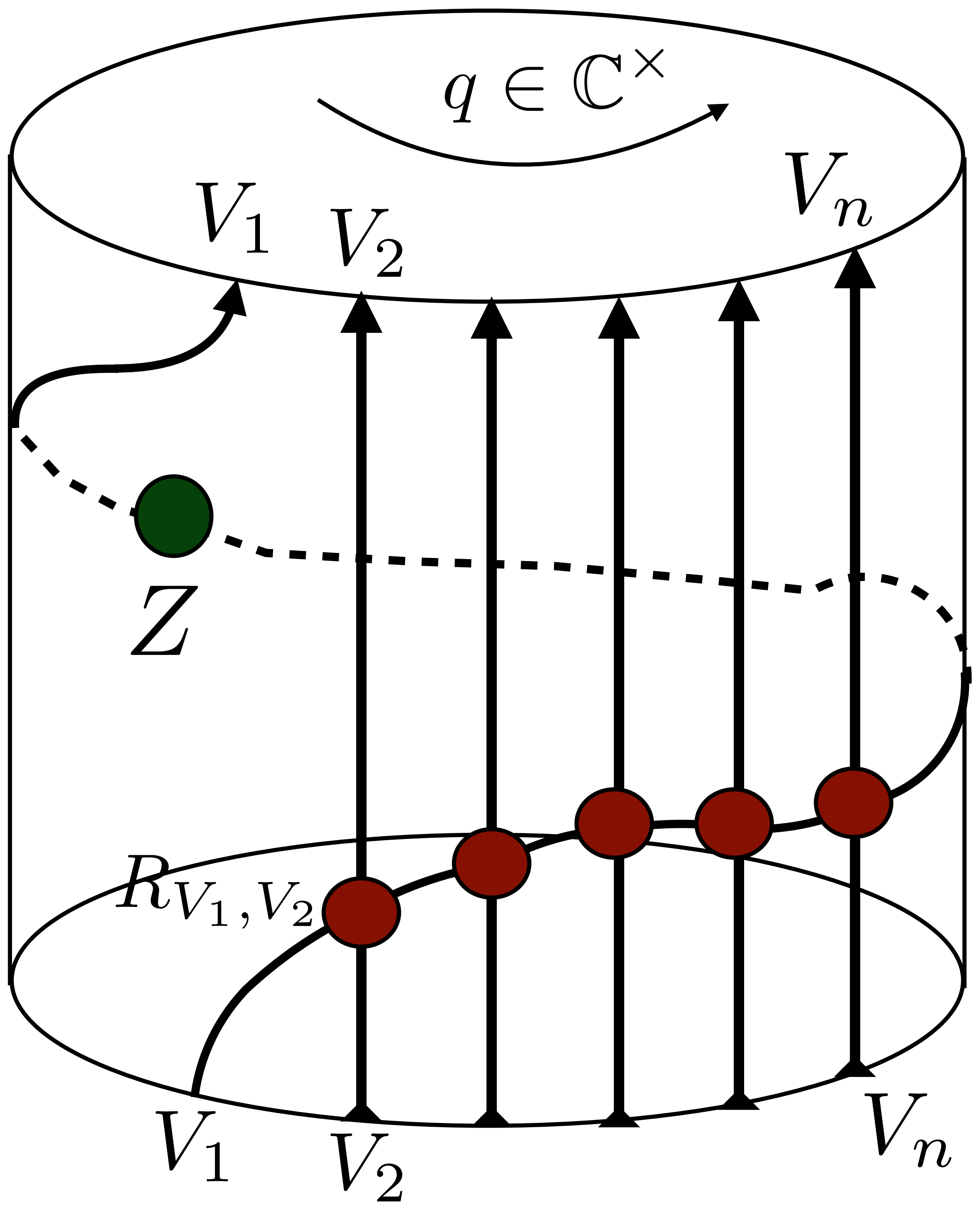
The XXZ Spin Chain

$\mathfrak{g} = \mathfrak{sl}_2$ spin-1/2 chain on n sites $V = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$

Consider Knizhnik-Zamolodchikov (qKZ) difference equation

[I. Frenkel Reshetikhin]

$$\Psi(qa_1, \dots a_n) = (Z \otimes 1 \otimes \cdots \otimes 1) R_{V_1,V_n} \cdots R_{V_1,V_2} \Psi(a_1, \dots a_n)$$



where

$$\Psi(a_1, \dots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

In the limit $q \to 1$

qKZ becomes an eigenvalue problem

Solutions of qKZ

[Aganagic Okounkov]

Schematic solution

$$\Psi_\alpha = \int \frac{d\mathbf{x}}{\mathbf{x}} f_\alpha(\mathbf{x}, a)\, \mathcal{K}(\mathbf{x}, z, a, q)$$

indexed by physical space

representation

universal kernel

$$\log \mathcal{K}(\mathbf{x}, z, a, q) \underset{q \to 1}{\sim} \frac{S(\mathbf{x}, z, a)}{\log q}$$

$$\frac{\partial S}{\partial x_i} = 0$$ Bethe equations for Bethe roots **x**

$$a_i \frac{\partial S}{\partial a_i} = \Lambda_i$$ Eigenvalues of qKZ operators

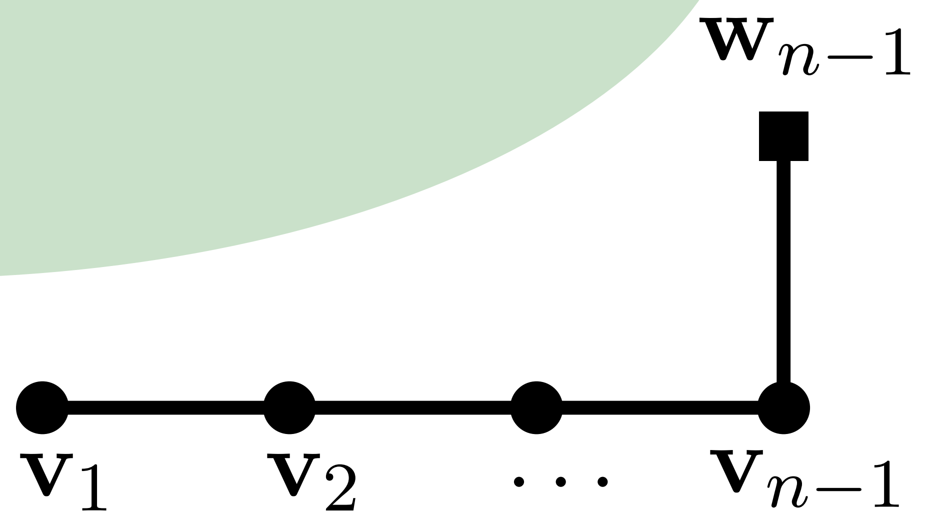
The map $\alpha \mapsto f_\alpha(\mathbf{x}^*)$ provides diagonalization

So we need to find `off shell' Bethe eigenfunctions $f_\alpha(\mathbf{x}, a)$

The Nekrasov-Shatashvili Correspondence

The answer will come from enumerative algebraic geometry inspired by physics

Hilbert space of states of quantum integrable system $\longleftrightarrow$ Equivariant K-theory of Nakajima quiver varitey



gauge group $G = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} U(v_i)$ $(v_1, v_2, \ldots)$ encode weight of rep α

Bethe roots **x** live in the maximal torus of G, by integrating over **x** we project on Weyl invariant functions of Bethe roots

Flavor group $G_F = \prod_i U(w_i)$ whose maximal torus gives parameters **a**

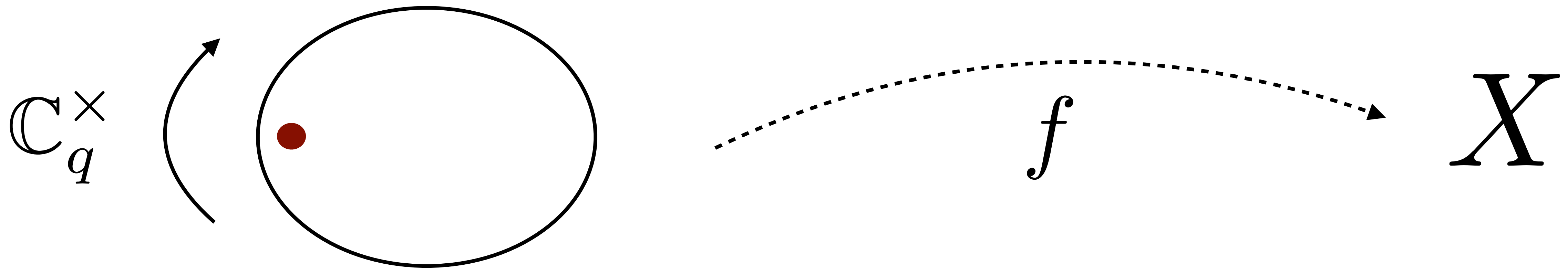
Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

Quantum K-theory of X

The quiver variety X = {Matter fields}/gauge group

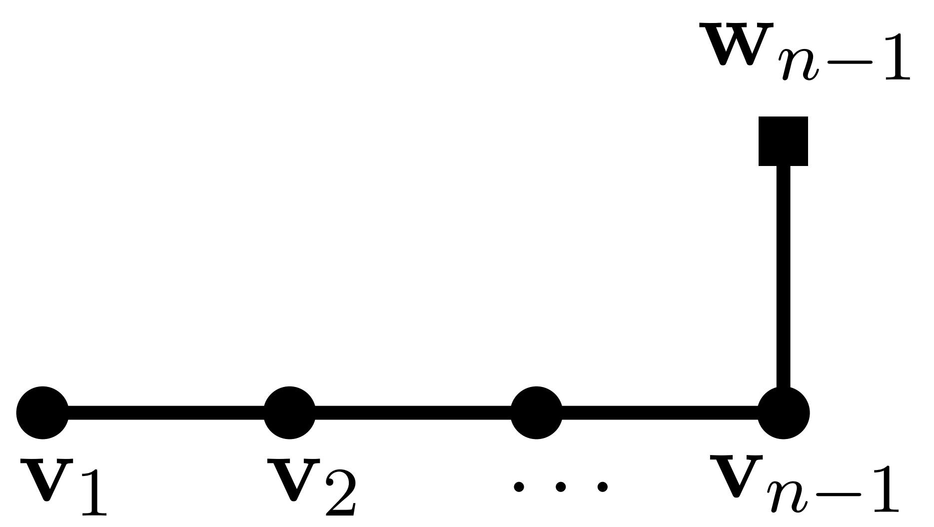
X is a module of a quantum group in the Nakajima correspondence construction

We will be computing integrals in K-theory of the space of quasimaps $f: \mathcal{C} - -- > X$ weighted by degree $\mathbf{z}^{\deg f}$ subject to equivariant action on the base nodal curve $\mathbb{C}_q^\times$



(cf Gromov-Witten invariants)

Nakajima Quiver Varieties



Rep(**v**,**w**) — linear space of quiver reps

Moment map $\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$

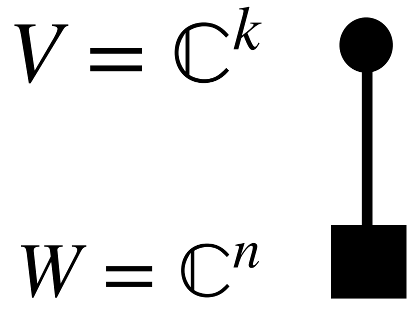
Quiver variety $X = \mu^{-1}(0)//_\theta G = \mu^{-1}(0)_{ss}/G$ $\quad G = \prod GL(\mathbf{v}_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus (a) $T = \mathbb{T}(\mathrm{Aut}(X))$

Tensorial polynomials of tautological bundles V_i, W_i and their duals generate classical T-equivariant K-theory ring of X

Ex: $T^*Gr_{k,n}$ $\quad \tau(V) = V^{\otimes 2} - \Lambda^3 V^*$



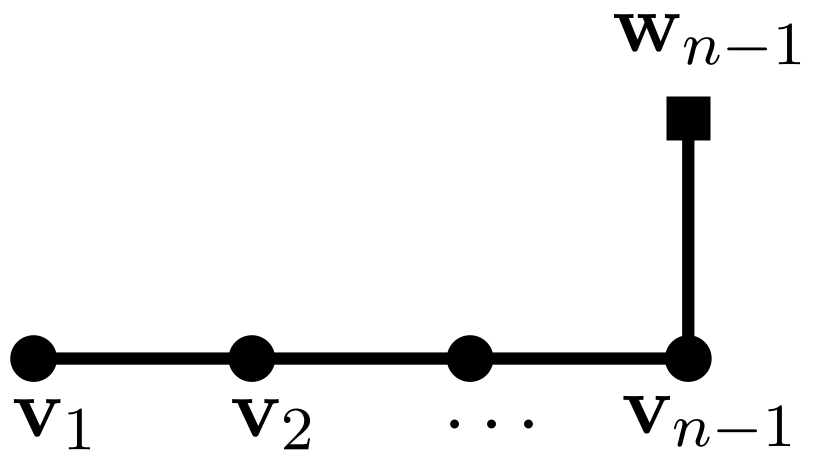
$\mathbf{v}_1 = k,\ \mathbf{w}_1 = n$ $\quad \tau(s_1, \cdots, s_k) = (s_1 + \cdots + s_k)^2 - \sum_{1 \le i_1 < i_2 < i_3 \le k} s_{i_1}^{-1} s_{i_2}^{-1} s_{i_3}^{-1}$

Quasimaps

[Ciocan-Fontanine, Kim, Maulik]
[Okounkov]

A **quasimap** $f : \mathcal{C} - - \rightarrow X$ is described by

- vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$, trivial bundles $\mathscr{W}_i$ of ranks $\mathbf{w}_i$
- section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$



$$\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$$

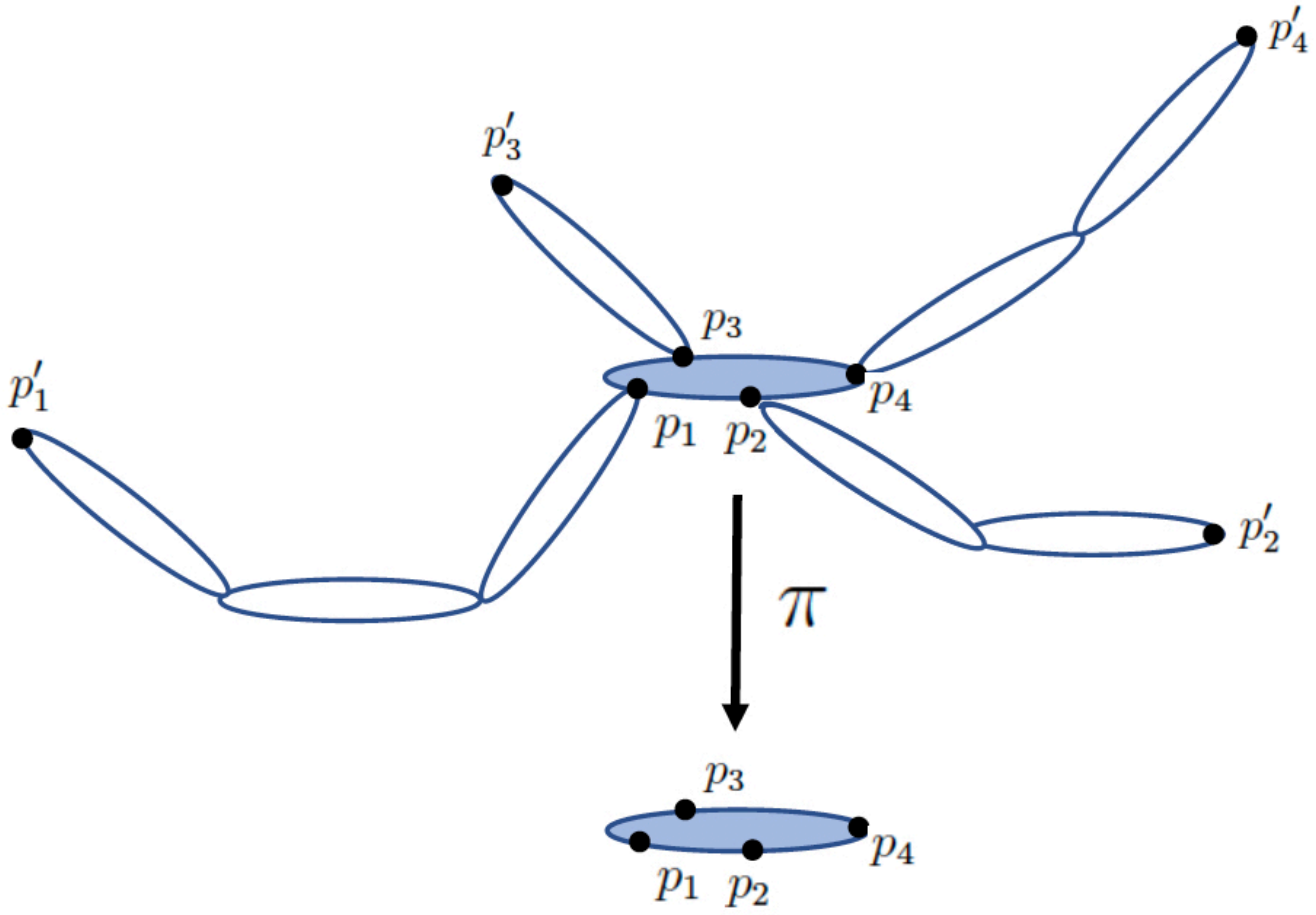
Evaluation map to quotient stack

$$\mathrm{ev}_p(f) = f(p) \in [\mu^{-1}(0)/G] \supset X$$

Quasimap is **stable** if $f(p) \in X$ for all but finitely many points — singularities

The moduli space of stable quasimaps $\mathbf{QM}^d(X)$

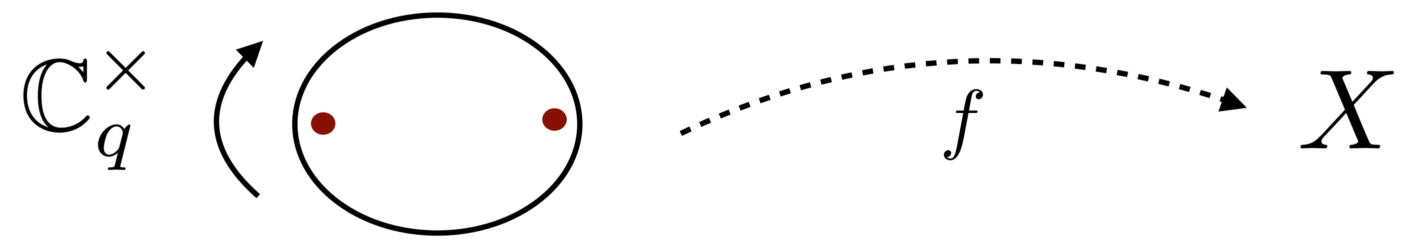
$\mathscr{V}_i$ and f vary



Vertex Function

[Okounkov]
[PK Pushkar Smirnov Zeitlin]

Quasimaps spaces admit action of $\mathbb{C}_q^\times$ on base $\mathbb{P}^1$ with two fixed points $p_1 = 0, \;\; p_2 = \infty$



Define **vertex function** for τ with quantum (Novikov) parameters **z**

$$\mathbf{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}(\widehat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \tau|_{p_1}, \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{nonsing}\, p_2})\boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}_q^\times}(X)_{loc}[[\boldsymbol{z}]]$$

fixed pts

$$K_T(X)_{loc} = K_T(X) \otimes_{\mathbb{Z}[a,\hbar]} \mathbb{Q}(a,\hbar)$$

Define **quantum K-theory** as a ring with multiplication

$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

Theorem: $QK(X)$ is a commutative associative unital algebra

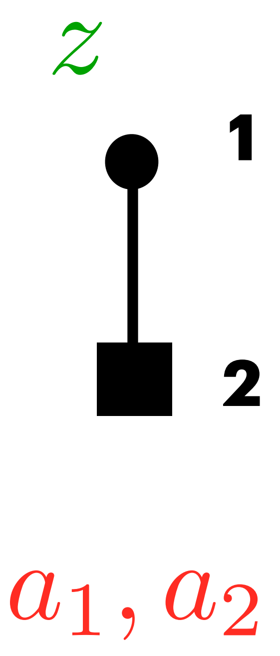
Vertex for $T^*\mathbb{P}^1$

Vertex function coefficient with trivial insertion

$$V^{(1)}_{\mathbf{p}} = \sum_{d>0} z^d \prod_{i=1}^{2} \frac{\left(\frac{q}{\hbar}\frac{a_{\mathbf{p}}}{a_i};q\right)_d}{\left(\frac{a_{\mathbf{p}}}{a_i};q\right)_d} = {}_2\phi_1\left(\hbar, \hbar\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, q\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; \frac{q}{\hbar}z\right)\,.$$

two fixed points

$\mathbf{p} = \{a_1\}$ and $\mathbf{p} = \{a_2\}$.



Truncation on V - Macdonald Polynomials!!

As a contour integral

$$V = \frac{e^{-\frac{\log z \cdot \log a_1 a_2}{\log q}}}{2\pi i} \int_C \frac{ds}{s}\, e^{\frac{\log z \cdot \log s}{\log q}} \frac{\varphi\left(\hbar\frac{s}{a_1}\right)}{\varphi\left(\frac{s}{a_1}\right)} \frac{\varphi\left(\hbar\frac{s}{a_2}\right)}{\varphi\left(\frac{s}{a_2}\right)}$$

[PK [arXiv:1805.00986] **Comm.Math.Phys.** (2021)]

Vertex functions are eigenfunctions of quantum tRS difference operators!

$$T_i(a)V(z,a) = e_i(z)V(z,a) \qquad T_i(z)V(z,a) = e_i(a)V(z,a)$$

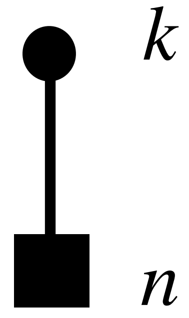
$$\hbar \to \hbar^{-1}$$

[PK Zeitlin [arXiv:1802.04463] Math.Res.Lett. **28** (2021) 435]

Bethe Equations for $T^*Gr_{k,n}$

The operator of quantum multiplication from saddle point approximation

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V_{\boldsymbol{p}}^{(\tau)}(z)}{V_{\boldsymbol{p}}^{(1)}(z)}$$



Theorem *The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(s_1, \cdots, s_k)$ evaluated at the solutions of the following equations:*

$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}\,, \quad i = 1 \cdots k.$$

Equivariant parameters a_i,

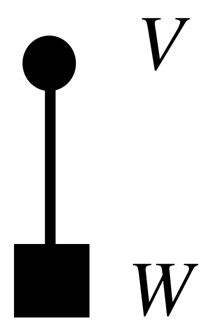
twist z,

Planck constant $\hbar$

Baxter Q-operator $Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$ has eigenvalue $Q(u) = \prod_{i=1}^{k} (u - s_i)$

The QQ-System for A_1

Short exact sequence of bundles



$$0 \to V \to W \to V^{\vee} \to 0$$

Eigenvalues of Q-operators

$$Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$$

$$\widetilde{Q}(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V^{\vee})(z) \circledast$$

Satisfy the QQ-relation

$$z\, \widetilde{Q}(\hbar u) Q(u) - \widetilde{Q}(u) Q(\hbar u) = \prod_{i=1}^{n} (u - a_i)$$

equivalent to the XXZ Bethe equations

QQ-System in General

Consider complex simple Lie algebra $\mathfrak{g}$ of rank r

Cartan matrix $a_{ij} = \langle \check{\alpha}_i, \alpha_j \rangle$

$$\widetilde{\xi}_i\, Q^i_-(u) Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u) Q^i_+(u) = \Lambda_i(u) \prod_{j>i} \left[Q^j_+(\hbar u) \right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(u) \right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Polynomials $Q_+(u)$ contain Bethe roots, $\Lambda(u)$ contain equivariant parameters

Polynomials $Q_-(u)$ are auxiliary

The Ubiquitous QQ-System

Bethe Ansatz equations for XXX, XXZ models — eigenvalues of Baxter operators

[Mukhin, Varchenko] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\hat{g})$

[Frenkel, Hernandez] ....

Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Nekrasov-Shatashvili] [Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

Spectral determinants in the QDE/IM Correspondence

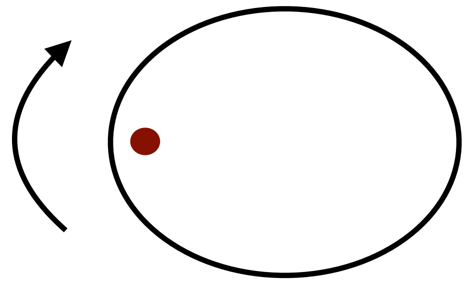
[Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

(G,q)-Opers

III. (G,q)-Connection

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$u \mapsto qu$$

q



G-simple simply-connected complex Lie group

Consider vector bundle $\mathscr{F}_G$ over $\mathbb{P}^1$

(G, q)-connection A is a meromorphic section of $Hom_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{F}_G, \mathscr{F}^q_G)$

Locally q-gauge transformation of the connection

$$A(u) \mapsto g(qu)A(u)g(u)^{-1} \qquad g(u) \in G(\mathbb{C}(u))$$

Compare with (standard) gauge transformations

$$\partial_u + A(u) \mapsto g(u)(\partial_u + A(u))g(u)^{-1} \qquad g(u) \in \mathfrak{g}(u)$$

(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathcal{F}_{B_-}$ is a reduction of $\mathcal{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathcal{F}_G \longrightarrow \mathcal{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally $$A(u) = n'(u)\prod_i (\phi_i(u)_i^{\check{\alpha}} s_i)\, n(u)$$

$$\phi_i(u) \in \mathbb{C}(u),\; n(u), n'(u) \in N_-(u) = [B_-(u), B_-(u)]$$

(SL(2),q)-Opers

Let $G = SL(2)$ The q-oper definition can be formulated as

Triple $(E, A, \mathscr{L})$

(E, A) is the $(SL(2), q)$ connection

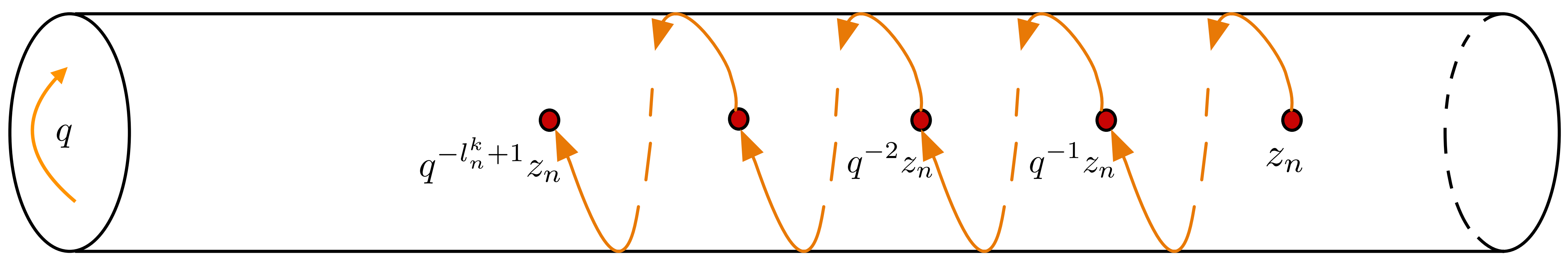
$\mathscr{L} \subset E$ is a line subbundle

The induced map $\bar{A} : \mathscr{L} \to (E/\mathscr{L})^q$ is an isomorphism

in a trivialization $\mathscr{L} =$ Span(s)

$$s(qu) \wedge A(u)s(u) \neq 0$$

Allow singularities $\quad s(qu) \wedge A(u)s(u) = \Lambda(u)$



$$\Lambda(u) = \prod_{l,j_l} (u - q^{j_l} a_l)$$

Add Twists $\quad Z = g(qu)A(u)g(u)^{-1}$

$$Z \in H \subset H(u) \subset G(u)$$

q-Opers, QQ-System & Bethe Ansatz

Chose trivialization of $\mathcal{L}$ $\quad s(u) = \begin{pmatrix} Q_+(u) \\ Q_-(u) \end{pmatrix}$ $\qquad$ Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition — SL(2) **QQ-system**

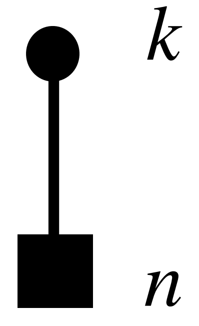
$$s(qu) \wedge A(u) s(u) = \Lambda(u) \longrightarrow \zeta\, Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(qu) Q_+(u) = \Lambda(u)$$

QQ-system to XXZ Bethe equations

$$Q_+(u) = \prod_{k=1}^{m} (u - s_k)$$

$$\prod_{l=1}^{n} \frac{s_i - q^{r_l} a_l}{s_i - a_l} = \zeta^2 q^k \prod_{j=1}^{k} \frac{q s_i - s_j}{s_i - q s_j}$$

$$i = 1, \dots, k$$



$$\hbar = q$$

q-Miura Transformation

Miura q-oper: $(E, A, \mathscr{L}, \hat{\mathscr{L}})$, where $(E, A, \mathscr{L})$ is a q-oper and $\hat{\mathscr{L}}$ is preserved by q-connection A

$$A(u) = \begin{pmatrix} g(u) & \Lambda(u) \\ 0 & g(u)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition $\quad A(u) = v(qu)Zv(u)^{-1} \qquad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

$$g(u) = \zeta \frac{Q_+(qu)}{Q_+(u)} \qquad v(u) = \begin{pmatrix} Q_+(u) & \zeta Q_-(u)Q_+(qu) - \zeta^{-1}Q_-(u)Q_+(qu) \\ 0 & Q_+(u) \end{pmatrix} \in B_+(u)$$

The q-oper condition becomes the **SL(2) QQ-system** $\quad \zeta\, Q_-(u)Q_+(qu) - \zeta^{-1}Q_-(qu)Q_+(u) = \Lambda(u)$

Difference Equation $\quad D_q(s) = As$

Scalar difference operator $\quad \left(D_q^2 - T(qu)D_q - \dfrac{\Lambda(qu)}{\Lambda(u)}\right) s_1 = 0$

tRS Hamiltonians

Recover 2-body tRS Hamiltonian from an (SL(2),q)-Oper

$$\det \begin{pmatrix} Q_+(u) & \zeta Q_+(qu) \\ Q_-(u) & \zeta^{-1} Q_-(qu) \end{pmatrix} = \Lambda(u)$$

Let $\quad Q_+(u) = u - p_+ \qquad Q_-(u) = u - p_-$

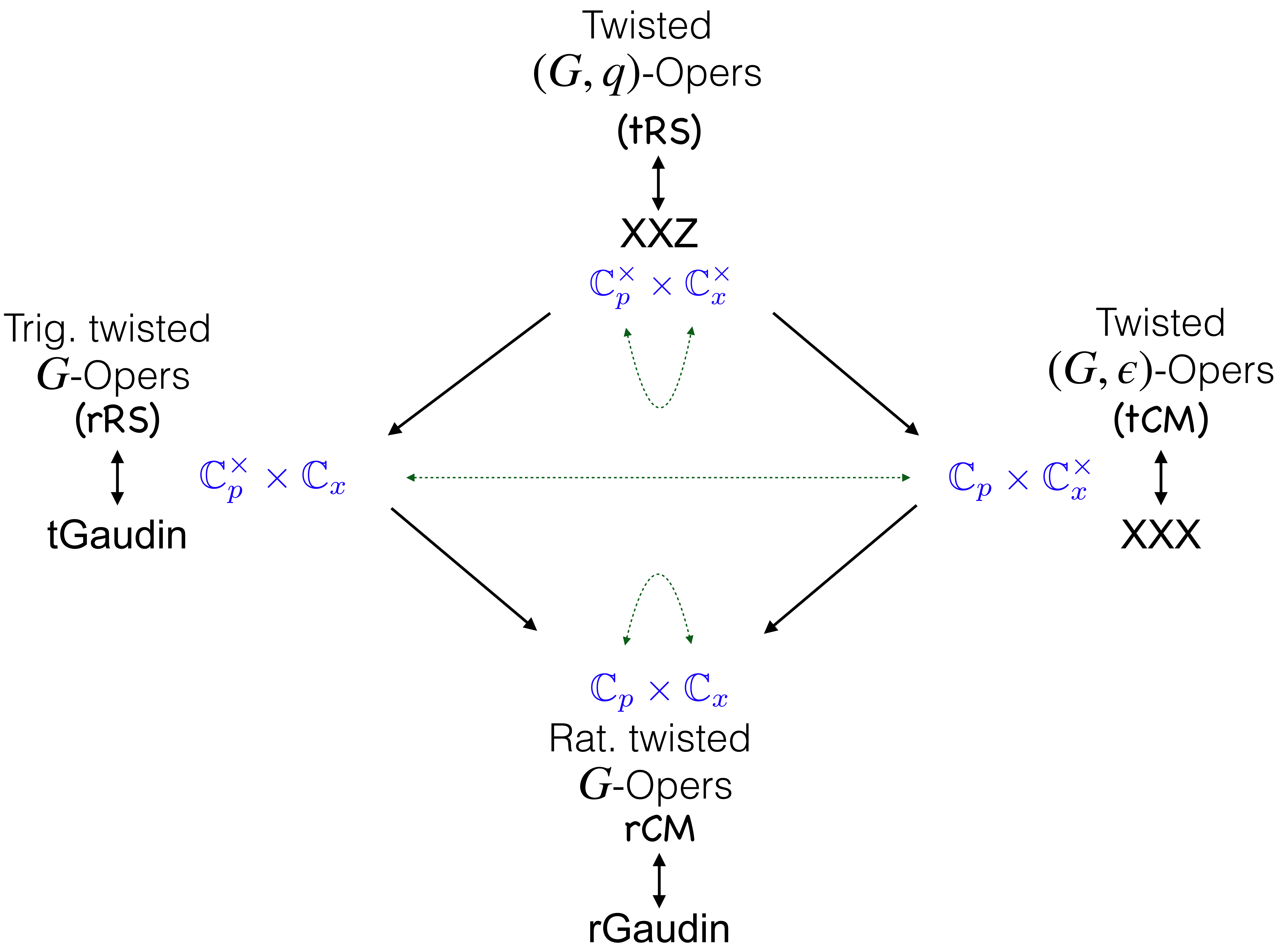
$$u^2 - u \left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \frac{q\zeta - q\zeta^{-1}}{\zeta^{-1} - \zeta} p_- \right] + p_+ p_- = (u - a_+)(u - a_-)$$

$$T_1 \qquad\qquad T_2$$

qOper condition yields
tRS Hamiltonians!

$$\det(u - T) = (u - a_+)(u - a_-)$$

Network of Dualities



Hierarchy of Models

[Mironov, Morozov, Gorsky...]
[Gorsky PK Koroteeva Shakirov]

p \ q	rational	trigonometric	elliptic
r	rational CMS	trigonometric CMS	elliptic CMS *quantum cohomology*
t	rational RS (dual trig. CMS)	trigonometric RS	elliptic RS *quantum K-theory*
e	dual elliptic CMS	dual elliptic RS	DELL *Elliptic Cohomology*

[PK Shakirov **LMP** 2020]

q-Opers and q-Langlands

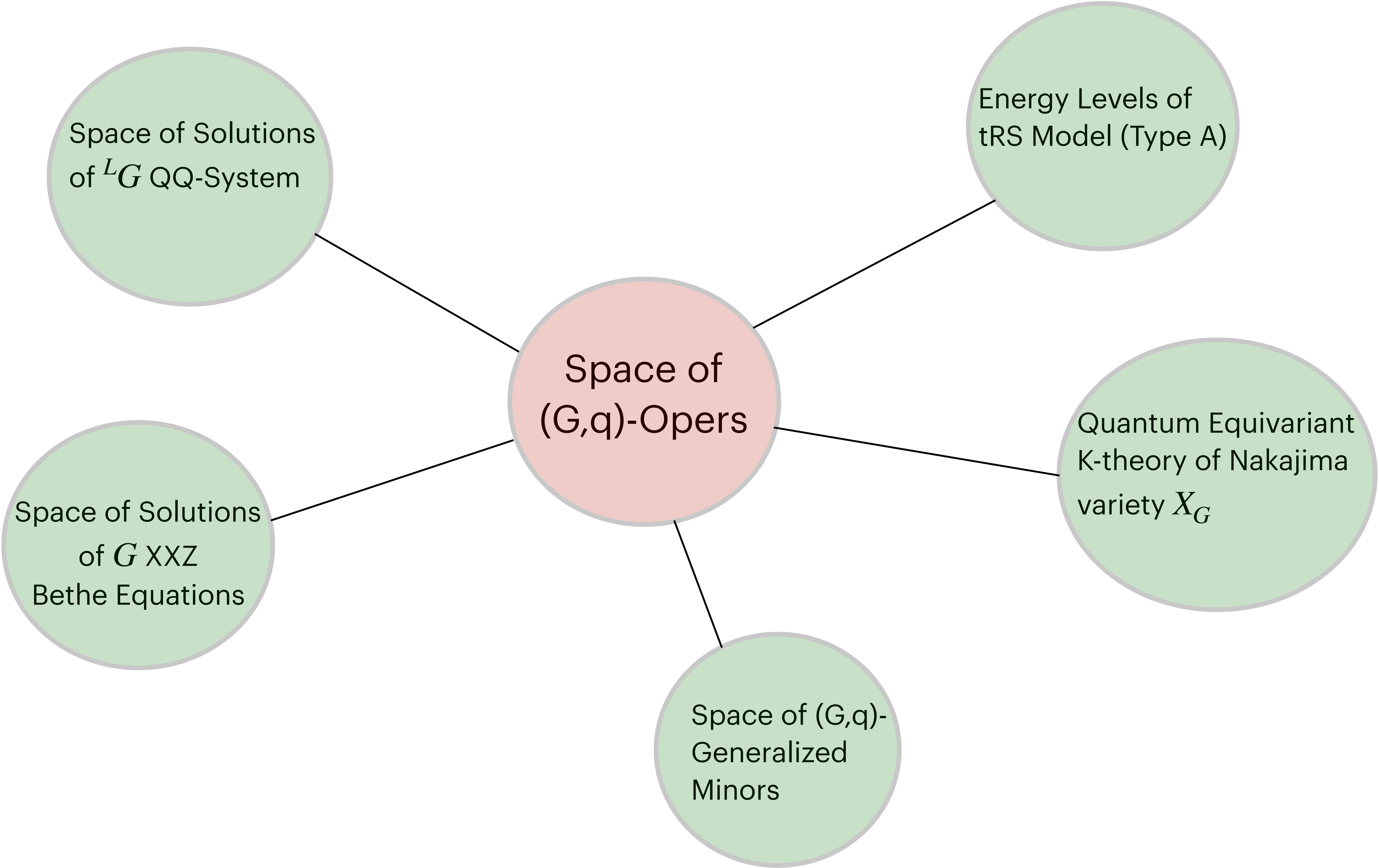
[Frenkel, PK, Zeitlin, Sage, 2021, to appear in **JEMS**]

Miura (G,q)-oper with singularities $$A(u)=\prod_i g_i(u)^{\check{\alpha}_i} e^{\frac{\Lambda_i(u)}{g_i(u)}e_i}$$

Theorem: There is a 1-to-1 correspondence between the set of nondegenerate Z-twisted (G,q)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

$$\widetilde{\xi}_i\, Q^i_-(u)Q^i_+(\hbar u)-\xi_i Q^i_-(\hbar u)Q^i_+(u)=\Lambda_i(u)\prod_{j>i}\left[Q^j_+(\hbar u)\right]^{-a_{ji}}\prod_{j<i}\left[Q^j_+(u)\right]^{-a_{ji}},\qquad i=1,\dots,r,$$

$$\widetilde{\xi}_i=\zeta_i\prod_{j>i}\zeta_j^{a_{ji}},\qquad \xi_i=\zeta_i^{-1}\prod_{j<i}\zeta_j^{-a_{ji}}$$

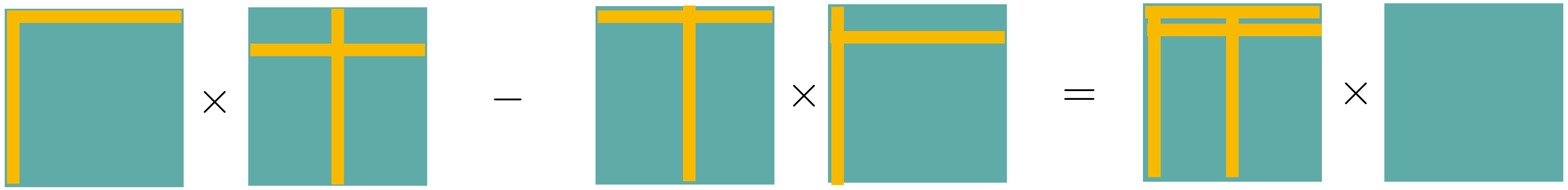

Space of Solutions
of ^LG QQ-System
Energy Levels of
tRS Model (Type A)
Space of
(G,q)-Opers
Quantum Equivariant
K-theory of Nakajima
variety X_G
Space of Solutions
of G XXZ
Bethe Equations
Space of (G,q)-
Generalized
Minors

IV. Cluster Algerbras

[PK, Zeitlin, 2022, to appear in **Crelle**]

The QQ-system $\xi_{i+1}\, Q^i_-(u)Q^i_+(u+\epsilon) - \xi_i Q^i_-(u+\epsilon)Q^i_+(u) = \Lambda_i(u)Q^{i+1}_+(u+\epsilon)Q^{i+1}_+(u)$

For $G = SL(n)$ obtain Lewis Carrol (Desnanot-Jacobi-Trudi) identity $M^1_1 M^2_i - M^1_i M^2_1 = M^{12}_{1i} M$



For general G obtain relation on generalized minors $\Delta^{\omega_i}(v(u)) = Q^i_+(u)$ [Fomin Zelevinsky]

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}\Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i}\Delta_{u\cdot\omega_i, vw_i\cdot\omega_i} = \prod_{j\neq i} \Delta^{-a_{ji}}_{u\cdot\omega_j, v\cdot\omega_j},$$

$u, v \in W_G$

q-Langlands Correspondence

[Aganagic Frenkel Okounkov]

Two types of solutions of the qKZ equation:

Analytic in chamber of equivariant parameters $\{a_i\}$— conformal blocks of $U_\hbar(\hat{g})$

Analytic in chamber of quantum parameters (twists) $\{\zeta_i\}$— conformal blocks for deformed W-algebra $W_{q,\hbar}({}^L\hat{g})$

The q-Langlands correspondence

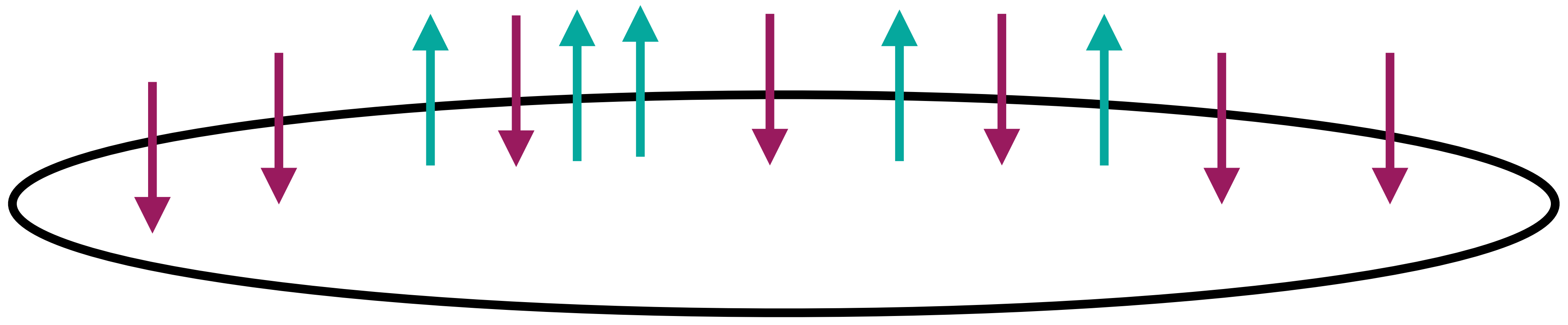
$$\begin{array}{ccc} U_\hbar(\hat{g}) & \longleftrightarrow & W_{q,\hbar}({}^L\hat{g}) \\ \downarrow & & \downarrow{\scriptstyle q \to 1} \\ \text{XXZ Bethe} & & (G, \hbar)\text{-oper} \end{array}$$

Equivalence of categories

$$\mathrm{Dmod}_\kappa(Bun_G) \longleftrightarrow \mathrm{Dmod}_{-\frac{1}{m\kappa}}(Bun_{{}^LG})$$

Quantum

QQ-Systems



SU(n) XXZ spin chain on n sites w/ **anisotropies** and **twisted periodic boundary conditions**

Planck's constant $\hbar$

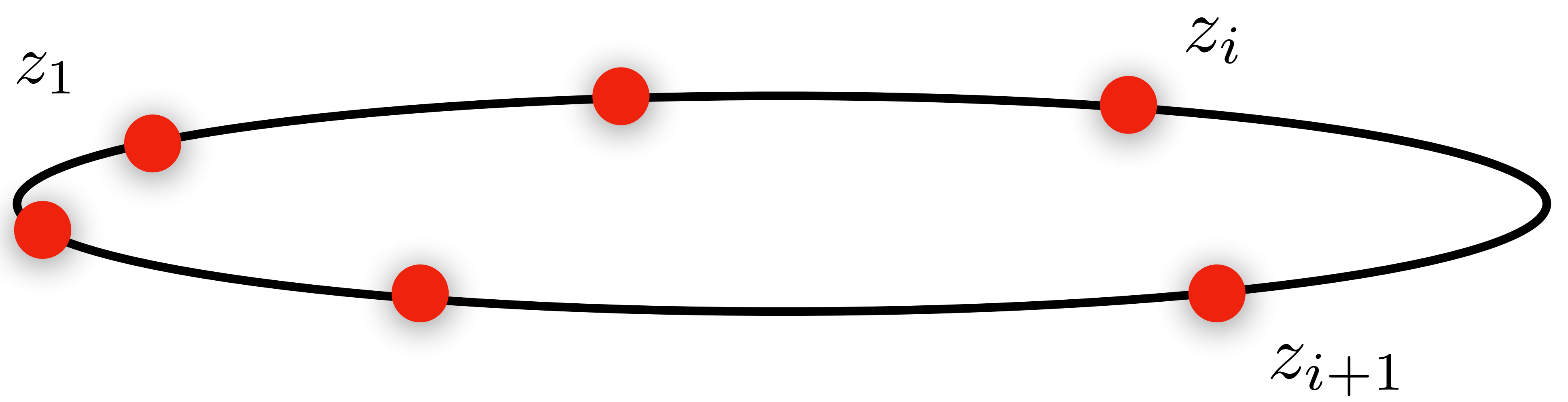
twist eigenvalues z_i

equivariant parameters (anisotropies) a_i

Bethe Ansatz Equations: $\exp \dfrac{\partial Y}{\partial \sigma_i} = 1$

Classical

q-Opers



n-particle trigonometric Ruijsenaars-Schneider model

Coupling constant $\hbar$

coordinates z_i

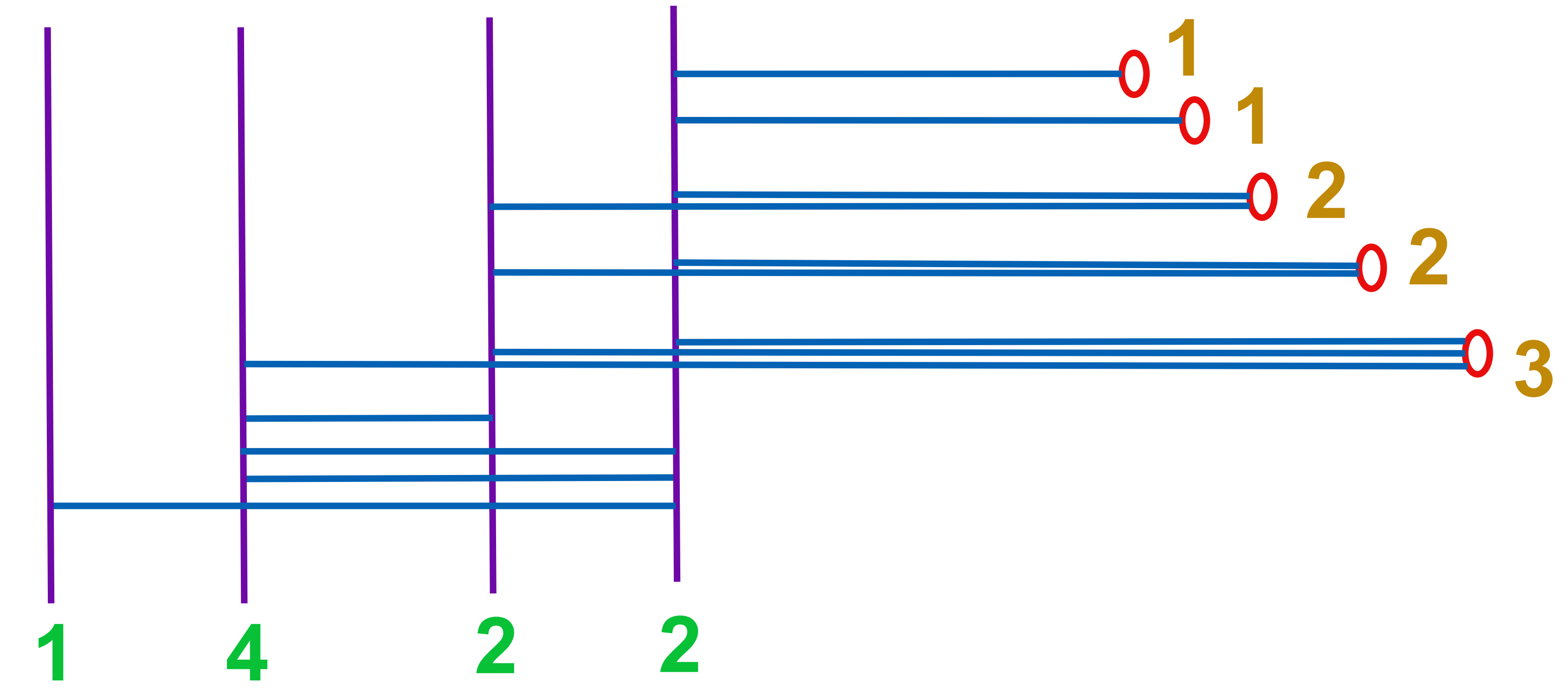
energy (eigenvalues of Hamiltonians) $e_i(a_i)$

Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \ldots, n$$

Quantum/Classical Duality & 3d Mirror Symmetry

[PK Gaiotto]
[PK Zeitlin]



tRS energy relations = XXZ Bethe equations

$$\det(u-T)=\prod_{i=1}^{N}(u-a_i)\,,\qquad \det(u-M)=\prod_{i=1}^{N}(u-\xi_i)$$

$\mathcal{L}_\mu$ Eigenvalues of M and Slodowy form on T

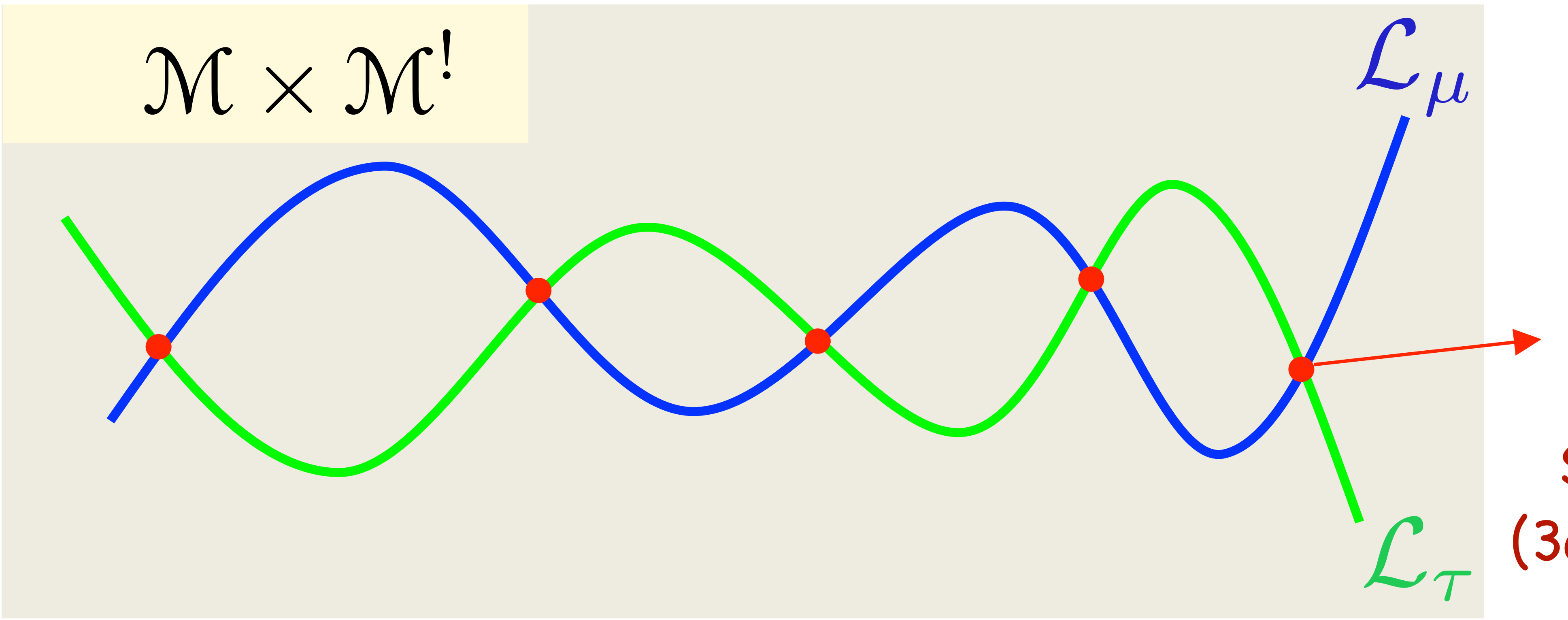
$\mathcal{L}_\tau$ Eigenvalues of T and Slodowy form on M

Solutions of Bethe equations — intersection points

Symplectic form $$\Omega=\sum_{i=1}^{N}\frac{dp_i^{\xi}}{p_i^{\xi}}\wedge\frac{d\xi_i}{\xi_i}-\frac{dp_i^{a}}{p_i^{a}}\wedge\frac{da_i}{a_i}$$

tRS momenta $$p_i^{\xi}=\exp\frac{\partial Y}{\partial \xi_i}\,,\qquad p_i^{a}=\exp\frac{\partial Y}{\partial a_i}$$

$$qMT-TM=u\otimes v^T$$



$Y=Y^!$

Symplectic duality
(3d mirror symmetry)