Opers — what they are and what are they good for

Peter Koroteev

Talk at University of Montreal 12/16/2022

Literature

[arXiv:2208.08031]
The Zoo of Opers and Dualities
P. Koroteev, A. M. Zeitlin

[arXiv:2108.04184] Crelle Journal
q-Opers, QQ-systems, and Bethe Ansatz II: Generalized Minors
P. Koroteev, A. M. Zeitlin

[arXiv:2105.00588]
3d Mirror Symmetry for Instanton Moduli Spaces
P. Koroteev, A. M. Zeitlin

[arXiv:2007.11786] J. Inst. Math. Jussieu
Toroidal q-Opers
P. Koroteev, A. M. Zeitlin

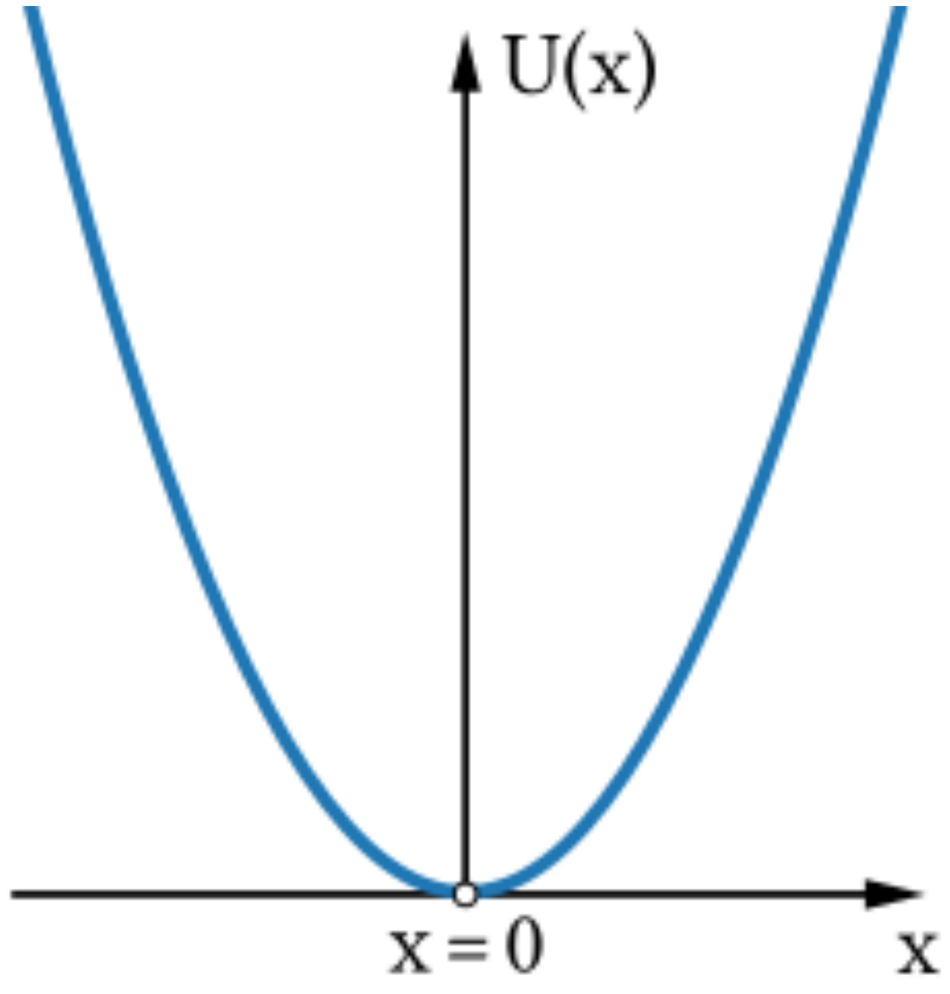
[arXiv:2002.07344] J. Europ. Math. Soc.
q-Opers, QQ-Systems, and Bethe Ansatz
E. Frenkel, P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1811.09937] Commun.Math.Phys. **381** (2021) 641
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality
P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1705.10419] Selecta Math. **27** (2021) 87
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

Symplectic Manifold

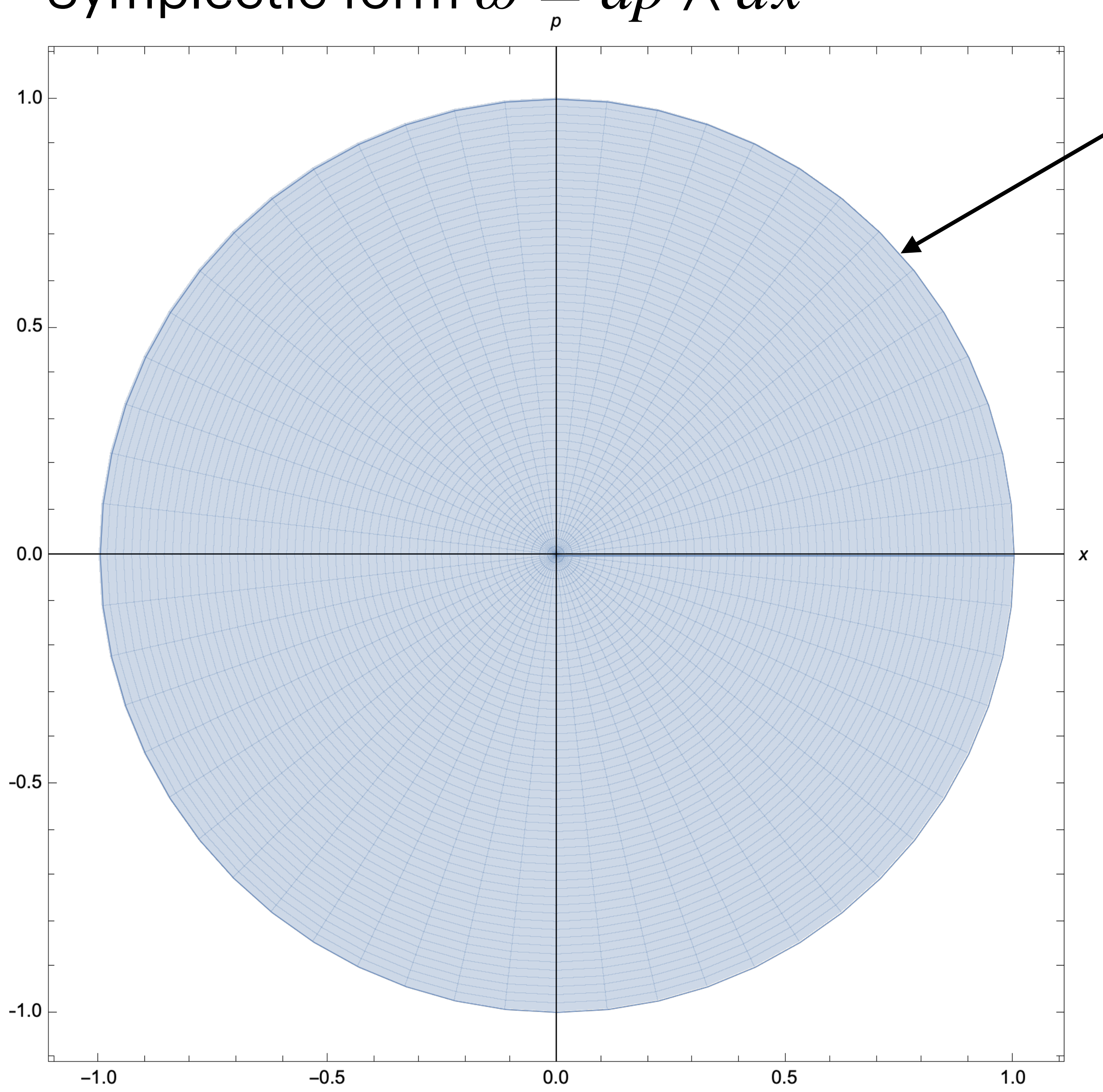
Harmonic oscillator $$H = \frac{p^2}{2} + \frac{x^2}{2}$$



Phase space — symplectic manifold $\mathscr{M}$

Symplectic form $\omega = dp \wedge dx$

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$



Lagrangian $\mathscr{L} \subset \mathscr{M}$ is a middle-dimensional submanifold and such that the restriction of the symplectic form on $\mathscr{L}$ vanishes

$$\omega|_{\mathcal{L}} = 0$$

Classical Integrability

Equations of motion

$$\frac{df}{dt} = \{H_1, f\}$$

Integrability — family of n conserved quantities which Poisson commute with each other

$$\{H_i, H_j\} = 0 \quad i, j = 1, \ldots, n$$

Liouville-Arnold Theorem

Compact Lagrangians $\mathscr{L}$: $\{H_i = E_i\}$ are isomorphic to tori

Evolution in the neighborhood of $\mathscr{L}$ is linearized in action/angle variables $\{I_i, \varphi_i\}_{i=1}^n$

$$\frac{d\varphi_i}{dt} = \omega_i, \qquad \frac{dI_i}{dt} = 0$$

Action/angle variables are hard to find

History (1960–current)

Many-body integrable systems — Calogero, Toda, Ruijsenaars (more on this later)

Continuous integrable models in (1+1)-dimensions: Korteweg-de-Vries, Intermediate Long-Wave, etc.

$$u_t = 6uu_x - u_{xxx}$$

They admit soliton solutions. Sectors with N solitons are described by finite N-body integrable systems

[my work on (1+1) hydro with Scirappa]

[arXiv:1510.00972] Lett.Math.Phys. **108** (2018) 45

[arXiv:1601.08238] J.Math.Phys. **57** (2016) 112302

Inverse scattering method — Lax pair data → action/angle variables

Quantization

Coordinates and momenta become operators

$$p, x \mapsto \hat{p}, \hat{x}$$

Poisson brackets associated to ω become commutators

$$\{A, B\}_{P.B.} \mapsto [A, B]$$

Heisenberg algebra

$$[\hat{p}, \hat{x}] = -i\hbar$$

$$\hat{x} f(x) = x f(x)$$

$$\hat{p} f(x) = -i\hbar f'(x)$$

Lagrangian constraint

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$

Replaced by operator

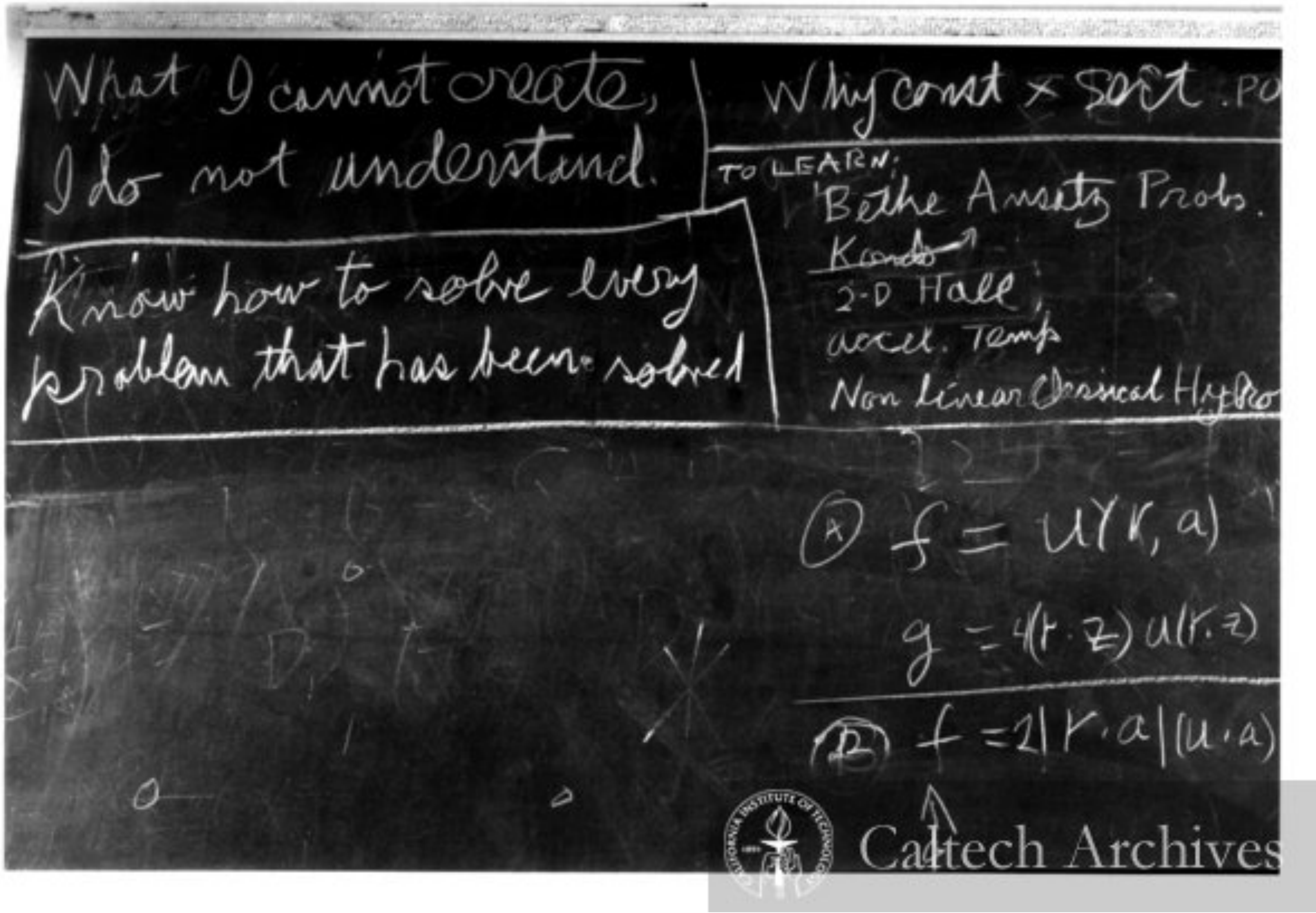
$$\left(\frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} - E\right) Z(x) = 0$$

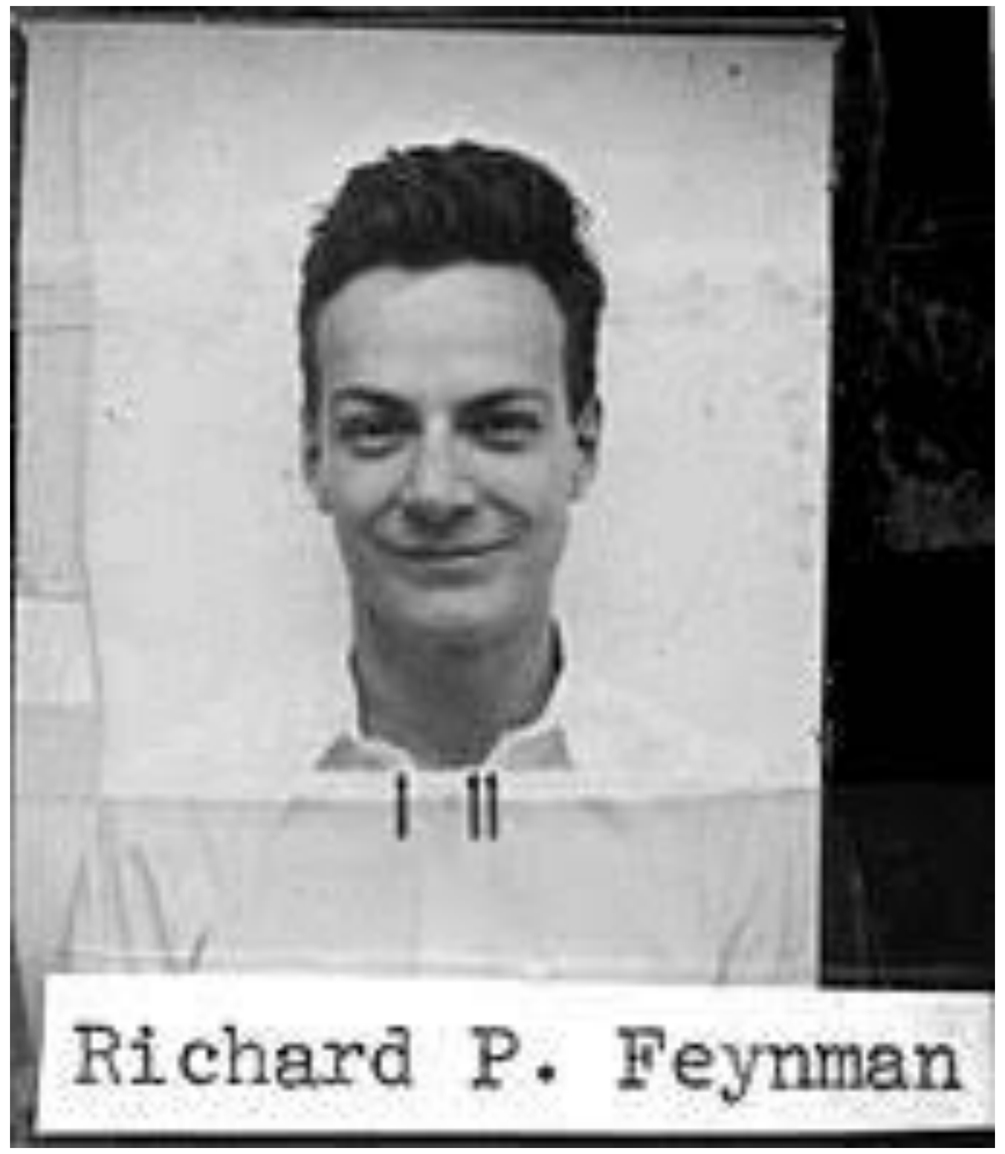
This ODE has square integrable solutions only
for special values of E

Integrability

$$[H_i, H_j] = 0 \qquad H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables — simultaneous diagonalization of H_i





I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz
and how mysteriously they jump out at you and work and you don't know why.
I am trying to understand all this better.

Motivation

Quantum/Classical Integrable Systems [PK, Gaiotto] [PK, Zeitlin] [Matsuo, Cherednik]

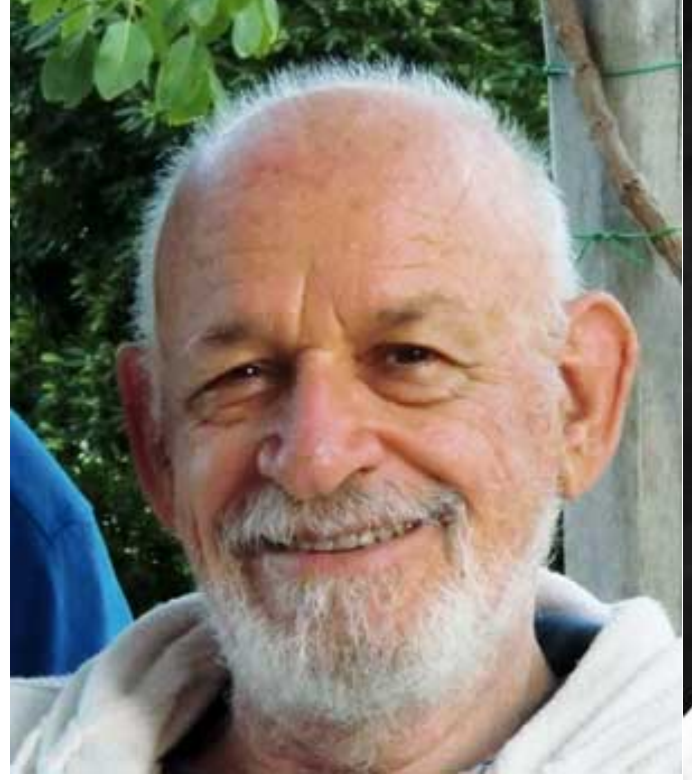
Quantum Geometry and Integrability [Okounkov et al] [Pushkar, Zeitlin, Smirnov] [PK, Pushkar, Smirnov, Zeitlin]

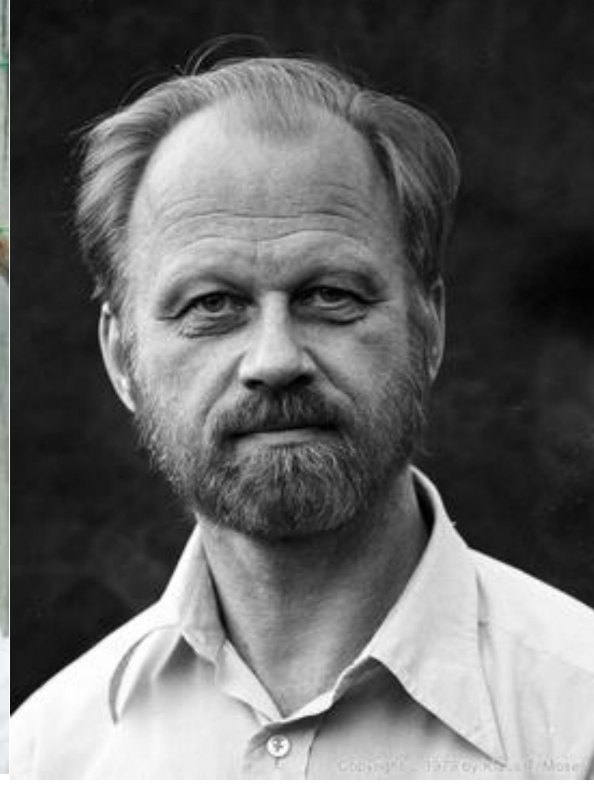
BPS/CFT Correspondence [Nekrasov Shatashvili]

Geometric q-Langlands Correspondence [Frenkel] [Aganagic, Frenkel, Okounkov]

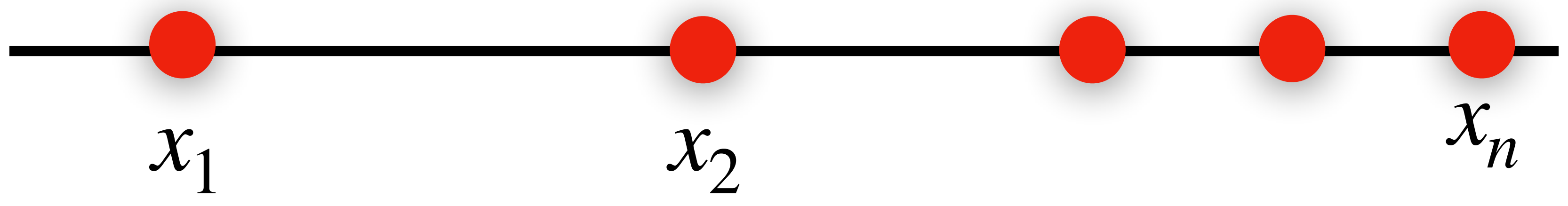
ODE/IM Correspondence [Bazhanov, Lukyanov, Zamolodchikov] [Dorey, Tateo]

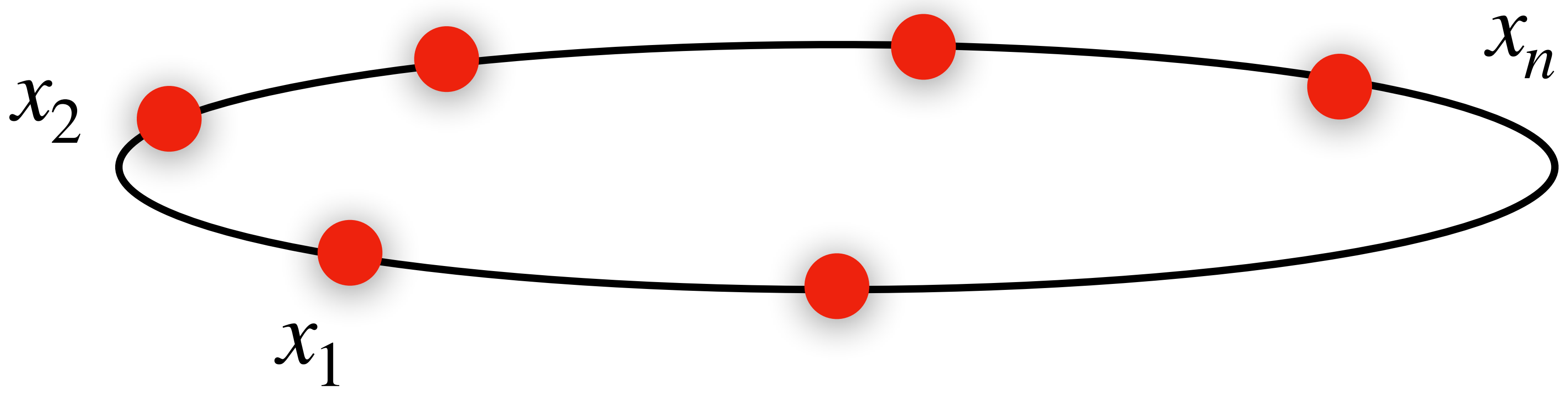
I. Many-Body Systems





Calogero in 1971 introduced a new integrable system. Moser in 1975 proved its integrability using Lax pair



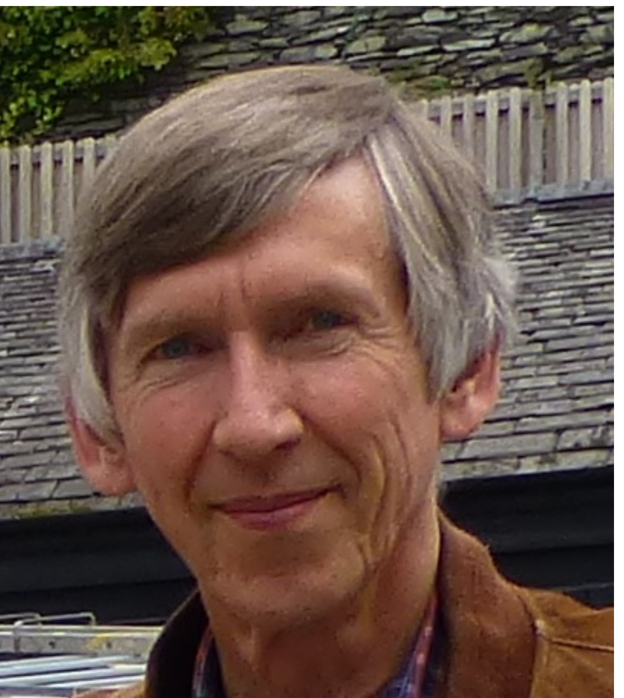


$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$

The **Calogero-Moser (CM)** system has several generalizations

rational CM → trigonometric CM → elliptic CM

$$V(x) \simeq \sum \frac{1}{(x_i - x_j)^2} \qquad V(x) \simeq \sum \frac{1}{\sinh(x_i - x_j)^2} \qquad V(x) \simeq \wp(x_j - x_i)$$



Another relativistic generalization called **Ruijsenaars-Schneider (RS)** family

rRS → tRS → eRS

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

In this talk, we'll describe geometry behind these models

Example: tRS Model with 2 Particles

Hamiltonians

$$T_1 = \frac{\xi_1 - t\xi_2}{\xi_1 - \xi_2} p_1 + \frac{\xi_2 - t\xi_1}{\xi_2 - \xi_1} p_2$$

$$T_2 = p_1 p_2$$

Symplectic form

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{d\xi_i}{\xi_i}$$

Integrals of motion

$$T_i = E_i$$

Coordinates ξ_i, momenta p_i coupling constant t, energies E_i

Quantization

$$p_i \xi_j = \xi_j p_i q^{\delta_{ij}} \qquad q \in \mathbb{C}^\times$$

tRS Momenta are shift operators

$$p_i f(\xi_i) = f(q\xi_i)$$

Eigenvalue Equations

$$T_i V = E_i V$$

Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

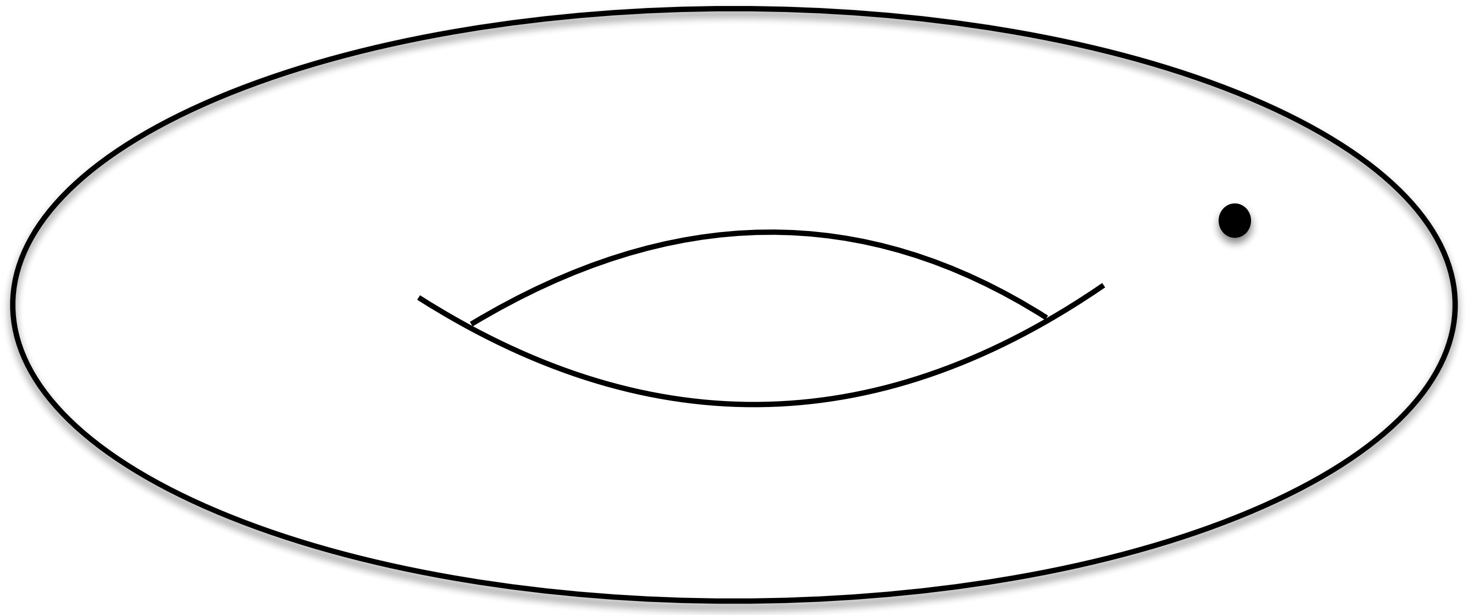
$$qMT - TM = u \otimes v^T$$

The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space $\mathscr{M}$**

Integrable Hamiltonians are ~TrT^k



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

$$C = \text{diag}(q, \ldots, q, q^{n-1})$$

Also can be understood as moduli space of flat connections on punctured torus

[my DAHA paper with Gukov, Nawata, Pei, Saberi [arXiv:2206.03565] to appear in **SpringerBriefs**]

Hierarchy of Models

[Mironov, Morozov, Gorsky...]
[Gorsky PK Koroteeva Shakirov]

p \ q	rational	trigonometric	elliptic
r	rational CMS	$\epsilon \to 0$ trigonometric CMS	$p \to 0$ elliptic CMS *quantum cohomology*
t	$R \to 0$ rational RS (dual trig. CMS)	$R \to 0$ $\epsilon \to 0$ trigonometric RS	$R \to 0$ $p \to 0$ elliptic RS *quantum K-theory*
e	dual elliptic CMS	$w \to 0$ dual elliptic RS	$w \to 0$ $p \to 0$ DELL *Elliptic Cohomology*

II. Quantum Integrability

Let $\mathfrak{g}$ Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}(t)$ loop algebra (Laurent poly valued in g)

Evaluation modules form a tensor category of $\hat{\mathfrak{g}}$

$$V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

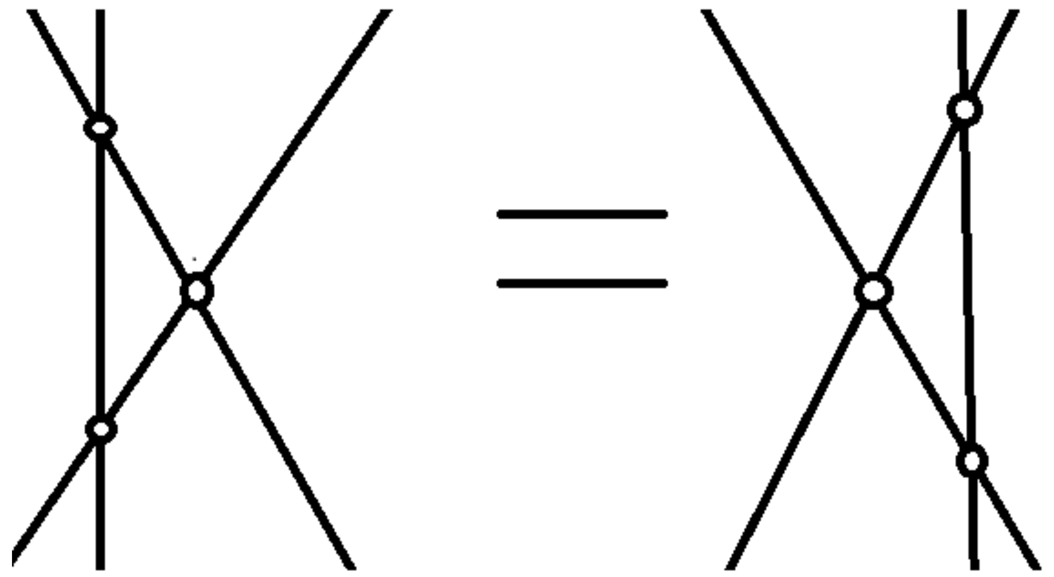
V_i are representations of $\mathfrak{g}$ a_i are special values of spectral parameter t

Quantum group is a noncommutative deformation $U_\hbar(\hat{\mathfrak{g}})$

with a nontrivial intertwiner — R-matrix

$$R_{V_1,V_2}(a_1/a_2) : V_1(a_1) \otimes V_2(a_2) \to V_2(a_2) \otimes V_1(a_1)$$

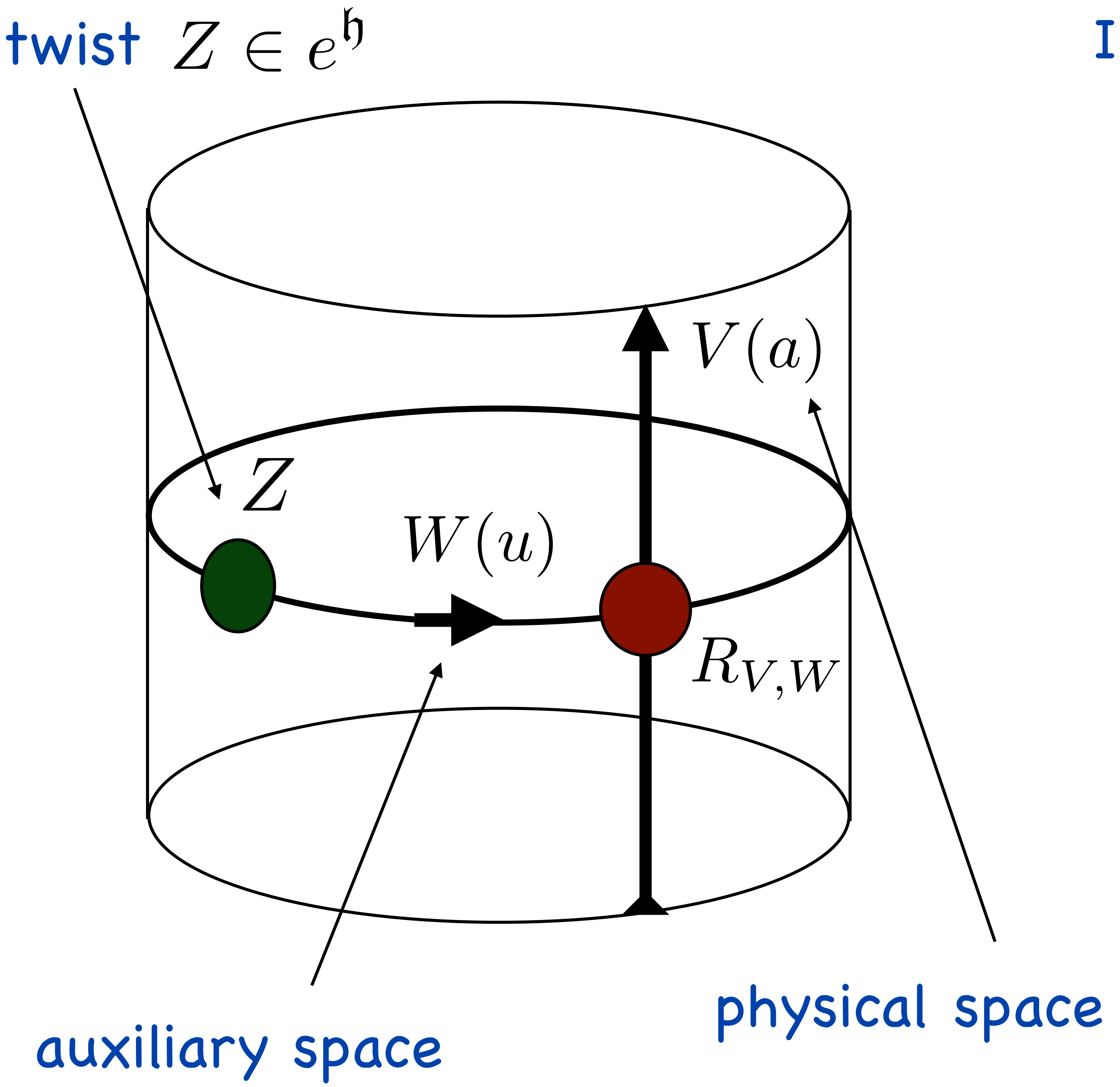
satisfying Yang-Baxter equation



Transfer Matrix

[Faddeev Reshetikhin Tachtajan]

The intertwiner represents an interaction vertex in integrable models. The quantum group is generated by matrix elements of R



Integrability comes from transfer matrix

$$T_W(u) = \mathrm{Tr}_{W(u)}\left((Z \otimes 1) T_{V,W}\right)$$

$$[T_W(u), T_W(u')] = 0$$

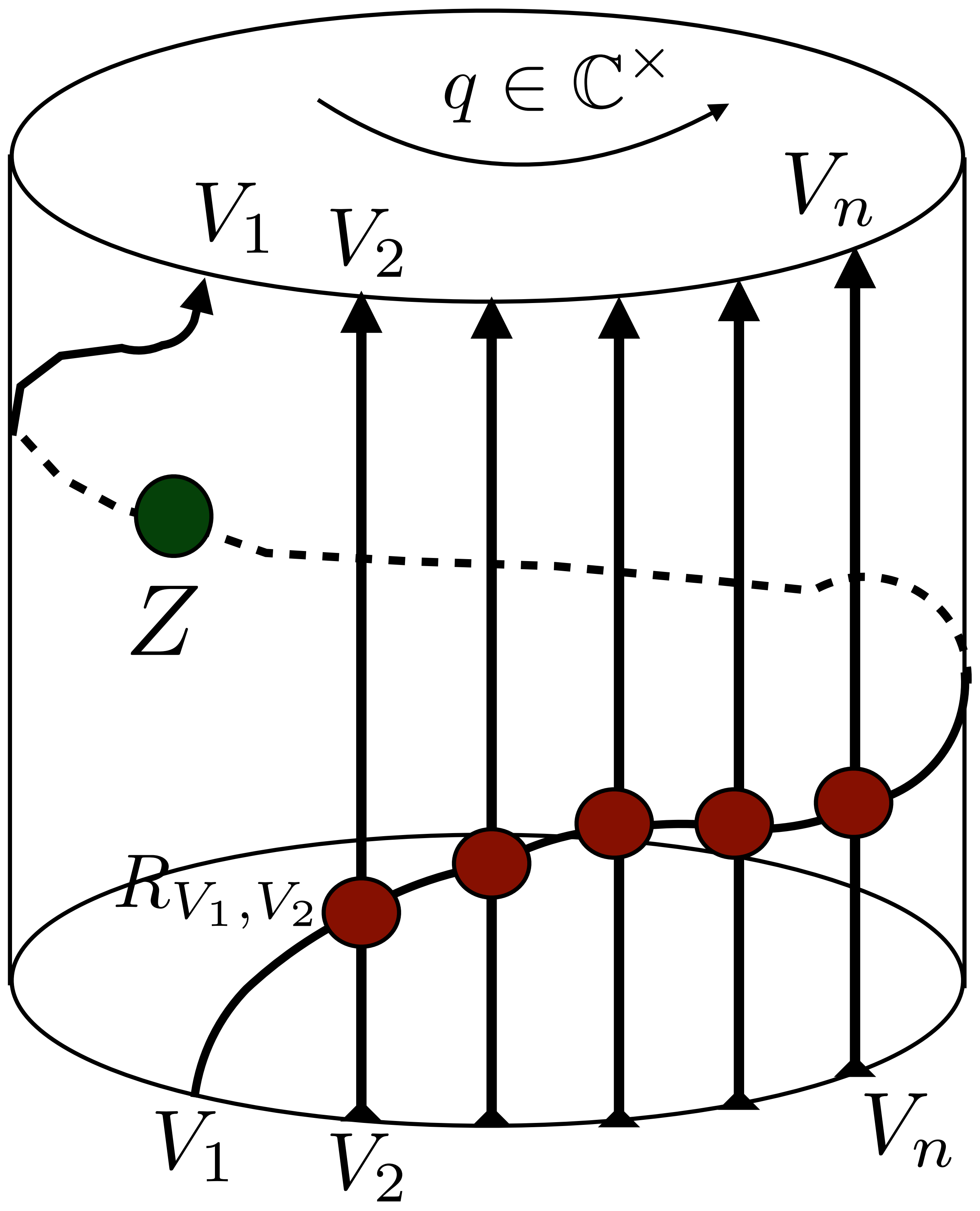
Transfer matrices are usually polynomials in u whose coefficients are the **integrals of motion**

The XXZ Spin Chain

$\mathfrak{g} = \mathfrak{sl}_2$ spin-1/2 chain on n sites $V = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$

Spectrum can be found using Bethe Ansatz techniques. However, if we want to understand the problem for more general algebras we need to think of the Knizhnik-Zamolodchikov difference equation (qKZ)

$$\Psi(qa_1, \dots a_n) = (Z \otimes 1 \otimes \cdots \otimes 1) R_{V_1,V_n} \cdots R_{V_1,V_2} \Psi(a_1, \dots a_n)$$



where

$$\Psi(a_1, \dots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

[I. Frenkel Reshetikhin]

In the limit $q \to 1$

qKZ becomes an eigenvalue problem

Solutions of qKZ

[Aganagic Okounkov]

Schematic solution

$$\Psi_\alpha = \int \frac{d\mathbf{x}}{\mathbf{x}}\, f_\alpha(\mathbf{x},a)\, \mathcal{K}(\mathbf{x},z,a,q)$$

indexed by physical space

representation

universal kernel

$$\log \mathcal{K}(\mathbf{x},z,a,q) \underset{q\to 1}{\sim} \frac{S(\mathbf{x},z,a)}{\log q}$$

$$\frac{\partial S}{\partial x_i} = 0$$ Bethe equations for Bethe roots **x**

$$a_i \frac{\partial S}{\partial a_i} = \Lambda_i$$ Eigenvalues of qKZ operators

The map $\alpha \mapsto f_\alpha(\mathbf{x}^*)$ provides diagonalization

So we need to find `off shell' Bethe eigenfunctions $f_\alpha(\mathbf{x},a)$

Nekrasov-Shatashvili Correspondence

The answer will come from enumerative algebraic geometry inspired by physics

Hilbert space of states of quantum integrable system

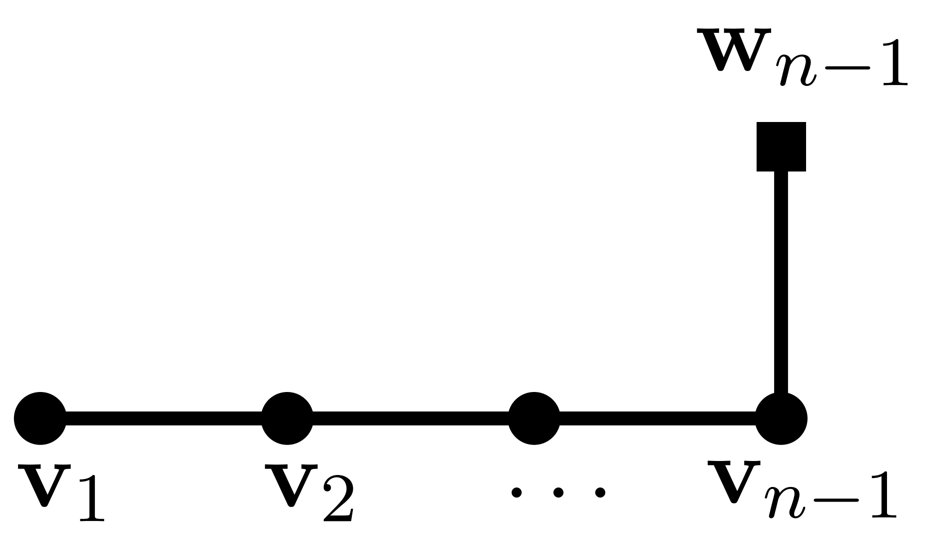
Equivariant K-theory of Nakajima quiver varitey (line operators in 3d SUSY gauge theory)

gauge group $G = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} U(v_i)$ $(v_1, v_2, \ldots)$ encode weight of rep α

Bethe roots **x** live in the maximal torus of G, by integrating over **x** we project on Weyl invariant functions of Bethe roots

Flavor group $G_F = \prod_i U(w_i)$ whose maximal torus gives parameters **a**

Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

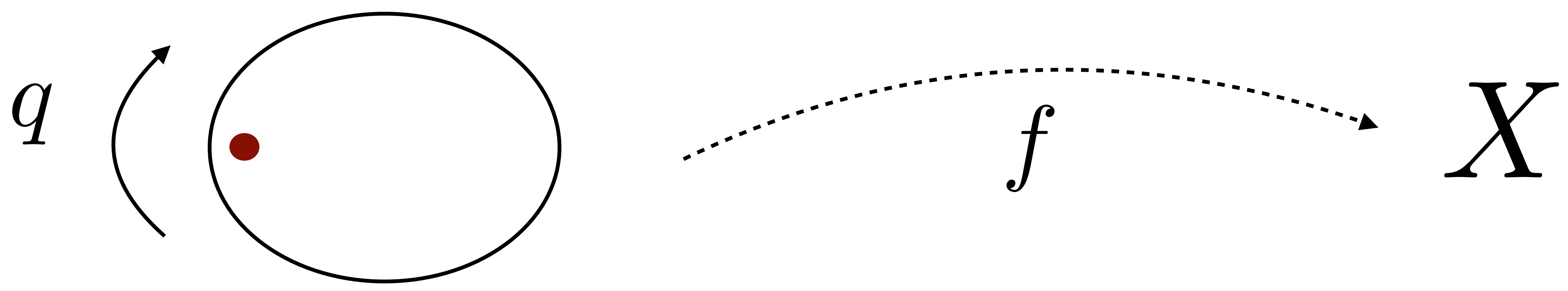


Quantum K-theory of X

The quiver variety X = {Matter fields}/gauge group

X is a module of some quantum group in Nakajima correspondence construction

We will be computing integrals in K-theory of the space of quasimaps $f : \mathcal{C} - -- > X$
weighted by degree $\mathbf{z}^{\deg f}$ subject to equivariant action on the base nodal curve $\mathbb{C}_q^\times$

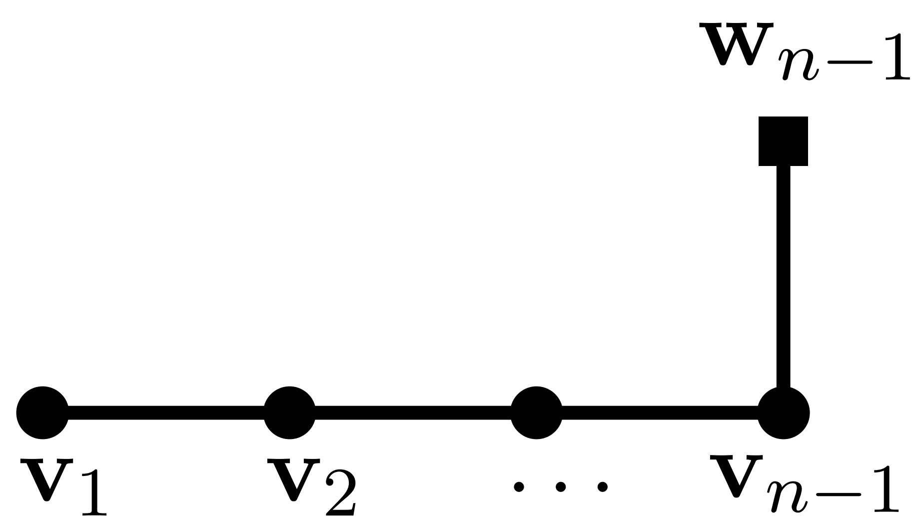


(cf Gromov-Witten invariants)

Quantum K-theory ring with quantum parameters **z** whose structure constants arise from 3-point correlators

Nakajima Quiver Varieties

Rep(**v**,**w**) — linear space of quiver reps



$\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$ moment map

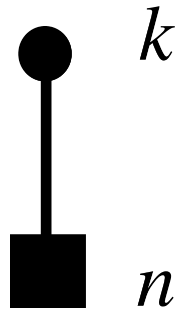
Nakajima quiver variety $X = \mu^{-1}(0)//_\theta G = \mu^{-1}(0)_{ss}/G$ $G = \prod GL(V_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus $T = \mathbb{T}(\mathrm{Aut}(X))$

Tensorial polynomials of tautological bundles V_i, W_i and their duals generate *classical T-equivariant K-theory* ring of X

Ex: T*Gr(k,n)



$\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$

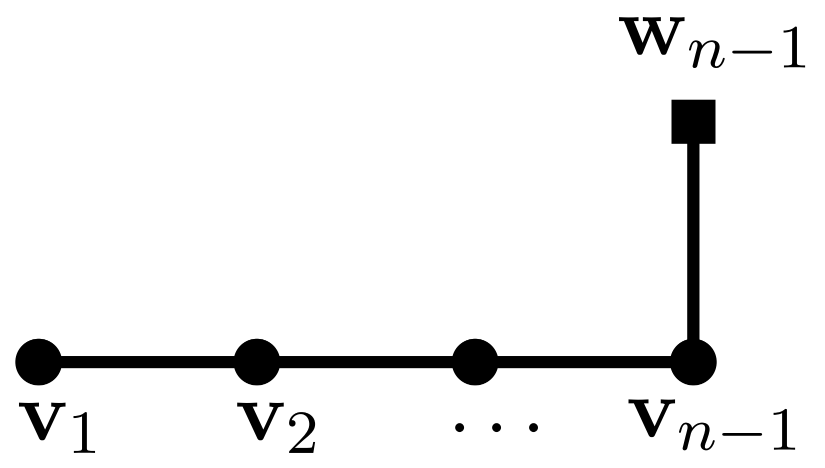
$\mathbf{v}_1 = k,\ \mathbf{w}_1 = n$

$$\tau(s_1, \cdots, s_k) = (s_1 + \cdots + s_k)^2 - \sum_{1\leq i_1<i_2<i_3\leq k} s_{i_1}^{-1} s_{i_2}^{-1} s_{i_3}^{-1}$$

Quasimaps

[Ciocan-Fontanine, Kim, Maulik]
[Okounkov]

Quasimap $f : \mathcal{C} - - \to X$ is described by collection of vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$ with section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$



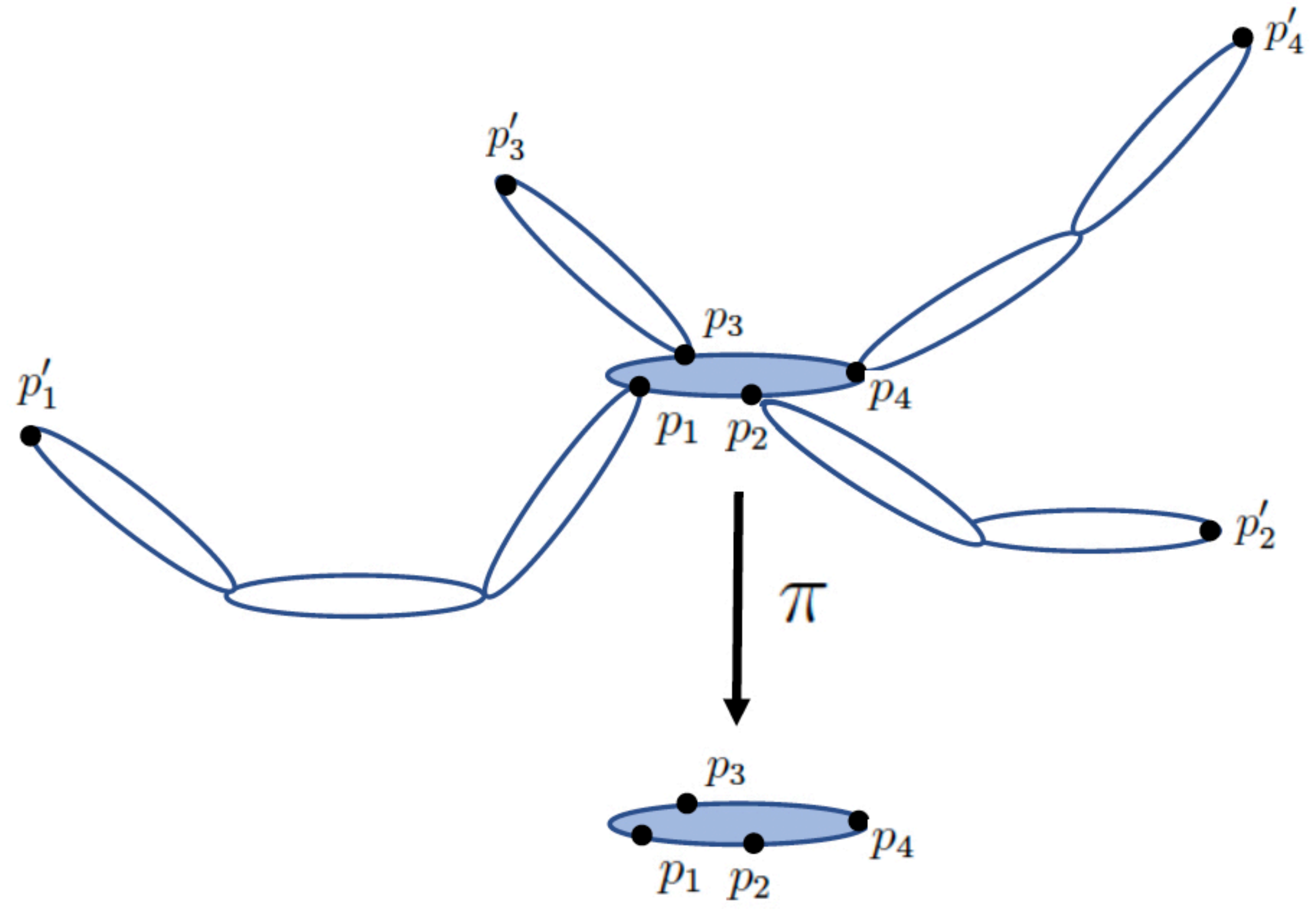
where $\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$

d_i degrees of $\mathscr{V}_i$.

Evaluation map to quotient stack

$$\mathrm{ev}_{p_i} : \mathsf{QM}^d_{\text{relative},p_1,\cdots,p_m} \to [\mu^{-1}(0)/G_{\mathsf{v}}]$$

$$\mathrm{ev}_p(C, p'_1, \ldots, p'_m, P, f, \pi) = f(p)$$



QM is nonsingular if $f(p) \in X$

for all but finitely many singular points

Vertex Function

[Okounkov]
[Pushkar Smirnov Zeitlin]

Spaces of quasimaps admit an action of an extra torus $\mathbb{C}_q^\times$ which scales the base $\mathbb{P}^1$ keeping two fixed points $p_1 = 0, \;\; p_2 = \infty$

Define **vertex function** with quantum (Novikov) parameters $z^{\mathbf{d}} = \prod_{i \in I} z_i^{d_i}$

$$\mathbf{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}(\widehat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \tau|_{p_1}, \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{nonsing}\, p_2}) \boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T} \times \mathbb{C}_q^\times}(X)_{loc}[[\boldsymbol{z}]]$$

descendent

Define **quantum K-theory** as a ring with multiplication

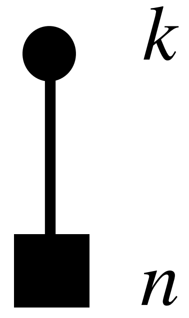
$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

Theorem: QK(X) is a commutative associative unital algebra

Bethe Equations for T*Gr(k,n)

Operator of quantum multiplication from saddle point approximation

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V_{\boldsymbol{p}}^{(\tau)}(z)}{V_{\boldsymbol{p}}^{(1)}(z)}$$



Theorem *The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(s_1,\cdots,s_k)$ evaluated at the solutions of the following equations:*

$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}\,, \quad i = 1\cdots k.$$

Equivariant parameters a_i,

twist z,

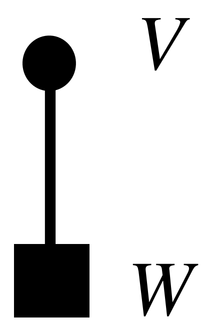
Planck constant $\hbar$

Baxter Q-operator $\quad Q(z) = \sum_{i=1}^{k} (-1)^i z^{k-i} (\Lambda^i V)(z) \otimes$

Has eigenvalue $\quad Q(z) = \prod_{i=1}^{k} (z - s_i)$

QQ-System for A_1

Short exact sequence of bundles



$$0 \to V \to W \to V^{\vee} \to 0$$

Eigenvalues of Q-operators

$$Q(z) = \sum_{i=1}^{k} (-1)^i z^{k-i} (\Lambda^i V)(z) \circledast$$

$$\widetilde{Q}(z) = \sum_{i=1}^{k} (-1)^i z^{k-i} (\Lambda^i V^{\vee})(z) \circledast$$

Satisfy the QQ-relation

$$z\widetilde{Q}(\hbar z) Q(z) - \widetilde{Q}(z) Q(\hbar z) = \prod_{i=1}^{n} (z - a_i)$$

Which is equivalent to the Bethe equations

QQ-System in General

Consider complex simple Lie algebra $\mathfrak{g}$ of rank r

Cartan matrix $a_{ij} = \langle \check{\alpha}_i, \alpha_j \rangle$

$$\widetilde{\xi}_i Q^i_-(z) Q^i_+(\hbar z) - \xi_i Q^i_-(\hbar z) Q^i_+(z) = \Lambda_i(z) \prod_{j>i} \Big[Q^j_+(\hbar z) \Big]^{-a_{ji}} \prod_{j<i} \Big[Q^j_+(z) \Big]^{-a_{ji}},$$
$$i = 1, \ldots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Polynomials $Q_+(z)$ contain Bethe roots, $\Lambda(z)$ contain equivariant parameters

Polynomials $Q_-(z)$ are auxiliary

The Ubiquitous QQ-System

Bethe Ansatz equations for XXX, XXZ models — eigenvalues of Baxter operators

[Mukhin, Varchenko] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\hat{g})$

[Frenkel, Hernandez] ....

Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Nekrasov-Shatashvili] [Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

Spectral determinants in the QDE/IM Correspondence

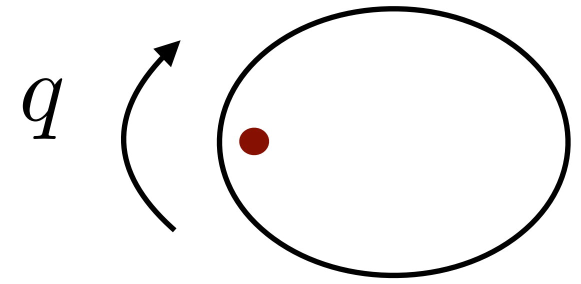
[Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

(G,q)-Opers

Quantum/Classical duality?

III. (G,q)-Connection

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto qz$$



G-simple simply-connected complex Lie group

Consider vector bundle $\mathscr{F}_G$ over $\mathbb{P}^1$

(G, q)-connection A is a meromorphic section of $Hom_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{F}_G, \mathscr{F}^q_G)$

Locally q-gauge transformation of the connection

$$A(z) \mapsto g(qz)A(z)g(z)^{-1} \qquad g(z) \in G(\mathbb{C}(z))$$

Compare with (standard) gauge transformations

$$\partial_z + A(z) \mapsto g(z)(\partial_z + A(z))g(z)^{-1} \qquad g(z) \in \mathfrak{g}(z)$$

(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathcal{F}_{B_-}$ is a reduction of $\mathcal{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathcal{F}_G \longrightarrow \mathcal{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally $$A(z) = n'(z)\prod_i(\phi_i(z)^{\check{\alpha}_i}\, s_i)n(z)$$

$\phi_i(z) \in \mathbb{C}(z)$ and $n(z), n'(z) \in N_-(z)$ $\qquad N_- = [B_-, B_-]$

(SL(2),q)-Opers

Let $G = SL(2)$ The q-oper definition can be reformulated as

Triple $(E, A, \mathscr{L})$

(E, A) is the $(SL(2), q)$ connection

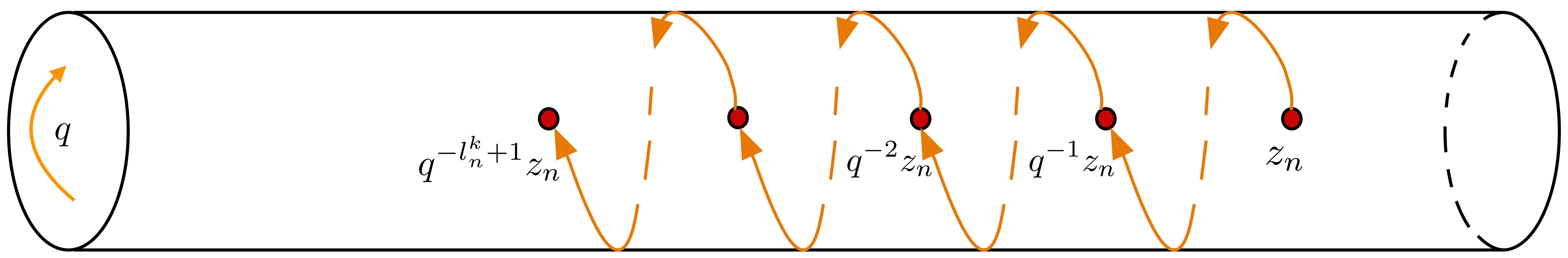
$\mathscr{L} \subset E$ is a line subbundle

The induced map $\bar{A} : \mathscr{L} \to (E/\mathscr{L})^q$ is an isomorphism

in a trivialization $\mathscr{L}$ = Span(s) $\quad s(qz) \wedge A(z)s(z) \neq 0$

Allow singularities $\quad s(qz) \wedge A(z)s(z) = \Lambda(z)$

$$\Lambda(z) = \prod_{p=1}^{L} \prod_{j_p=0}^{r_p-1} (z - q^{-j_p} z_p)$$



Add Twists $\quad Z = g(qz)A(z)g(z)^{-1}$ $\quad Z \in H \subset H(z) \subset G(z)$

q-Opers, QQ-System, and Bethe Ansatz

Chose trivialization of $\mathcal{L}$ $\quad s(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix}$

Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition — SL(2) **QQ-system**

$$s(qz) \wedge Z s(z) = \Lambda(z) \longrightarrow \zeta Q_-(z) Q_+(zq) - \zeta^{-1} Q_-(zq) Q_+(z) = \Lambda(z)$$

Roots of Q_+ $\quad Q_+(z) = \prod_{k=1}^{m} (z - w_k)$

From QQ-system to XXZ Bethe equations

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \qquad k = 1, \dots, m.$$

$$q^r \prod_{p=1}^{L} \frac{w_k - q^{1-r_p} z_p}{w_k - q z_p} = -\zeta^2 q^m \prod_{j=1}^{m} \frac{q w_k - w_j}{w_k - q w_j}, \qquad k = 1, \dots, m$$

$$\hbar = q$$

q-Miura Transformation

Miura (SL(2),q)-oper is a quadruple $(E, A, \mathcal{L}, \hat{\mathcal{L}})$ where $(E, A, \mathcal{L})$ is an (SL(2),q)-oper and $\hat{\mathcal{L}}$ is preserved by the q-connection *A*

$$A(z) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition $$A(z) = v(zq)Zv(z)^{-1}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

$$g(z) = \zeta \frac{Q_+(qz)}{Q_+(z)}$$

$$v(z) = \begin{pmatrix} Q_+(z)^{-1} & \xi Q_+(qz)Q_-(z) - \xi^{-1}Q_+(z)Q_-(qz) \\ 0 & Q_+(z) \end{pmatrix} \in B_+(z)$$

The q-oper condition becomes the **SL(2) QQ-system** $$\zeta Q_-(z)Q_+(zq) - \zeta^{-1}Q_-(zq)Q_+(z) = \Lambda(z)$$

Difference Equation $$D_q(s) = As.$$

Scalar difference operator $$\left(D_q^2 - T(qz)D_q - \frac{\Lambda(qz)}{\Lambda(z)} \right) s_1 = 0$$

tRS Hamiltonians

Recover 2-body tRS Hamiltonian from a simple q-Oper

Let $\quad Q_- = z - p_- \quad$ and $\quad Q_+ = c(z - p_+)$

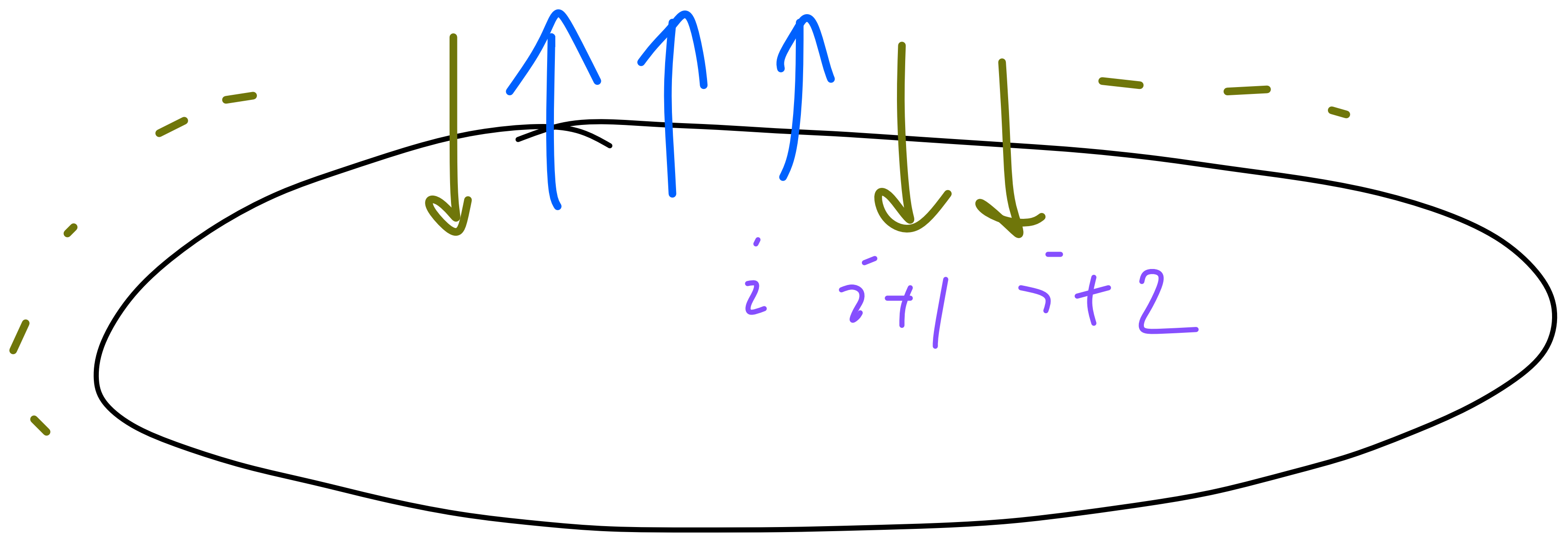
$$z^2 - \frac{z}{q}\Bigg[\underbrace{\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}}p_+ + \frac{q\zeta - \zeta^{-1}}{\zeta - \zeta^{-1}}p_-}_{T_1}\Bigg] + \underbrace{\frac{p_+p_-}{q}}_{T_2} = (z - z_+)(z - z_-)$$

qOper condition yields
tRS Hamiltonians!

$$\det(z - L_{tRS}) = (z - z_+)(z - z_-)$$

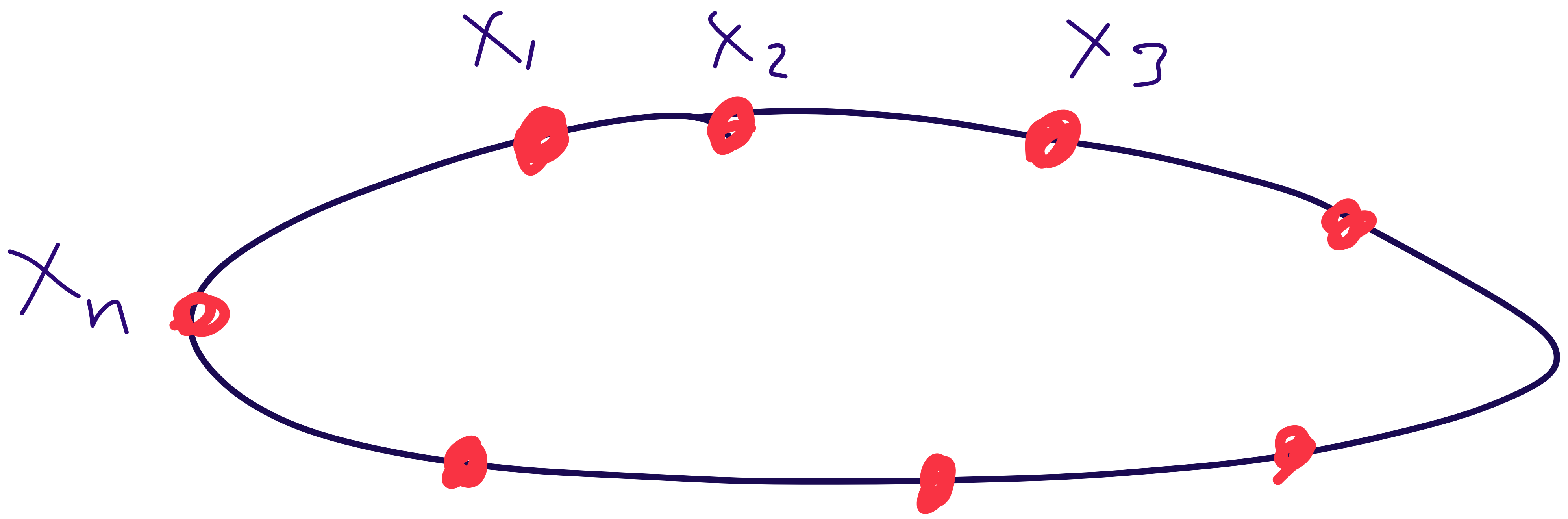
Quantum

QQ-Systems



Classical

q-Opers



SU(**n**) XXZ spin chain on n sites w/ ***anisotropies*** and ***twisted periodic boundary conditions***

Planck's constant $\hbar$

twist eigenvalues z_i

equivariant parameters (anisotropies) a_i

Bethe Ansatz Equations: $\dfrac{\partial Y}{\partial \sigma_i} = 0$

$$\frac{\zeta_i}{\zeta_{i+1}} \prod_{\beta=1}^{\mathbf{v}_{i-1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2}\sigma_{i-1,\beta}}{\sigma_{i-1,\beta} - \hbar^{1/2}\sigma_{i,\alpha}} \cdot \prod_{\beta\neq\alpha}^{\mathbf{v}_i} \frac{\hbar\sigma_{i,\alpha} - \sigma_{i,\beta}}{\hbar\sigma_{i,\beta} - \sigma_{i,\alpha}} \cdot \prod_{\beta=1}^{\mathbf{v}_{i+1}} \frac{\sigma_{i,\alpha} - \hbar^{1/2}\sigma_{i+1,\beta}}{\sigma_{i+1,\beta} - \hbar^{1/2}\sigma_{i,\alpha}} = (-1)^{\delta_i}$$

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{dz_i}{z_i}$$

n-particle trigonometric Ruijsenaars-Schneider model

$$[T_i, T_j] = 0$$

Coupling constant $\hbar$

$$T_1 = \sum_{i=1}^{n} \prod_{j\neq i} \frac{\hbar z_i - z_j}{z_i - z_j} p_i$$

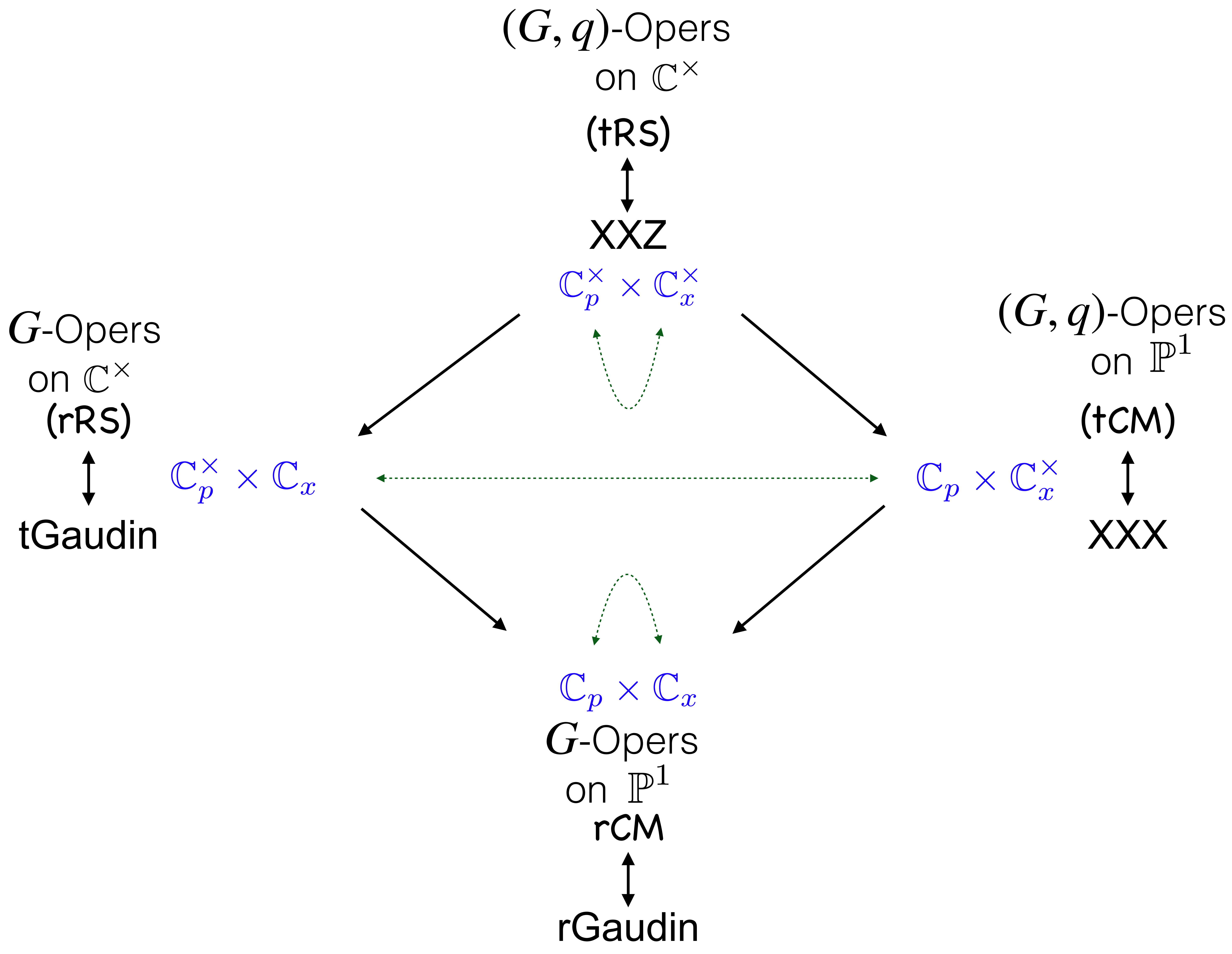
coordinates z_i

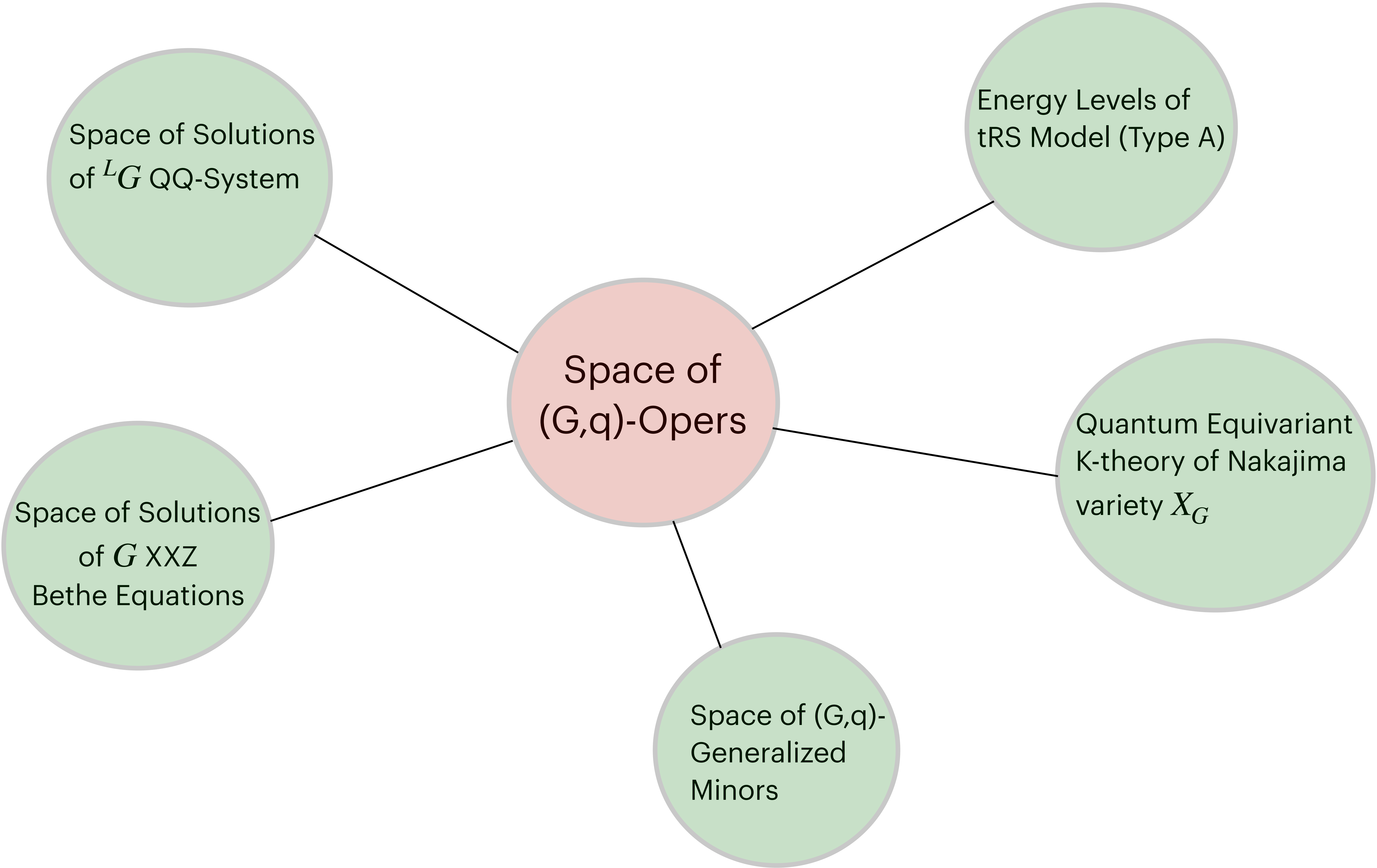
energy (eigenvalues of Hamiltonians) $e_i(a_i)$

Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \ldots, n$$

Network of Dualities




Space of Solutions of ${}^{L}G$ QQ-System
Energy Levels of tRS Model (Type A)
Space of (G,q)-Opers
Quantum Equivariant K-theory of Nakajima variety X_G
Space of Solutions of G XXZ Bethe Equations
Space of (G,q)-Generalized Minors

q-Opers and q-Langlands

[Frenkel, PK, Zeitlin, Sage, 2021, to appear in JEMS]

Miura (G,q)-oper with singularities
$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i} \; e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \qquad g_i(z) \in \mathbb{C}(z)^\times$$

Theorem: There is a one-to-one correspondence between the set of nondegenerate Z-twisted (G,q)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

$$\widetilde{\xi}_i Q^i_-(z) Q^i_+(qz) - \xi_i Q^i_-(qz) Q^i_+(z) =$$
$$\Lambda_i(z) \prod_{j>i} \Big[Q^j_+(qz) \Big]^{-a_{ji}} \prod_{j<i} \Big[Q^j_+(z) \Big]^{-a_{ji}}, \qquad i = 1, \ldots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Proof uses
$$v(z) = \prod_{i=1}^r y_i(z)^{\check{\alpha}_i} \prod_{i=1}^r e^{-\frac{Q^i_-(z)}{Q^i_+(z)} e_i} \ldots, \qquad g_i(z) = \zeta_i \frac{Q^i_+(qz)}{Q^i_+(z)}$$

Cluster Algebras

[PK, Zeitlin, 2022, to appear in Crelle]

The QQ-system $\xi_{i+1}Q_i^+(z+\epsilon)Q_i^-(z) - \xi_i Q_i^+(z)Q_i^-(z+\epsilon) = (\xi_{i+1}-\xi_i)\Lambda_i(z)Q_{i-1}(z)Q_{i+1}(z)$

For $G = SL(n)$ obtain Lewis Carrol identity

$$M_1^1 M_i^2 - M_i^1 M_1^2 = M_{1i}^{12} M$$

For general G obtain relation on generalized minors $\Delta^{\omega_i}(v^{-1}(z)) = Q_+^i(z)$

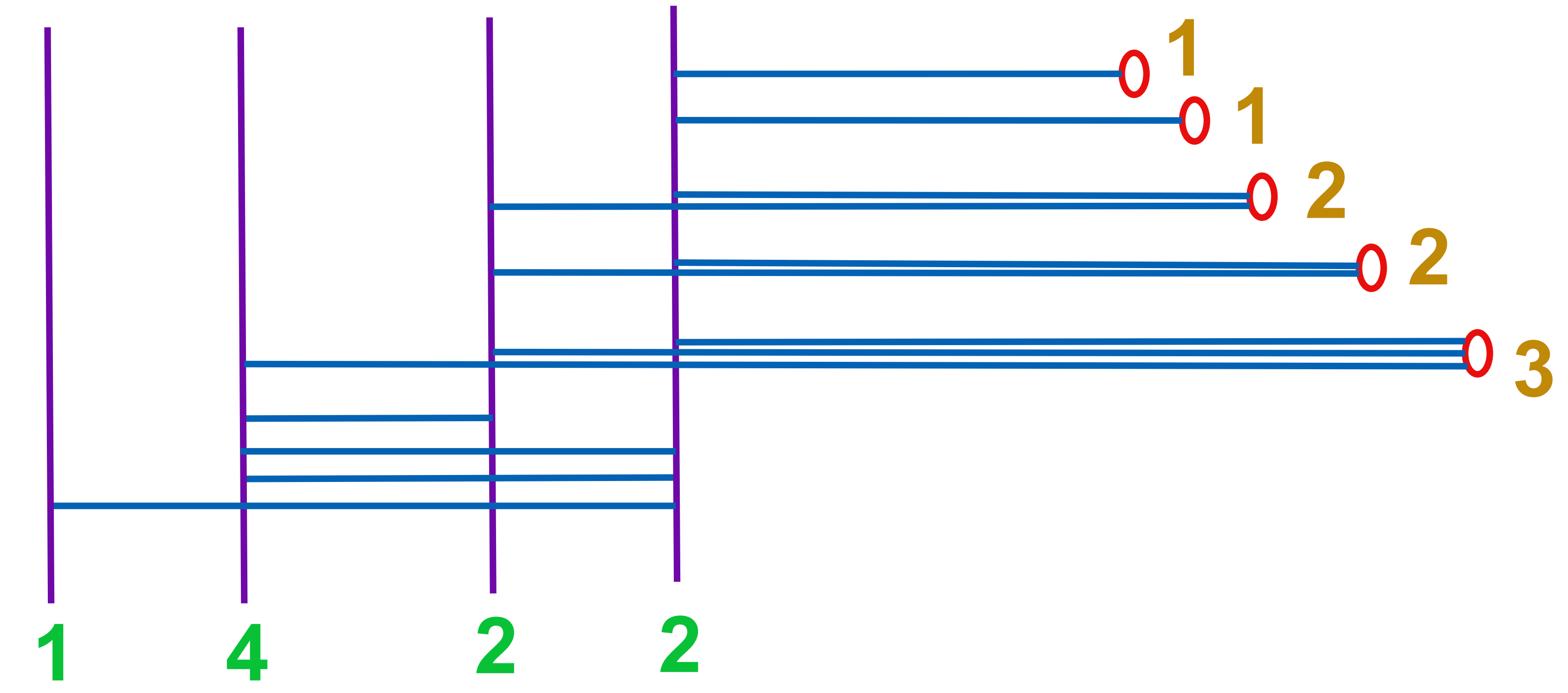
[Fomin Zelevinsky]

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}\Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i}\Delta_{u\cdot\omega_i, vw_i\cdot\omega_i} = \prod_{j\neq i}\Delta_{u\cdot\omega_j, v\cdot\omega_j}^{-a_{ji}},$$

$u, v \in W_G$

Quantum/Classical Duality & 3d Mirror Symmetry

[PK Gaiotto]
[PK Zeitlin]



Symplectic form $$\Omega = \sum_{i=1}^{N} \frac{dp_i^{\xi}}{p_i^{\xi}} \wedge \frac{d\xi_i}{\xi_i} - \frac{dp_i^{a}}{p_i^{a}} \wedge \frac{da_i}{a_i}$$

tRS momenta $$p_i^{\xi} = \exp \frac{\partial Y}{\partial \xi_i}\,, \qquad p_i^{a} = \exp \frac{\partial Y}{\partial a_i}$$

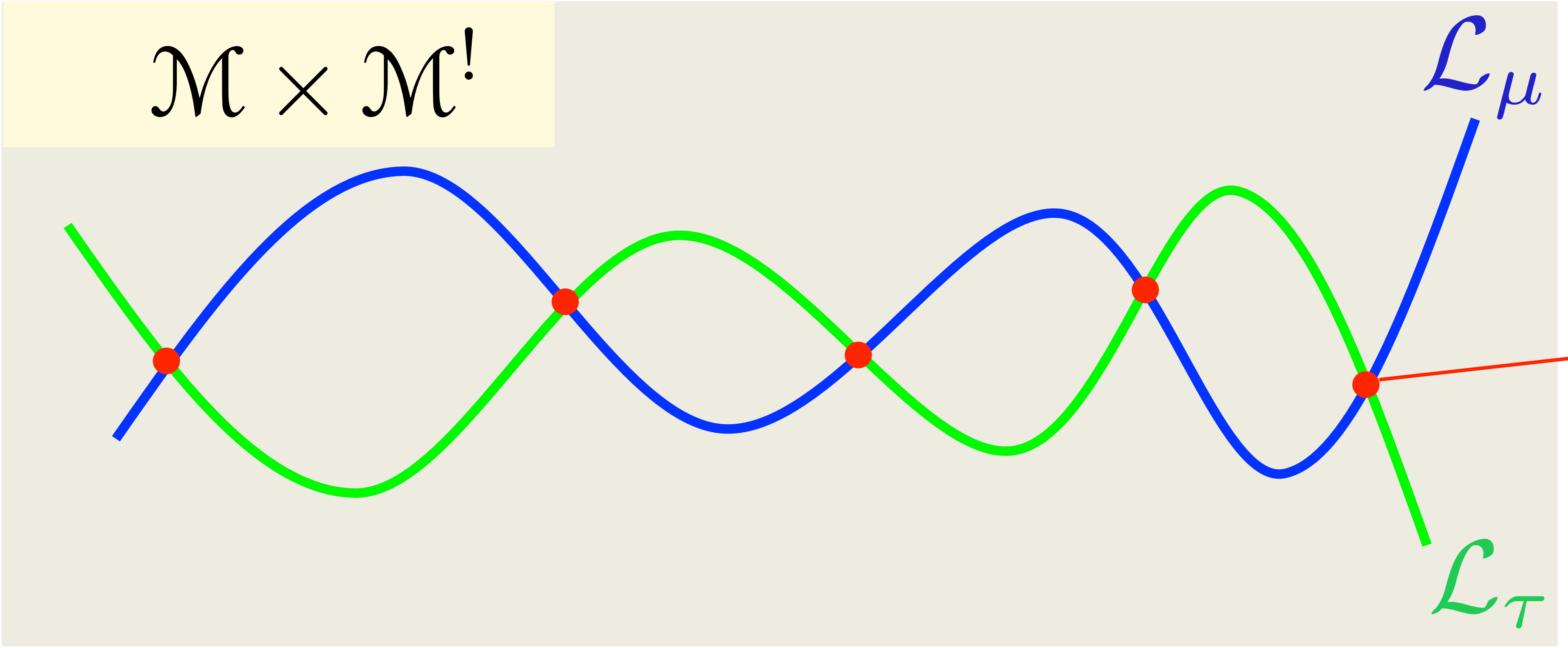
tRS energy relations = XXZ Bethe equations

$$qMT - TM = u \otimes v^T$$

$$\det(u - T) = \prod_{i=1}^{N} (u - a_i)\,, \qquad \det(u - M) = \prod_{i=1}^{N} (u - \xi_i)$$

$\mathcal{L}_\mu$ Eigenvalues of M and Slodowy form on T

$\mathcal{L}_\tau$ Eigenvalues of T and Slodowy form on M



$Y = Y^!$

3d mirror symmetry

Solutions of Bethe equations — intersection points

q-Langlands Correspondence

[Aganagic Frenkel Okounkov]

Two types of solutions of the qKZ equation:

Analytic in chamber of equivariant parameters $\{a_i\}$— conformal blocks of $U_\hbar(\widehat{g})$

Analytic in chamber of quantum parameters (twists) $\{\zeta_i\}$— conformal blocks for deformed W-algebra $W_{q,\hbar}({}^L\widehat{g})$

The q-Langlands correspondence

$$\begin{tikzcd} U_\hbar(\widehat{g}) \arrow[d] \arrow[r, leftrightarrow] & W_{q,\hbar}({}^L\widehat{g}) \arrow[d, "q \to 1"] \\ \text{XXZ Bethe} & (G,\hbar)\text{-oper} \end{tikzcd}$$

Equivalence of categories

$$\mathrm{Dmod}_\kappa(Bun_G) \longleftrightarrow \mathrm{Dmod}_{-\frac{1}{m\kappa}}(Bun_{{}^LG})$$

Merci Beaucoup!