Quantum Geometry & Integrable Systems

Peter Koroteev

Talk at Heriot-Watt University 3/06/2023

My Background

- I work in Representation Theory and Algebraic Geometry with applications to Mathematical Physics, in particular, to Integrable Systems
- A theoretical physicist by training, I have now almost completely switched to pure math. Still, I try to write one or two papers per year in hep-th
- The term `*Physical Mathematics*' (in a nutshell — using string theory/QFT intuition to prove math theorems) is perhaps the most precise two-word description of my research

Current Research

- Integrable Systems from Algebraic Geometry
  Enumerative counts for Nakajima quiver varieties, Opers, Geometric Langlands Correspondence.

- Geometric Representation Theory
  Quantization by Branes. Algebras from deformation quantization of some nice families of hyperKähler spaces.

- Physics and Mathematics of $\mathcal{N} = 2$ gauge theories and their stringy origins
  The BPS/CFT correspondence

Early Career Work

- Cosmology
- Nonperturbative aspects of Supersymmetric Quantum Field Theory
- Condensed Matter applications
- Resurgence in QFT

Papers on AG&Integrability

[arXiv:23xx.xxxxx]
The qDE/IM Correspondence
E. Frenkel, P. Koroteev, A. M. Zeitlin

[arXiv:2208.08031]
The Zoo of Opers and Dualities
P. Koroteev, A. M. Zeitlin

[arXiv:2108.04184] **Crelle Journal**
q-Opers, QQ-systems, and Bethe Ansatz II: Generalized Minors
P. Koroteev, A. M. Zeitlin

[arXiv:2105.00588]
3d Mirror Symmetry for Instanton Moduli Spaces
P. Koroteev, A. M. Zeitlin

[arXiv:2007.11786] **J. Inst. Math. Jussieu**
Toroidal q-Opers
P. Koroteev, A. M. Zeitlin

[arXiv:2002.07344] **JEMS**
q-Opers, QQ-Systems, and Bethe Ansatz
E. Frenkel, P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1805.00986] **Commun.Math.Phys. 381 (2021) 175**
A-type Quiver Varieties and ADHM Moduli Spaces
P. Koroteev

[arXiv:1811.09937] **Commun.Math.Phys. 381 (2021) 641**
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality
P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1802.04463] **Math.Res.Lett. 28 (2021) 435**
qKZ/tRS Duality via Quantum K-Theoretic Counts
P. Koroteev, A. M. Zeitlin

[arXiv:1705.10419] **Selecta Math. 27 (2021) 87**
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

Classical Integrability

- Classical integrable systems of n d.o.f. have n integrals of motion that are in involution with each other $\{H_i, H_j\}_{\text{PB}} = 0$.

- Examples include many-body systems like Calogero, Ruijsenaars, DELL, etc $\sum \frac{p_i^2}{2m} + \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}$, and continuous (1+1) dimensional models like KdV, Intermediate Long Wave, etc.

[arXiv:1510.00972] Lett.Math.Phys. **108** (2018) 45
[arXiv:1601.08238] J.Math.Phys. **57** (2016) 112302

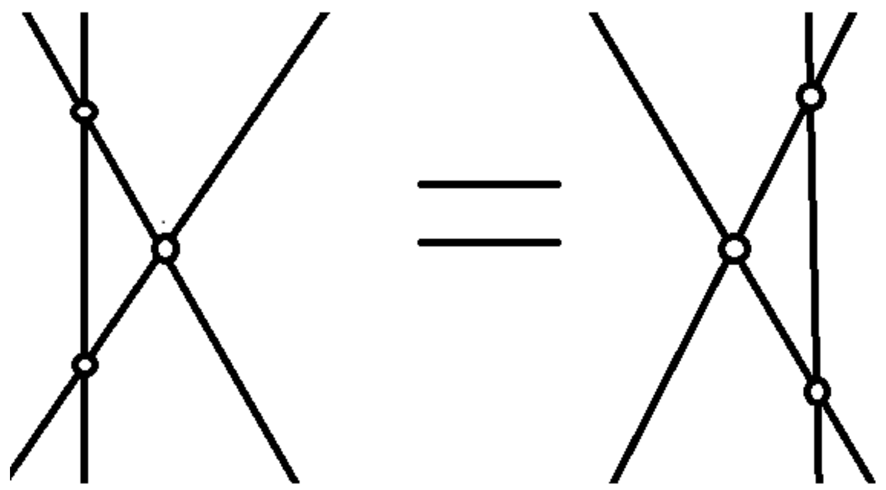
- The former can be defined algebraically. The latter admit soliton solutions and are connected to the former. Both were shown to be connected to the Seiberg-Witten solution of $\mathcal{N} = 2$ theories and to geometry

- Compact Lagrangians $\{H_i = E_i\}$ are isomorphic to tori and evolution in their vicinity is linear (Liouville-Arnold)

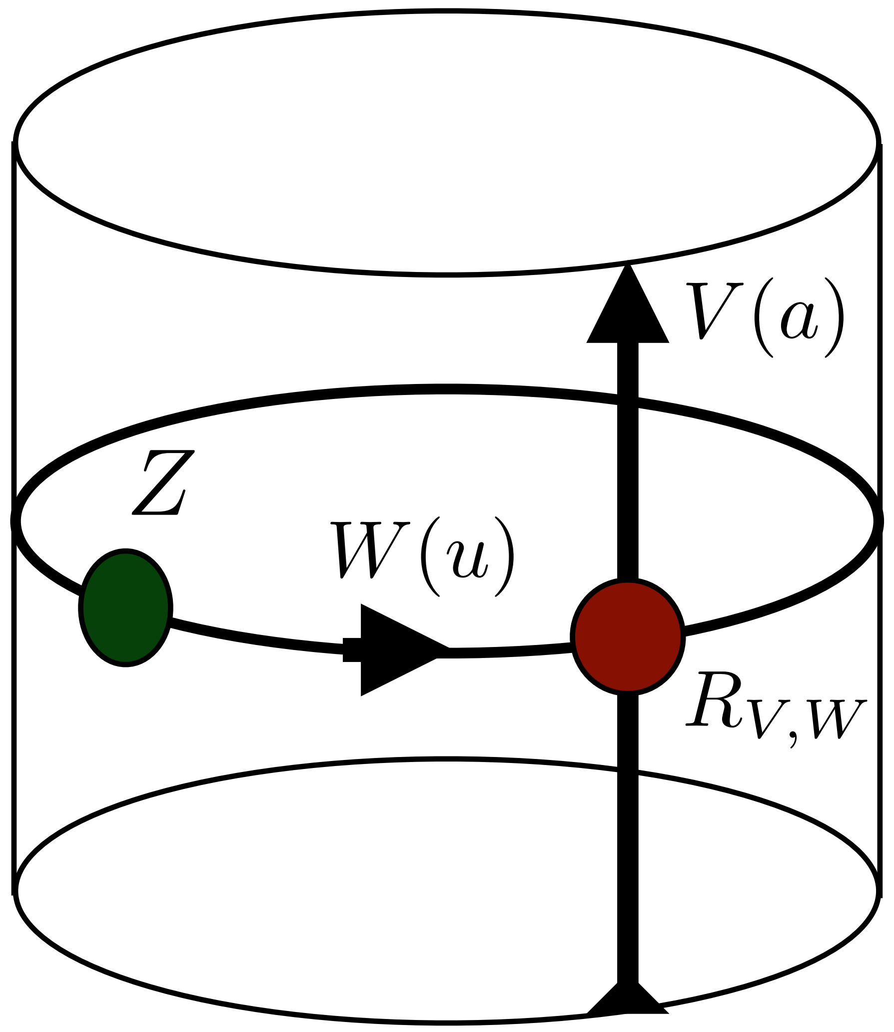
Quantum Integrability

Quantum group $U_\hbar(\hat{\mathfrak{g}})$ is a noncommutative deformation of the loop group with a nontrivial intertwiner — R-matrix

Yang-Baxter equation



$$R_{V_1,V_2}(a_1/a_2):\ V_1(a_1)\otimes V_2(a_2)\to V_2(a_2)\otimes V_1(a_1)$$

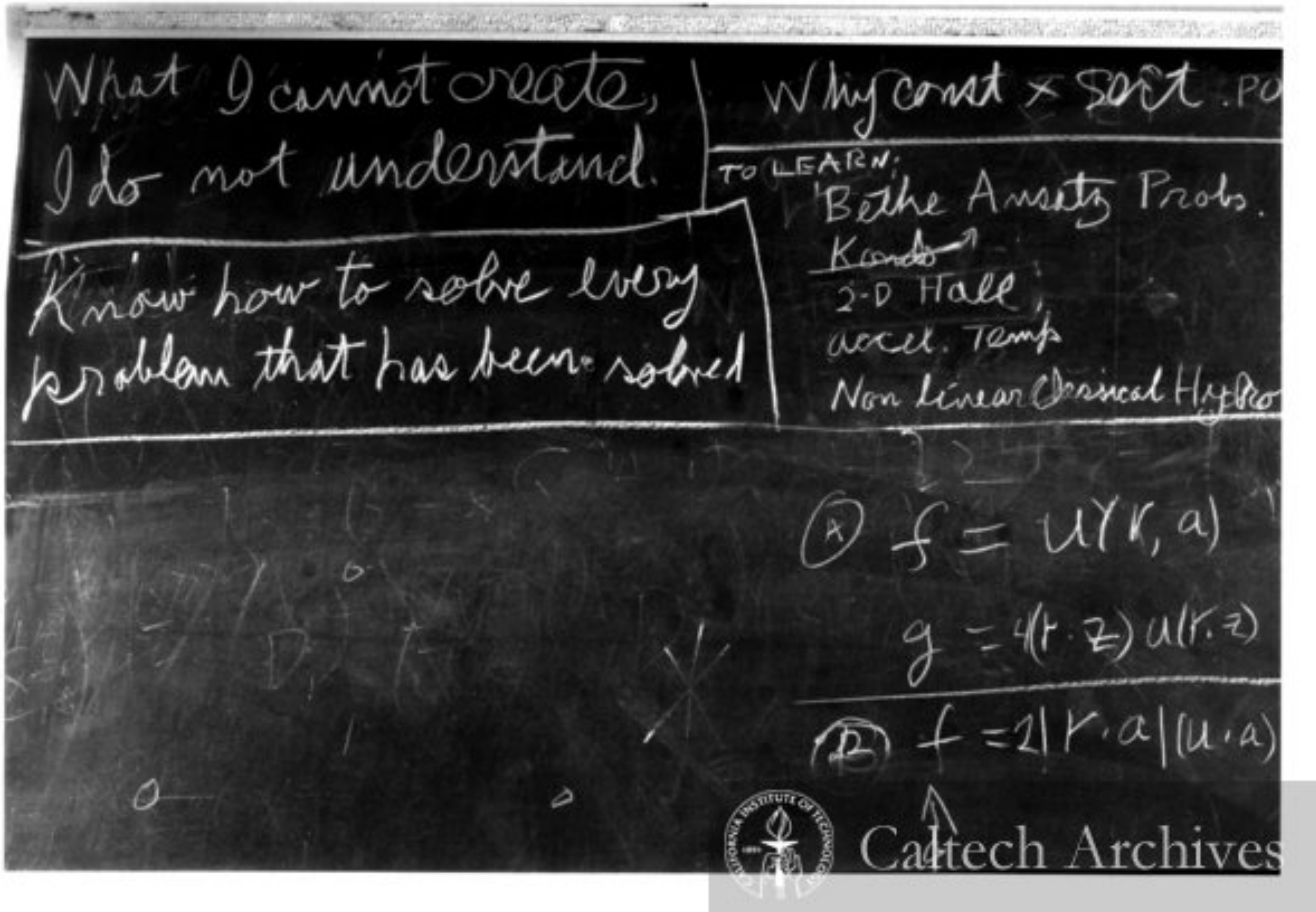


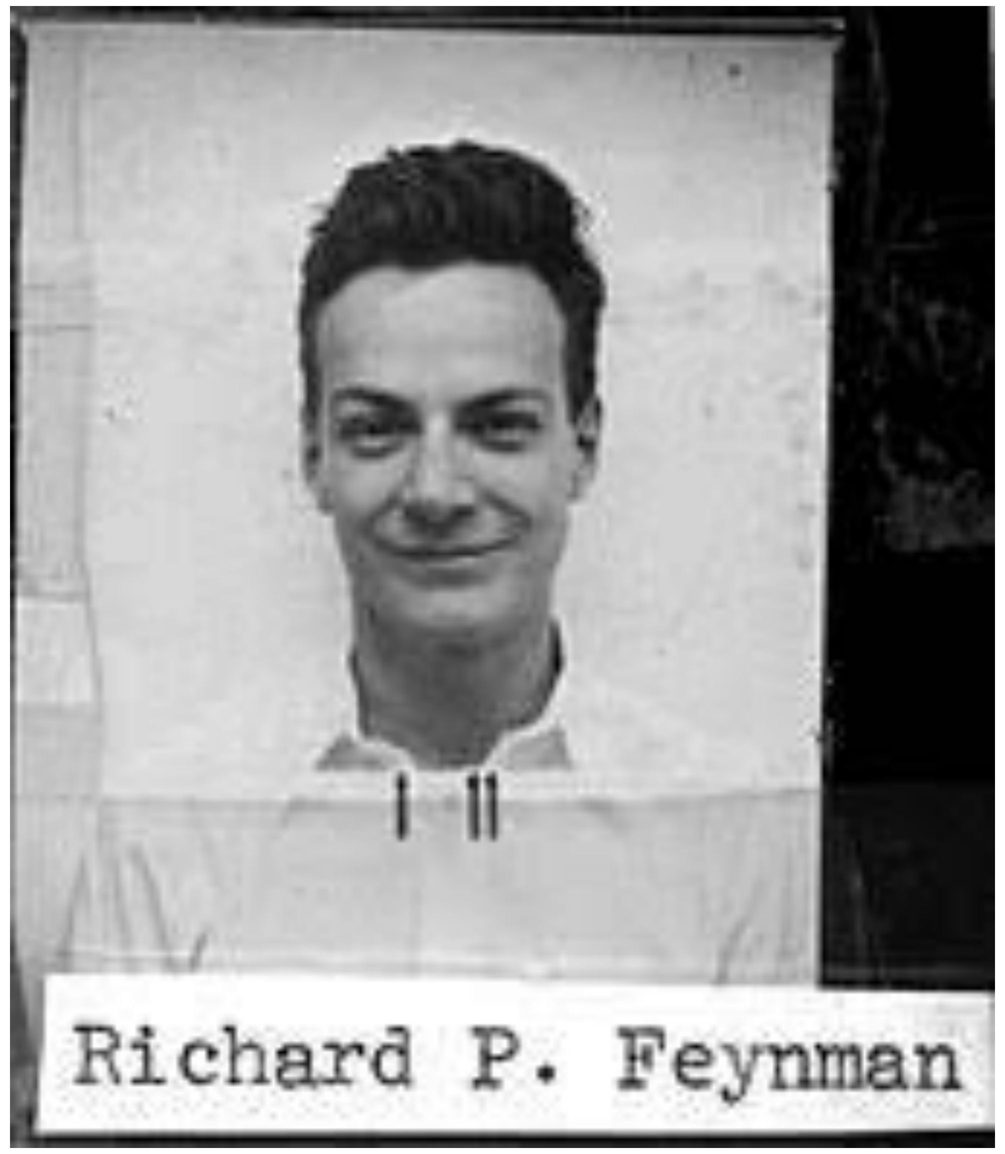
Integrability comes from transfer matrices which generates Bethe algebra

$$T_W(u) = Tr_{W(u)}((Z\otimes 1)R_{V,W}) \qquad [T_W(u),T_W(u')]=0$$

Transfer matrices are usually polynomials in u whose coefficients are the **integrals of motion**

Classical IS can be quantized using methods of physics — Omega background [Nekrasov], Quantization by branes [Gukov, Witten]

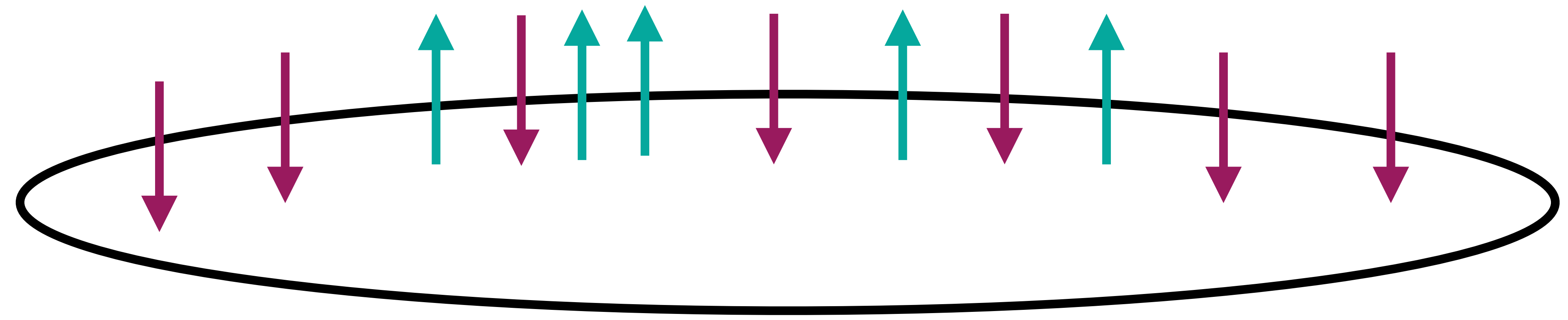




I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why.
I am trying to understand all this better.

Quantum

QQ-Systems



XXZ Spin chain with **anisotropies** and **twisted periodic boundary conditions**

Planck's constant $\hbar$

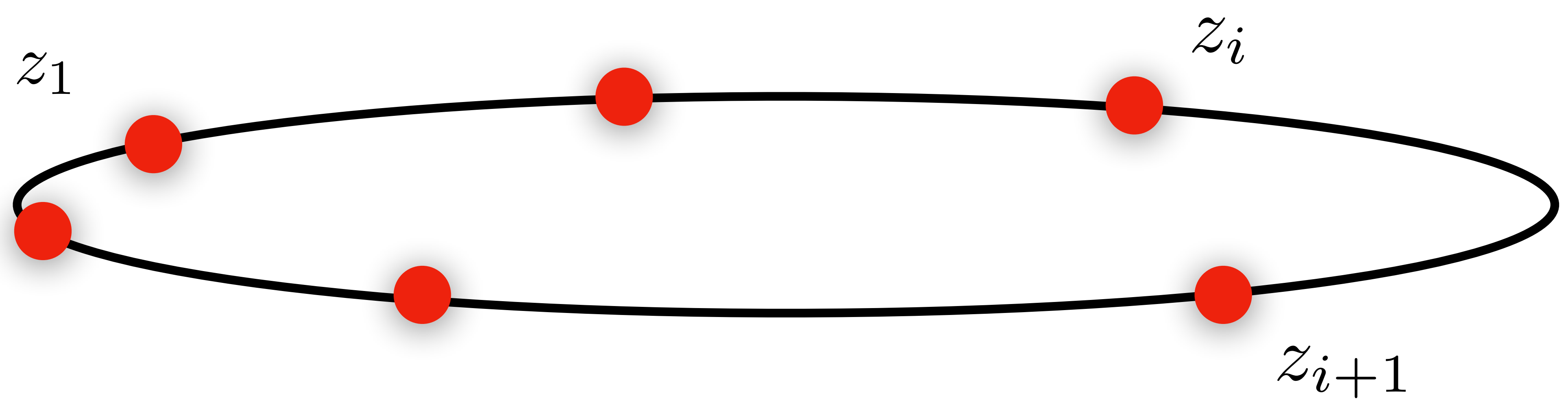
twist eigenvalues z_i

equivariant parameters (anisotropies) a_i

Bethe Ansatz Equations: $\exp \frac{\partial Y}{\partial \sigma_i} = 1$

Classical

q-Opers



n-particle trigonometric Ruijsenaars-Schneider model

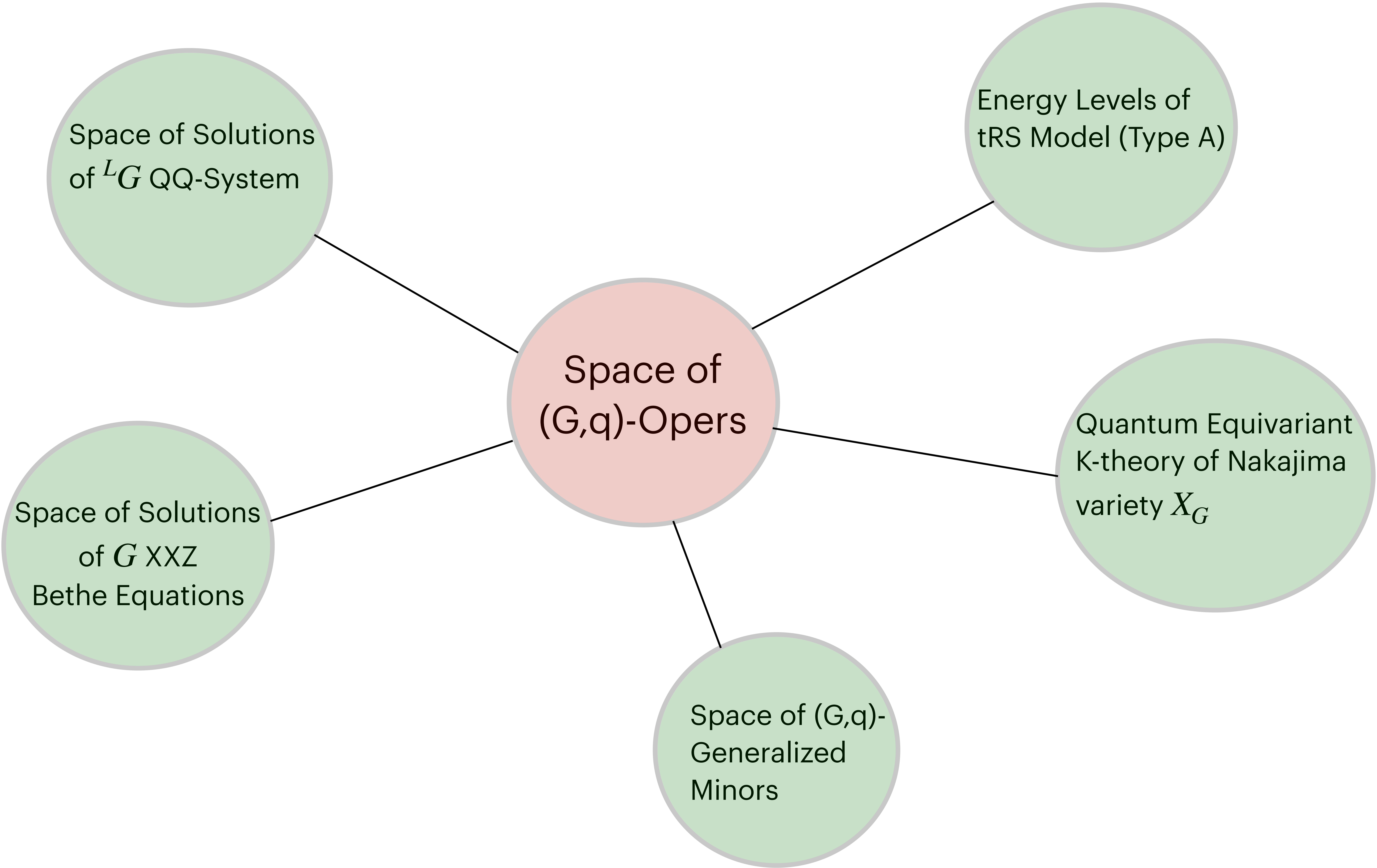
Coupling constant $\hbar$

coordinates z_i

energy (eigenvalues of Hamiltonians) $e_i(a_i)$

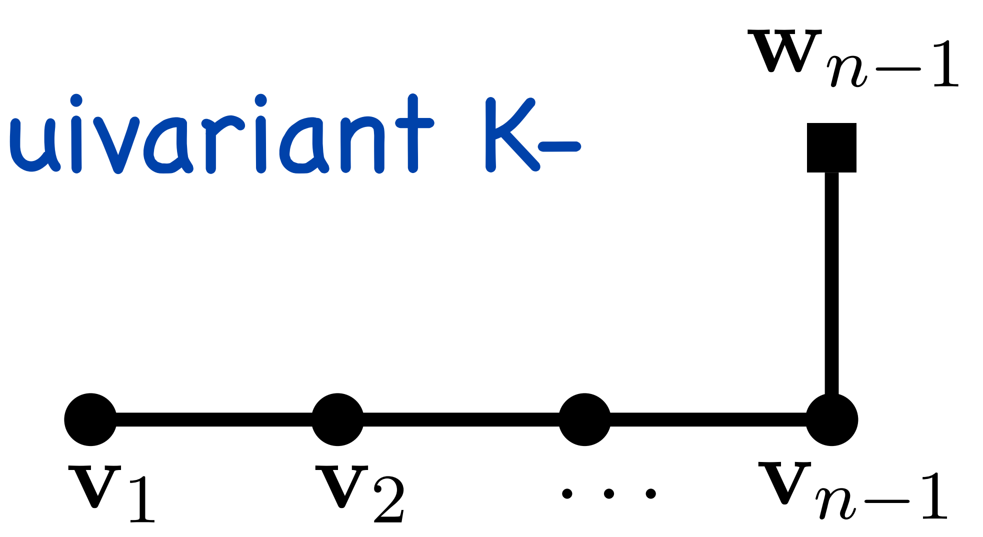
Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \dots, n$$


Space of Solutions of ^{L}G QQ-System
Energy Levels of tRS Model (Type A)
Space of (G,q)-Opers
Quantum Equivariant K-theory of Nakajima variety X_G
Space of Solutions of G XXZ Bethe Equations
Space of (G,q)-Generalized Minors

The Gauge/Bethe Correspondence

[Nekrasov Shatashvili]
[Aganagic Okounkov]



Hilbert space of states of a quantum integrable system is identified with equivariant K-theory of Nakajima quiver variety

gauge group $G = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} U(v_i)$ $(v_1, v_2, \ldots)$ encode weight of a representation

Bethe roots **s** live in the maximal torus of G, by integrating over **s** we project on Weyl invariant functions thereof

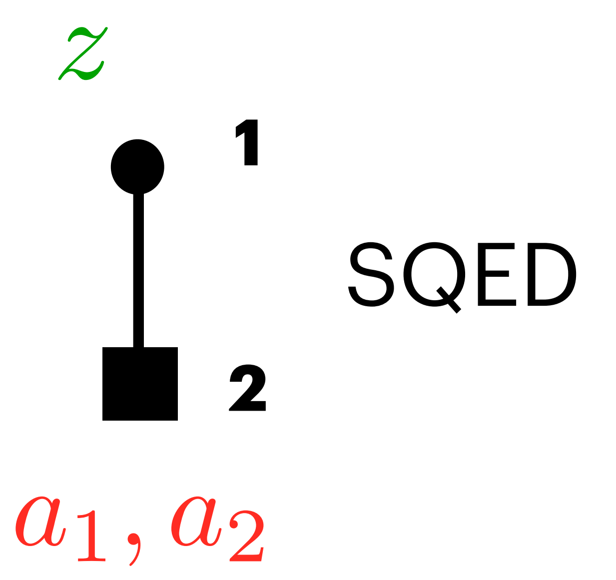
Flavor group $G_F = \prod_i U(w_i)$ whose maximal torus gives parameters **a**

Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

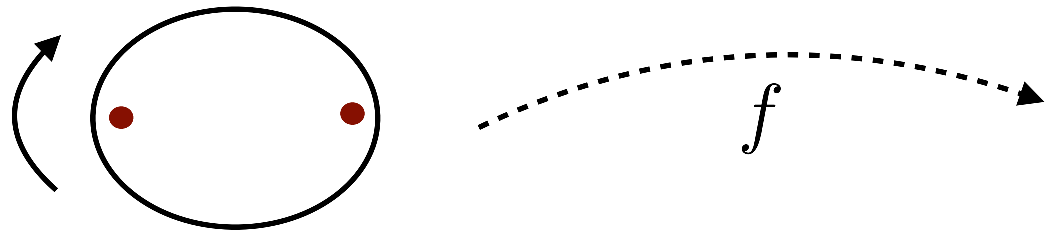
Quantum K-theory

Classical K-theory of a quiver variety is generated by tensorial polynomials of tautological bundles and their duals

For quantum deformation parameterized by z we study quasimaps from $\mathbb{P}^1$ to X



$p_1 = 0, \;\; p_2 = \infty$

$\mathbb{C}_q^{\times}$ 

X

Vertex functions (vortex partition functions) are eigenfunctions of quantum tRS difference operators (Ward identities)!

$$T_i(a)V(z,a) = e_i(z)V(z,a) \qquad \hbar \to \hbar^{-1} \qquad T_i(z)V(z,a) = e_i(a)V(z,a)$$

3d Mirror symmetry

[PK Zeitlin [arXiv:1802.04463] Math.Res.Lett. **28** (2021) 435]

Saddle point approximation yields Bethe equations

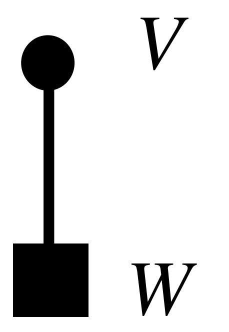
$q \to 1$

$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}\,, \quad i = 1\cdots k.$$

The QQ-System

Baxter Q-operator $Q(u)=\sum_{i=1}^{k}(-1)^k u^{k-i}(\Lambda^i V)(z)\circledast$ has eigenvalue $Q(u)=\prod_{i=1}^{k}(u-s_i)$

Short exact sequence of bundles for $T^*Gr_{k,n}$ $\quad 0\to V\to W\to V^\vee\to 0$



Eigenvalues of operators Q and $\widetilde{Q}$ (generated by $V^\vee$) satisfy the QQ-relation

$$z\,\widetilde{Q}(\hbar u)Q(u)-\widetilde{Q}(u)Q(\hbar u)=\prod_{i=1}^{n}(u-a_i)$$

which is equivalent to Bethe equations

Also:

Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

Relations between generalized minors (Jacobi-like identities)

[Fomin, Zelevinski] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\hat{g})$

[Frenkel, Hernandez] ....

Spectral determinants in the QDE/IM correspondence

[Frenkel, PK, Zeitlin, to appear][Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

Describes (q-)oper bundles

[Frenkel, PK, Zeitlin, Sage]

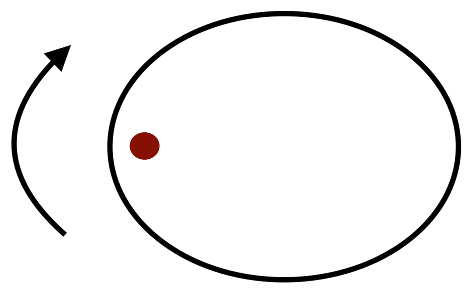
(G,q)-Opers

[PK, Sage, Zeitlin, Commun.Math.Phys. **381** (2021) 641]

$M_q : \mathbb{P}^1 \to \mathbb{P}^1$

$u \mapsto qu$

q



Principal bundle $\mathscr{F}_G$ over $\mathbb{P}^1$

(G,q)-connection A is a meromorphic section of $Hom_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{F}_G, \mathscr{F}_G^q)$

q-gauge transformation $A(u) \mapsto g(qu)A(u)g(u)^{-1} \quad g(u) \in G(\mathbb{C}(u))$

(SL(2),q)-oper

Triple $(E, A, \mathscr{L})$

(E, A) is the $(SL(2), q)$ connection

$\mathscr{L} \subset E$ is a line subbundle

The induced map $\bar{A} : \mathscr{L} \to (E/\mathscr{L})^q$ is an isomorphism

in a trivialization $\mathscr{L} =$ Span(s)

$$s(qu) \wedge A(u)s(u) \neq 0$$

Chose trivialization of $\mathscr{L}$ $\quad s(u) = \begin{pmatrix} Q(u) \\ \widetilde{Q}(u) \end{pmatrix}$

Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition with $A(u) = Z$ — SL(2) QQ-system

$$z\,\widetilde{Q}(\hbar u)Q(u) - \widetilde{Q}(u)Q(\hbar u) = \prod_{i=1}^{n}(u - a_i)$$

$q = \hbar$

q-Opers, QQ-System & Bethe Ansatz

[Frenkel, PK, Sage, Zeitlin to appear in **JEMS**]

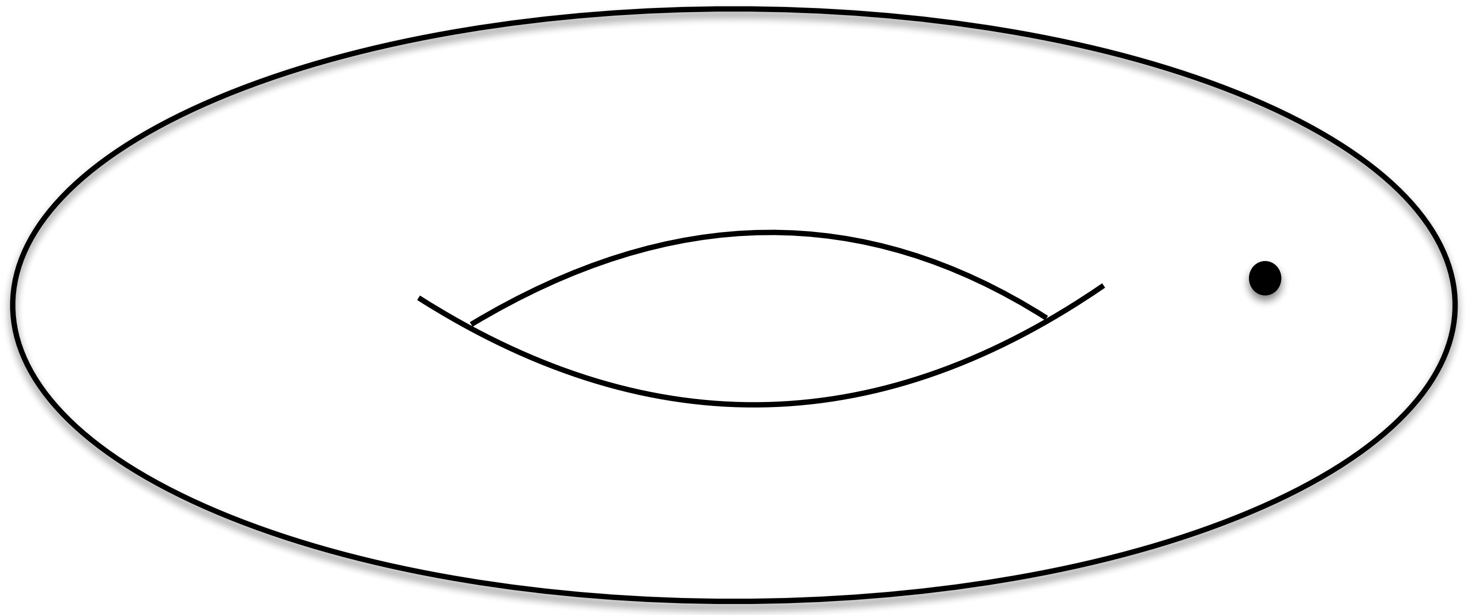
Theorem: There is a 1-to-1 correspondence between the set of nondegenerate Z-twisted (G,q)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

Book | May 2023

Branes and DAHA Representations

- **Authors:** Du Pei , Ingmar Saberi , Peter Koroteev , Satoshi Nawata , Sergei Gukov

Geometric representation theory of double affine Hecke algebra (DAHA) in terms of Hitchin moduli space of once-punctured torus



Deformation Quantization

Spherical $\mathfrak{sl}_2$ DAHA

(line ops in $\mathcal{N} = 2^*$ theory)

$$qxy - yx = (q - q^{-1})z + \text{cyclic}$$

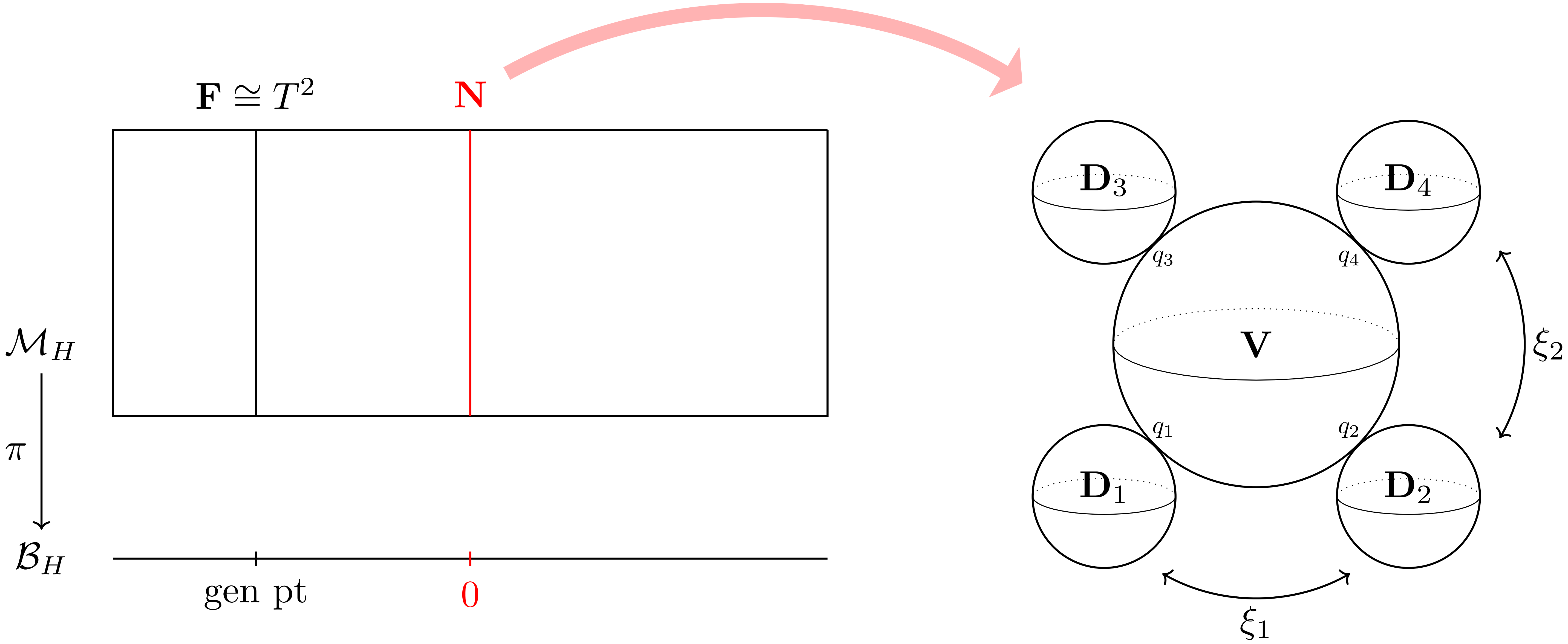
$$\rho : \pi_1(C_p) \to \mathrm{SL}(2, \mathbb{C})$$

$$x = \mathrm{Tr}(\rho(\mathfrak{m})), \;\; y = \mathrm{Tr}(\rho(\mathfrak{l})), \;\; \text{and } z = \mathrm{Tr}\big(\rho(\mathfrak{m}\mathfrak{l}^{-1})\big)$$

Wilson   't Hooft   Dyonic

Categorification

$$\mathbf{Hom}(\mathcal{B}_{cc}, -) : D^b\mathbf{ABrane}(\mathfrak{X}) \longrightarrow D^b\mathbf{Rep}(\mathscr{O}^q(\mathfrak{X}))$$



Teaching at University of California

- I taught math and physics and various levels: undergraduate math courses at UC Berkeley and UC Davis

University of California, Berkeley
Math-55 *Discrete Mathematics*. 2022
Math-54 *Linear Algebra*. 2021
Math-53 *Multivariable Calculus*. 2020
Math-142 *Elementary Algebraic Topology*. 2019
My favorite course!! Math-H185 *Honors Introduction to Complex Analysis*. 2019
Math-185 *Introduction to Complex Analysis*. 2019

University of California, Davis
MAT-125A *Real Analysis*. Spring quarter 2016
MAT-108 *Introduction to Abstract Math*. Winter quarter 2016
MAT-016A *Short Calculus*. Spring quarter 2017
MAT-25 *Advanced Calculus*. Winter quarter 2018
MAT-16B *Calculus*. Spring quarter 2018
MAT-21B *Calculus*. Fall quarter 2018

Earlier Teaching Activities

- Physics teaching assistant at University of Minnesota
- Mentoring USPhO&USMO students
- Organizing school olympiads during my undergrad years
- Running Summer school for advanced high school students in STEM fields

Teaching Philosophy

- Creating an adequate grading scheme
- Splitting tests (midterms/quizzes) throughout the term. It has been known by brain researchers that learning is more efficient if the new material is presented in smaller portions and then students are tested immediately after. Last-minute (day/week) learning does not develop long-term memory
- A graduate-level course should be designed to prepare students for their independent research in mathematics. Ideally, after having completed a course, a grad student should be capable of investigating the topic on there own
- You don’t know it until you teach it!

Berkeley/Stanford Math Circle

- The Bay Area has a long history of mathematical education with UC Berkeley, UC Davis, USCF, and Stanford around
- There are math circle programs for grade school students
- I teach in Berkeley and Stanford Math circles in both elementary and upper divisions
- I also help with math Olympiads in Berkeley and Stanford
- Olympiads/Circles, etc in Scotland/UK? I am ready to contribute

Some Math Circle Problems

Show that the set of all distinct partitions of n is in bijection with the set of all odd partitions of n

Find the number of ways to put in a row of length **n** dominoes of sizes 2×1 and 1×1

Can a 10×10 square board be paved with the 4×1 rectangular plates?