Opers & Integrability

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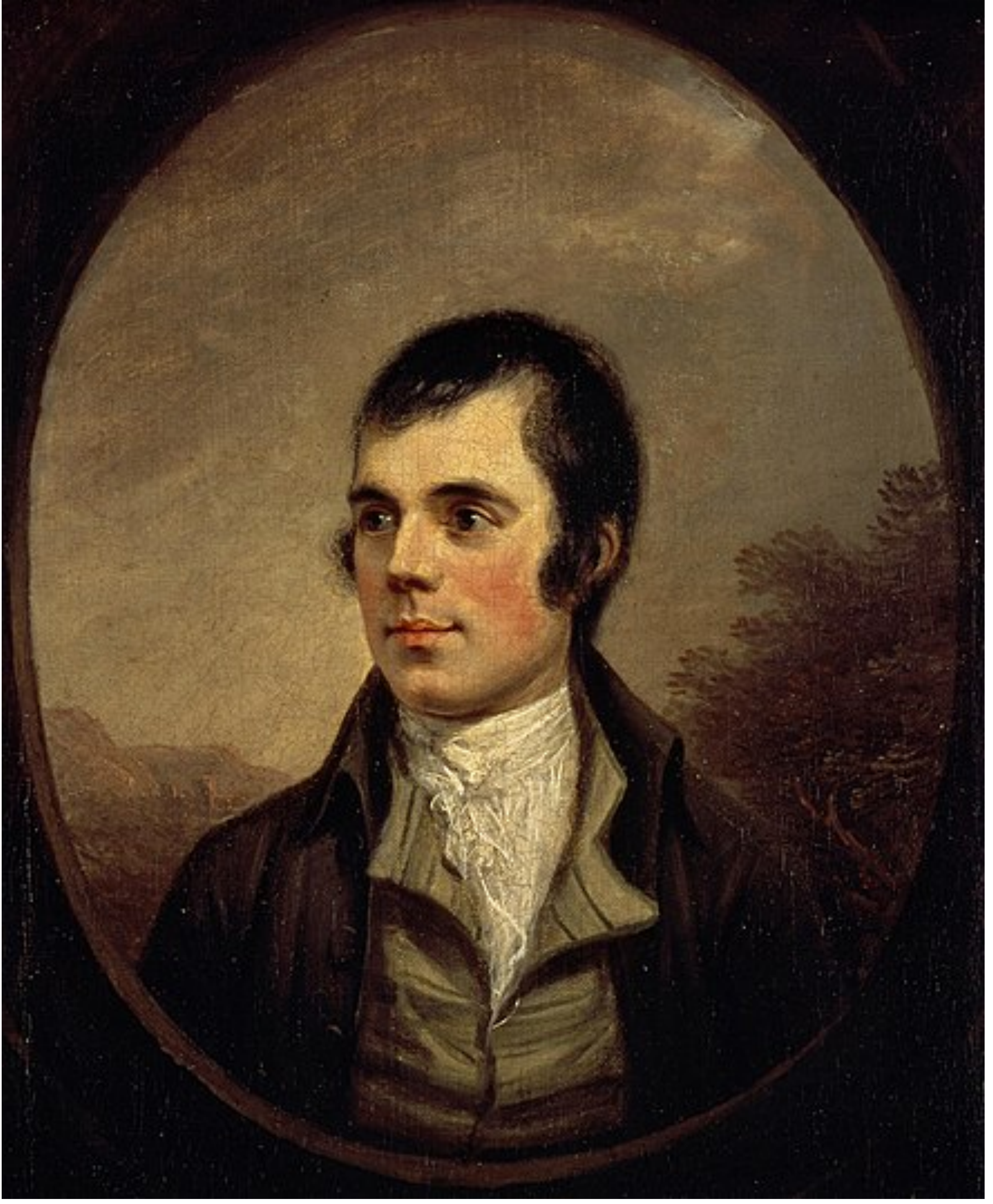
Talk at ISLAND workshop, Highlands, Scotland 6/26/2023



My heart's in the Highlands, my heart is not here,
My heart's in the Highlands, a-chasing the deer;
Chasing the wild-deer, and following the roe,
My heart's in the Highlands, wherever I go

Classical Integrability

Equations of motion

$$\frac{df}{dt} = \{H_1, f\}$$

Integrability — family of n conserved quantities which Poisson commute with each other

$$\{H_i, H_j\} = 0 \quad i, j = 1, \ldots, n$$

Liouville-Arnold Theorem

Compact Lagrangians $\mathscr{L}$: $\{H_i = E_i\}$ are isomorphic to tori

Evolution in the neighborhood of $\mathscr{L}$ is linearized in action/angle variables $\{I_i, \varphi_i\}_{i=1}^{n}$

$$\frac{d\varphi_i}{dt} = \omega_i, \qquad \frac{dI_i}{dt} = 0$$

Action/angle variables are hard to find

Examples

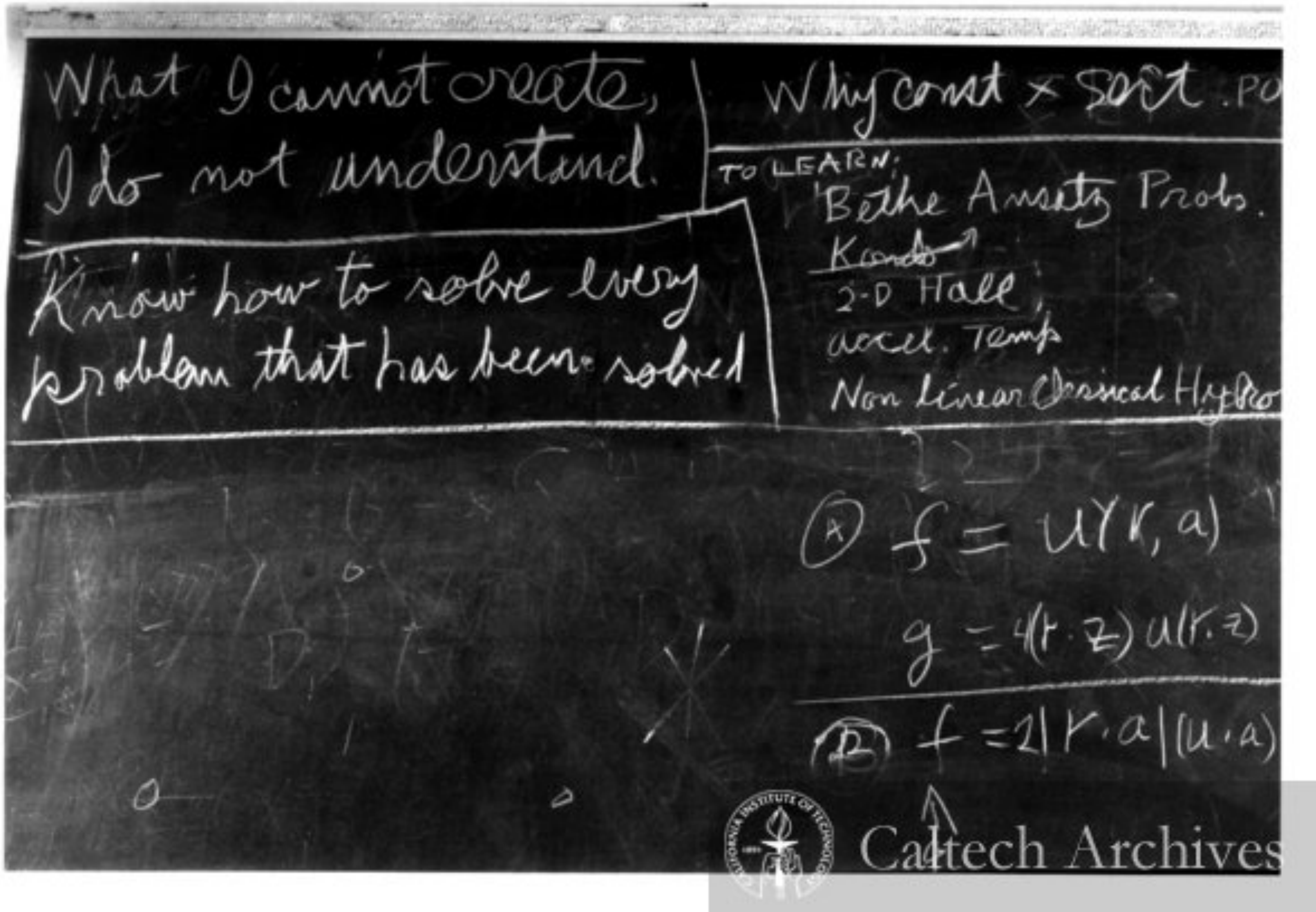
Many-body integrable systems — Calogero, Toda, Ruijsenaars (more on this later)

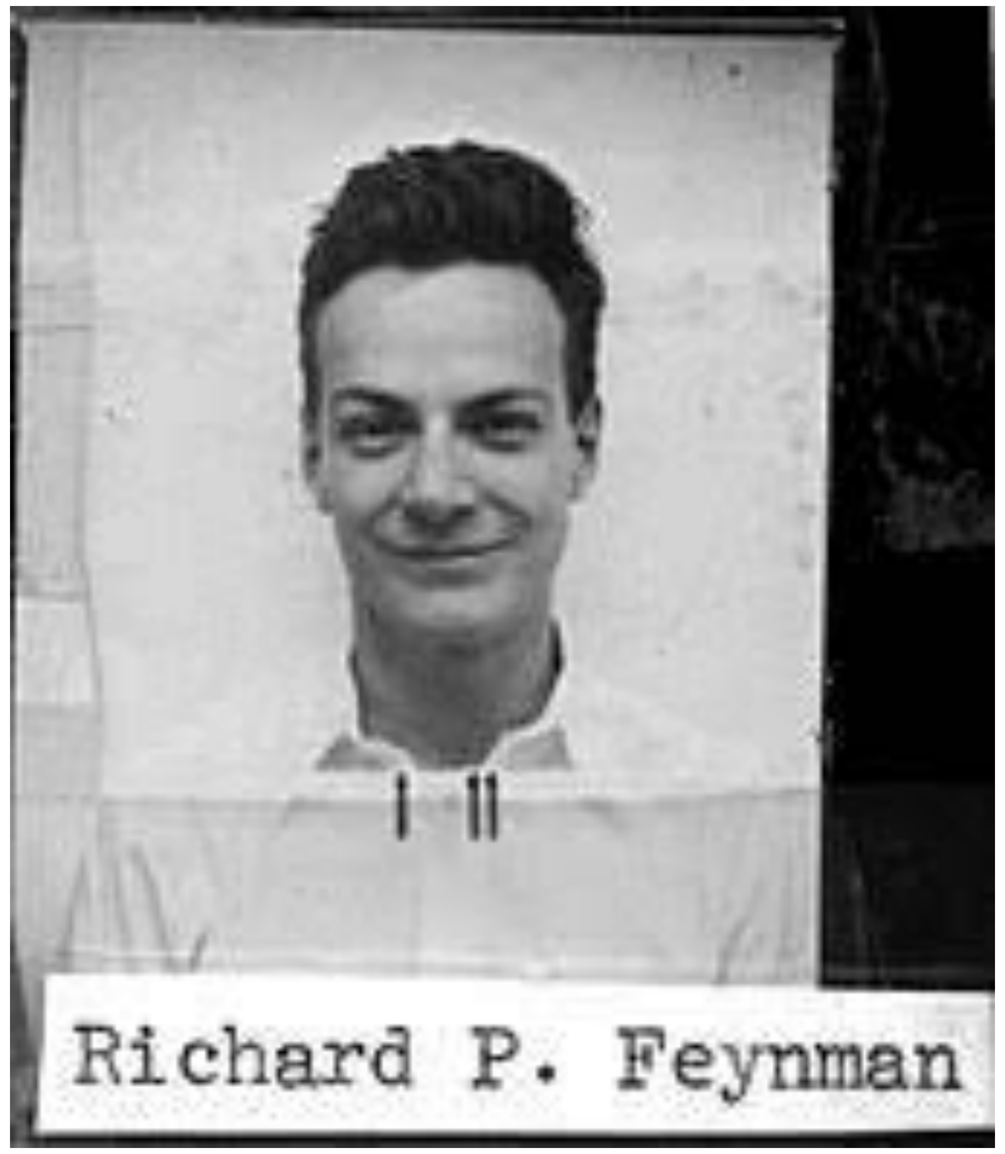
Continuous integrable models in (1+1)-dimensions: Korteweg-de-Vries, Intermediate Long-Wave, etc.

$$u_t = 6uu_x - u_{xxx}$$

They admit soliton solutions. Sectors with N solitons are described by finite N-body integrable systems

Inverse scattering method — Lax pair data $\rightarrow$ action/angle variables

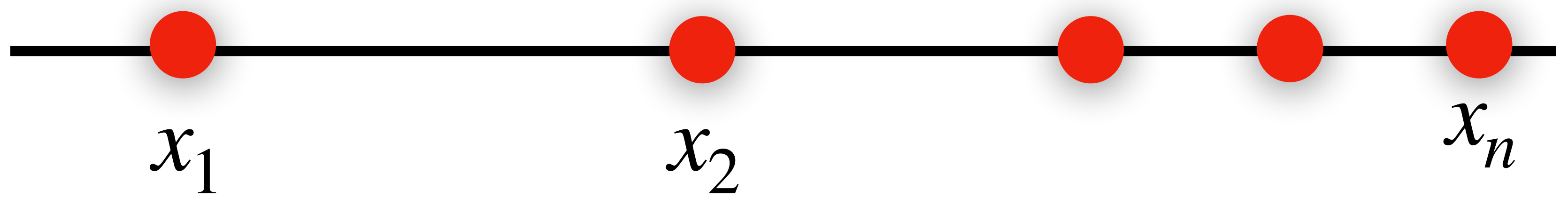


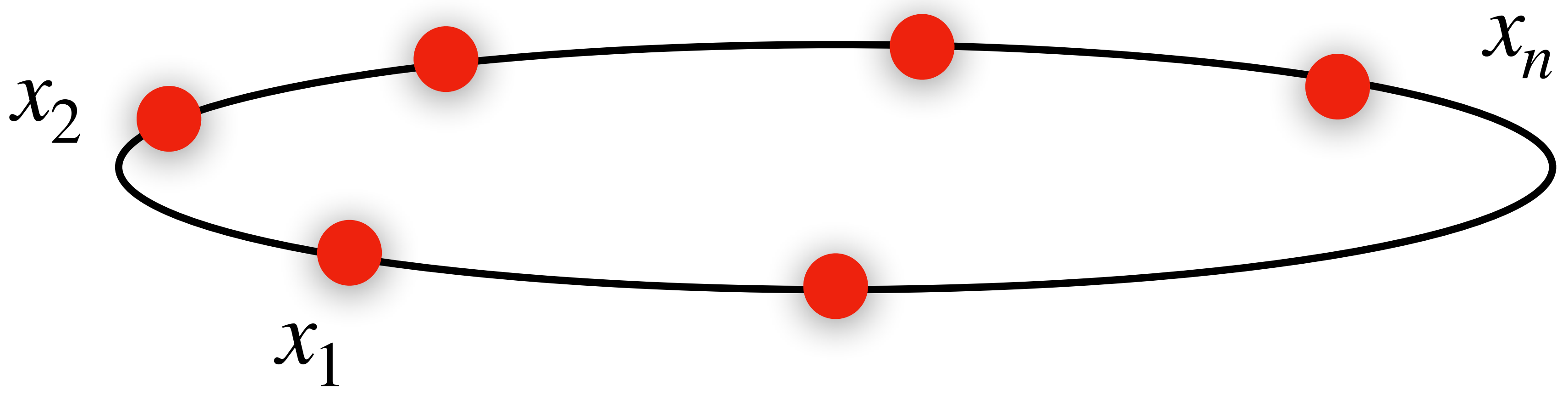


I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better.

I. Many-Body Systems

Calogero in 1971 introduced a new integrable system. Moser in 1975 proved its integrability using Lax pair





$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$

The **Calogero-Moser (CM)** system has several generalizations: rational CM $\rightarrow$ trigonometric CM $\rightarrow$ elliptic CM

$$V(x) \simeq \sum \frac{1}{(x_i - x_j)^2} \qquad V(x) \simeq \sum \frac{1}{\sinh(x_i - x_j)^2} \qquad V(x) \simeq \wp(x_j - x_i)$$

Another relativistic generalization called **Ruijsenaars-Schneider (RS)** family

rRS $\rightarrow$ tRS $\rightarrow$ eRS

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

Example: tRS Model with 2 Particles

Hamiltonians

$$T_1 = \frac{\xi_1 - \hbar\xi_2}{\xi_1 - \xi_2} p_1 + \frac{\xi_2 - \hbar\xi_1}{\xi_2 - \xi_1} p_2$$

$$T_2 = p_1 p_2$$

Coordinates ξ_i, momenta p_i

coupling constant $\hbar$, energies E_i

Symplectic form

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{d\xi_i}{\xi_i}$$

Integrals of motion

$$T_i = E_i$$

Quantization

$$p_i \xi_j = \xi_j p_i q^{\delta_{ij}} \qquad q \in \mathbb{C}^\times$$

tRS Momenta are shift operators

$$p_i f(\xi_i) = f(q\xi_i)$$

Eigenvalue Equations

$$T_i V = E_i V$$

Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

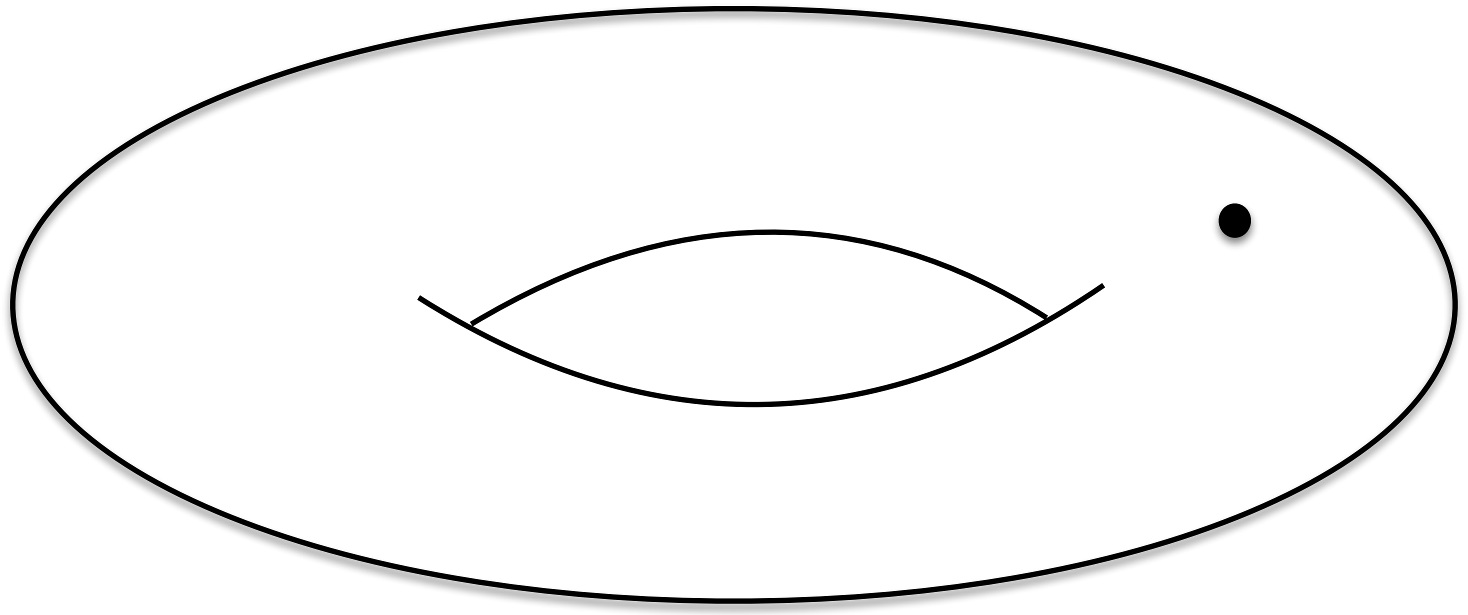
$$\hbar MT - TM = u \otimes v^T$$

The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space** $\mathscr{M}$

Flat connections on punctured torus



$$\mathcal{M}_n = \{A, B, C\}/GL(n; \mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

Integrable Hamiltonians are ~TrT^k

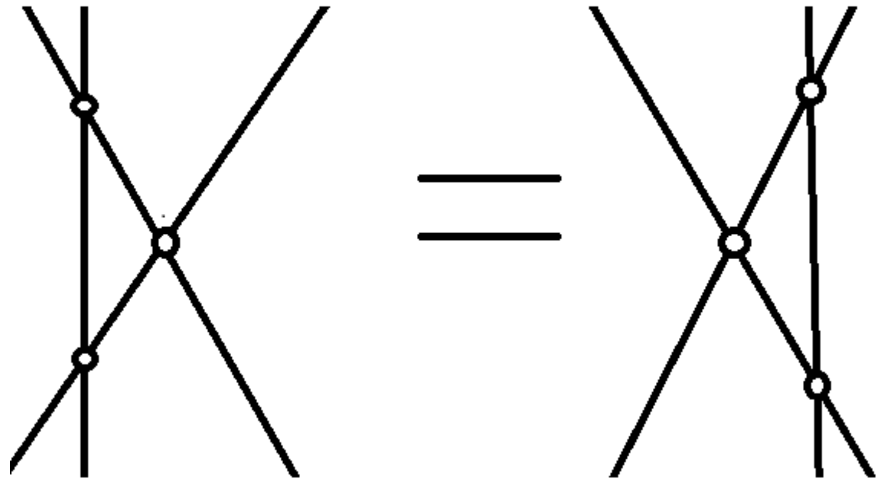
T-Lax matrix

[my DAHA paper with Gukov, Nawata, Pei, Saberi
[arXiv:2206.03565] **SpringerBriefs** (2023)]

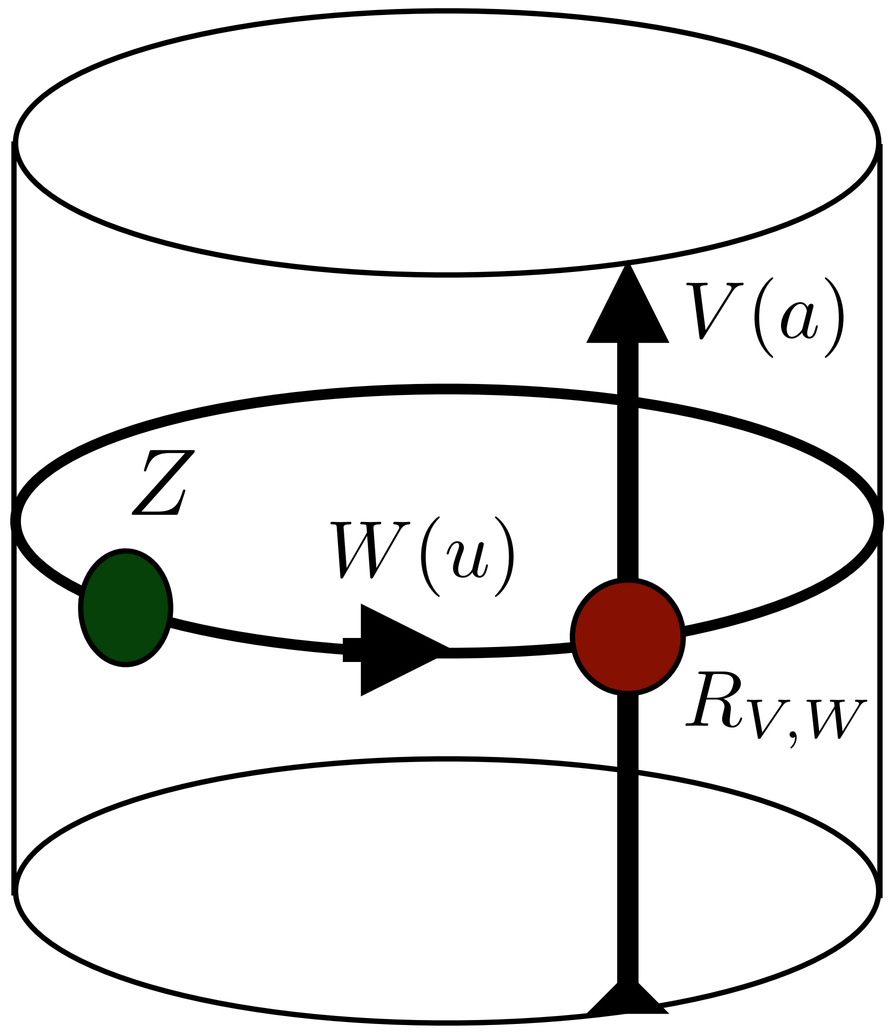
II. Quantum Integrability

Quantum group $U_\hbar(\hat{\mathfrak{g}})$ is a noncommutative deformation of the loop group with a nontrivial intertwiner — R-matrix

Yang-Baxter equation



$$R_{V_1,V_2}(a_1/a_2):\ V_1(a_1)\otimes V_2(a_2)\to V_2(a_2)\otimes V_1(a_1)$$



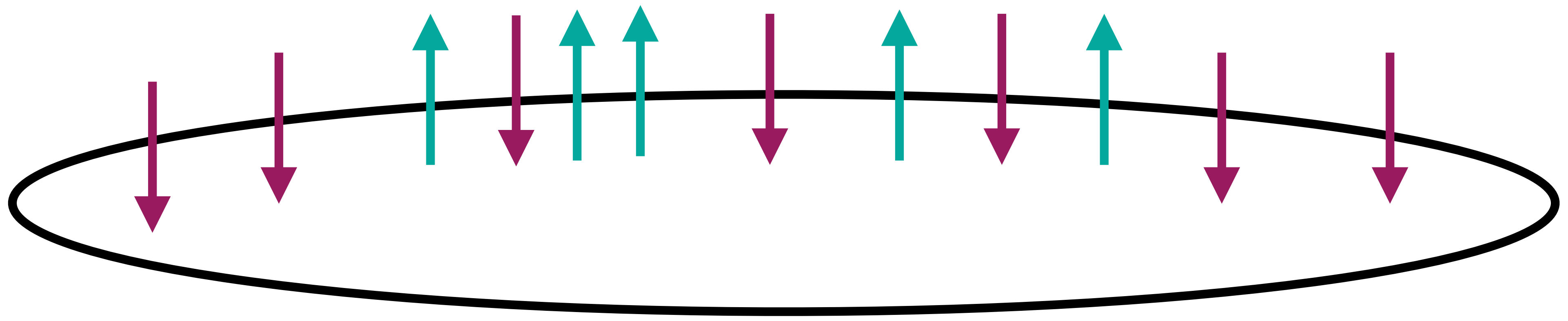
Integrability comes from transfer matrices which generates Bethe algebra

$$T_W(u)=Tr_{W(u)}((Z\otimes 1)R_{V,W}) \qquad [T_W(u),T_W(u')]=0$$

Transfer matrices are usually polynomials in u whose coefficients are the **integrals of motion**

Classical IS can be quantized using methods of physics — Omega background [Nekrasov], Quantization by branes [Gukov, Witten]

Quantum



SU(**n**) XXZ spin chain on n sites w/ **anisotropies** and **twisted periodic boundary conditions**

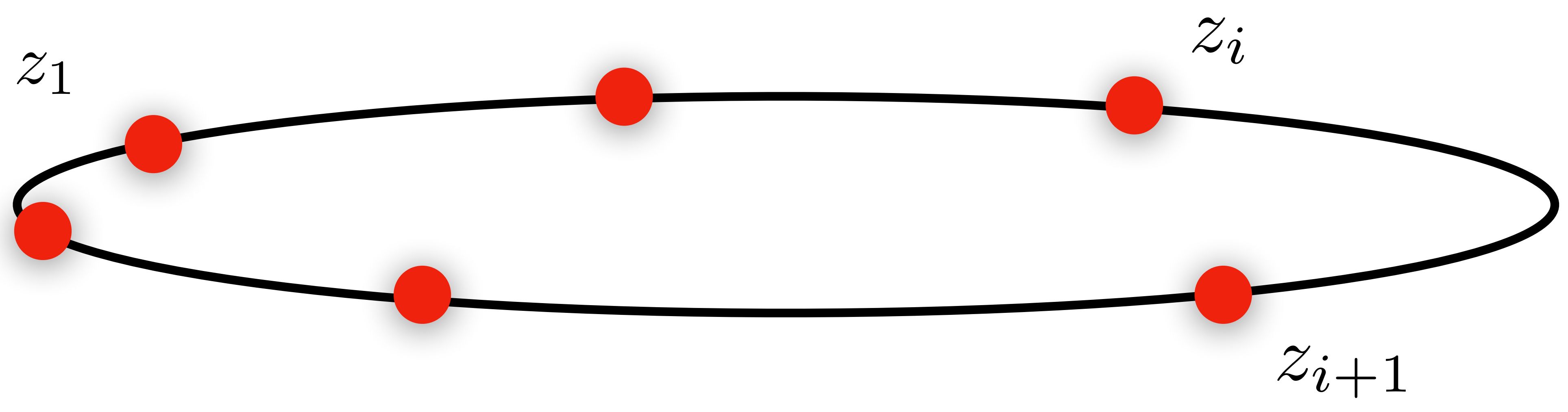
Planck's constant $\hbar$

twist eigenvalues z_i

equivariant parameters (anisotropies) a_i

Bethe Ansatz Equations: $\exp \frac{\partial Y}{\partial \sigma_i} = 1$

Classical



n-particle trigonometric Ruijsenaars-Schneider model

Coupling constant $\hbar$

coordinates z_i

energy (eigenvalues of Hamiltonians) $e_i(a_i)$

Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \dots, n$$

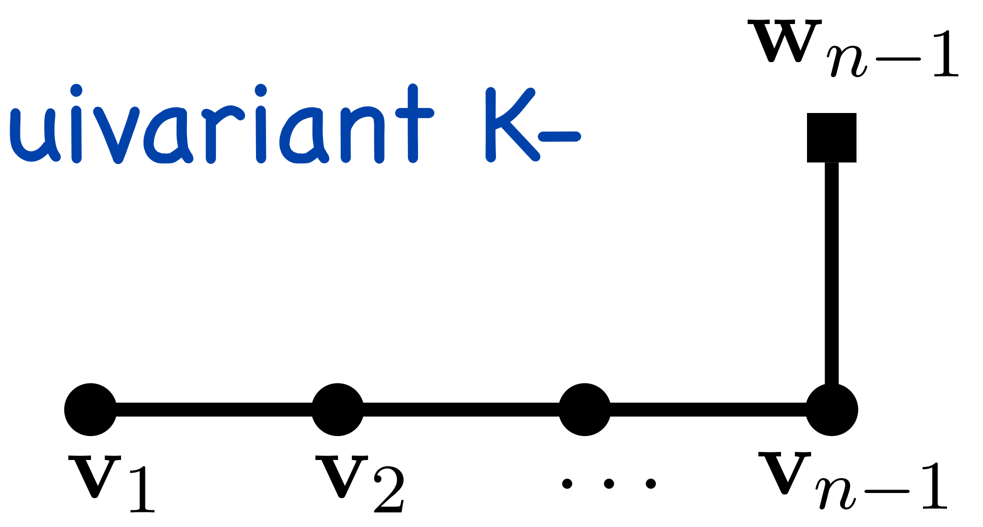
The Quantum/Classical Duality

Why do we expect quantum and classical models to be related to each other?

1) Enumerative Algebraic Geometry Motivated by Physics
2) Geometric Langlands — Opers

The Gauge/Bethe Correspondence

[Nekrasov Shatashvili]
[Aganagic Okounkov]



Hilbert space of states of a quantum integrable system is identified with equivariant K-theory of Nakajima quiver variety

gauge group $G = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} U(v_i)$ $(v_1, v_2, \ldots)$ encode weight of a representation

Bethe roots **s** live in the maximal torus of G, by integrating over **s** we project on Weyl invariant functions thereof

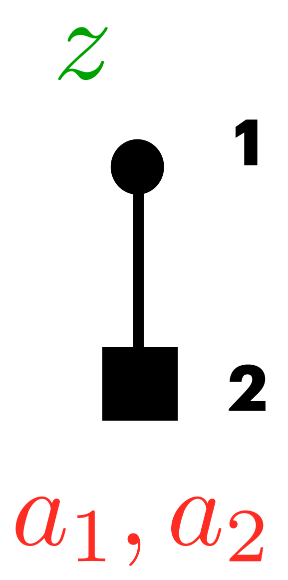
Flavor group $G_F = \prod_i U(w_i)$ whose maximal torus gives parameters **a**

Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

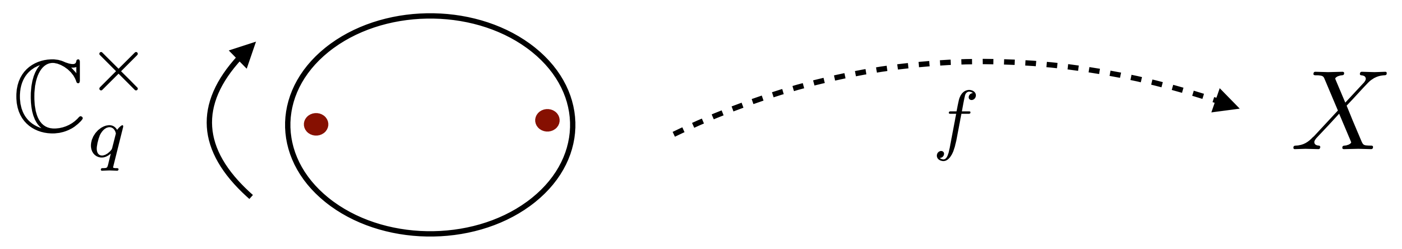
Quantum K-theory

Classical K-theory of a quiver variety is generated by tensorial polynomials of tautological bundles and their duals

For quantum deformation parameterized by z we study quasimaps from $\mathbb{P}^1$ to X



$p_1 = 0, \ \ p_2 = \infty$



Vertex functions are eigenfunctions of quantum tRS difference operators in equivariant parameters and in twist parameters!

$$T_i(a)V(z,a) = e_i(z)V(z,a) \qquad \hbar \to \hbar^{-1} \qquad T_i(z)V(z,a) = e_i(a)V(z,a)$$

3d Mirror symmetry

[PK Zeitlin [arXiv:1802.04463] Math.Res.Lett. **28** (2021) 435]

Saddle point approximation yields Bethe equations

$q \to 1$

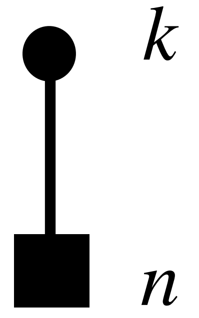
$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}\,, \quad i = 1 \cdots k.$$

Bethe Equations for $T^*Gr_{k,n}$

[Pushkar Smirnov Zeitlin]

Operator of quantum multiplication

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V_{\boldsymbol{p}}^{(\tau)}(z)}{V_{\boldsymbol{p}}^{(1)}(z)}$$



Theorem *The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(s_1, \cdots, s_k)$ evaluated at the solutions of the following equations:*

$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\, \hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}, \quad i = 1 \cdots k.$$

Equivariant parameters a_i,

twist z,

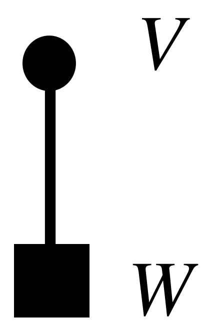
Planck constant $\hbar$

Baxter Q-operator $Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$ has eigenvalue $Q(u) = \prod_{i=1}^{k} (u - s_i)$

The QQ-System for A_1

Short exact sequence of bundles

$$0 \to V \to W \to V^\vee \to 0$$



Eigenvalues of Q-operators

$$Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$$

$$\widetilde{Q}(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V^\vee)(z) \circledast$$

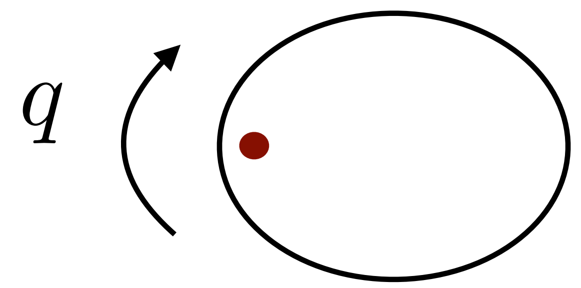
Satisfy the QQ-relation

$$z\, \widetilde{Q}(\hbar u) Q(u) - \widetilde{Q}(u) Q(\hbar u) = \prod_{i=1}^{n} (u - a_i)$$

equivalent to the XXZ Bethe equations

III. (G,q)-Connection

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$u \mapsto qu$$



G-simple simply-connected complex Lie group

Consider vector bundle $\mathscr{F}_G$ over $\mathbb{P}^1$

(G,q)-connection A is a meromorphic section of $Hom_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{F}_G, \mathscr{F}_G^q)$

Locally q-gauge transformation of the connection

$$A(u) \mapsto g(qu)A(u)g(u)^{-1} \qquad g(u) \in G(\mathbb{C}(u))$$

Compare with (standard) gauge transformations

$$\partial_u + A(u) \mapsto g(u)(\partial_u + A(u))g(u)^{-1} \qquad g(u) \in \mathfrak{g}(u)$$

(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathfrak{F}_G, A, \mathfrak{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathfrak{F}_{B_-}$ is a reduction of $\mathfrak{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathfrak{F}_G \longrightarrow \mathfrak{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally $$A(u) = n'(u) \prod_i (\phi_i(u)_i^{\check{\alpha}} s_i)\, n(u)$$

$$\phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N_-(u) = [B_-(u), B_-(u)]$$

(SL(2),q)-Opers

Let $G = SL(2)$ The q-oper definition can be formulated as

Triple $(E, A, \mathscr{L})$

(E, A) is the $(SL(2), q)$ connection

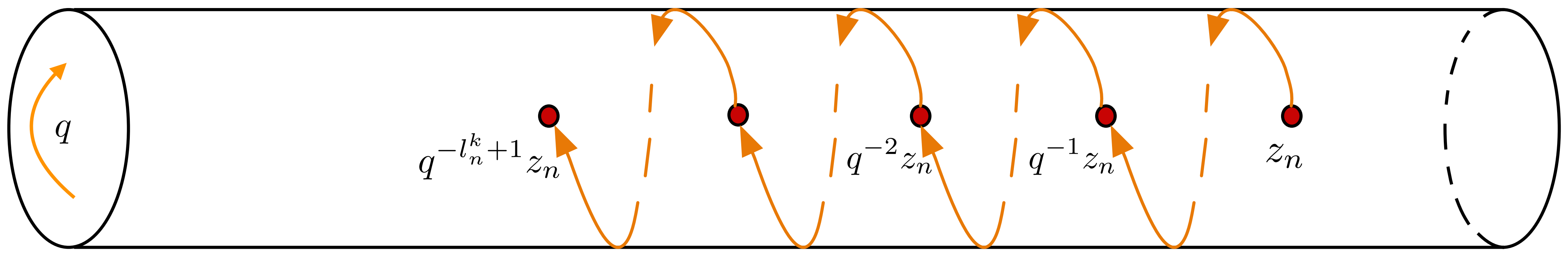
$\mathscr{L} \subset E$ is a line subbundle

The induced map $\bar{A} : \mathscr{L} \to (E/\mathscr{L})^q$ is an isomorphism

in a trivialization $\mathscr{L} = \text{Span}(s)$

$$s(qu) \wedge A(u)s(u) \neq 0$$

Allow singularities $\quad s(qu) \wedge A(u)s(u) = \Lambda(u)$



$$\Lambda(u) = \prod_{l, j_l} (u - q^{j_l} a_l)$$

Add Twists $\quad Z = g(qu)A(u)g(u)^{-1}$

$$Z \in H \subset H(u) \subset G(u)$$

q-Opers, QQ-System & Bethe Ansatz

Chose trivialization of $\mathcal{L}$ $\quad s(u) = \begin{pmatrix} Q_+(u) \\ Q_-(u) \end{pmatrix}$ $\qquad$ Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition — SL(2) **QQ-system**

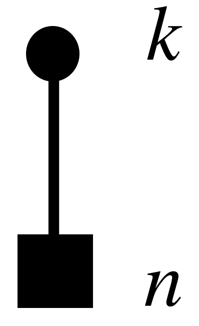
$$s(qu) \wedge A(u)s(u) = \Lambda(u) \longrightarrow \zeta\, Q_-(u)Q_+(qu) - \zeta^{-1}Q_-(qu)Q_+(u) = \Lambda(u)$$

QQ-system to XXZ Bethe equations

$$Q_+(u) = \prod_{k=1}^{m}(u - s_k)$$

$$\prod_{l=1}^{n} \frac{s_i - q^{r_l} a_l}{s_i - a_l} = \zeta^2 q^k \prod_{j=1}^{k} \frac{q s_i - s_j}{s_i - q s_j}$$

$$i = 1, \ldots, k$$



$\hbar = q$

q-Miura Transformation

Miura q-oper: $(E, A, \mathscr{L}, \hat{\mathscr{L}})$, where $(E, A, \mathscr{L})$ is a q-oper and $\hat{\mathscr{L}}$ is preserved by q-connection A

$$A(u) = \begin{pmatrix} g(u) & \Lambda(u) \\ 0 & g(u)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition $\quad A(u) = v(qu) Z v(u)^{-1} \qquad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

$$g(u) = \zeta \frac{Q_+(qu)}{Q_+(u)}$$

$$v(u) = \begin{pmatrix} Q_+(u) & \zeta Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(u) Q_+(qu) \\ 0 & Q_+(u) \end{pmatrix} \in B_+(u)$$

The q-oper condition becomes the **SL(2) QQ-system** $\quad \zeta\, Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(qu) Q_+(u) = \Lambda(u)$

Difference Equation $\quad D_q(s) = As$

Scalar difference operator $\quad \left(D_q^2 - T(qu) D_q - \frac{\Lambda(qu)}{\Lambda(u)} \right) s_1 = 0$

tRS Hamiltonians

Recover 2-body tRS Hamiltonian from an (SL(2),q)-Oper

$$\det \begin{pmatrix} Q_+(u) & \zeta Q_+(qu) \\ Q_-(u) & \zeta^{-1} Q_-(qu) \end{pmatrix} = \Lambda(u)$$

Let $\quad Q_+(u) = u - p_+ \qquad Q_-(u) = u - p_-$

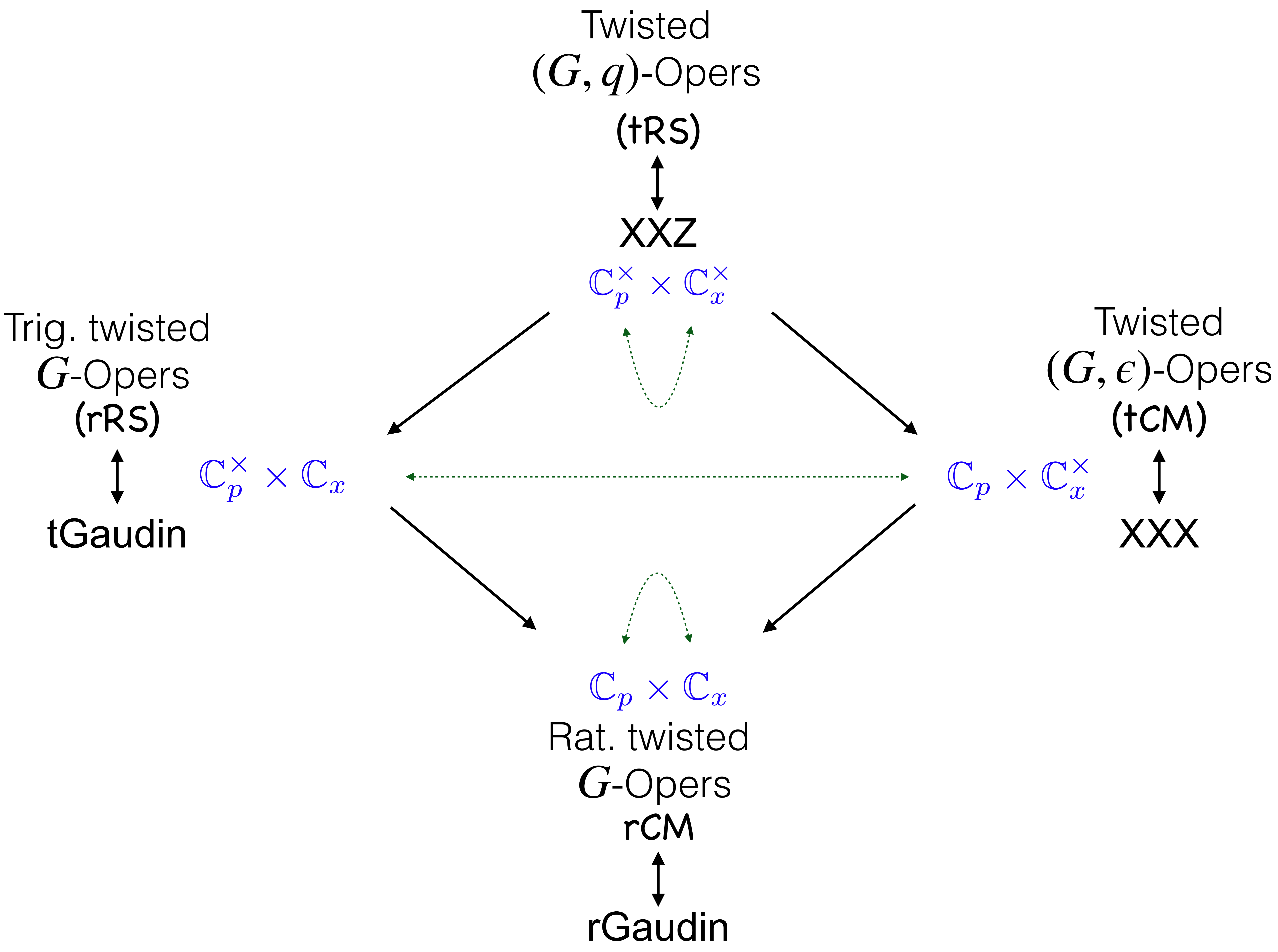
$$u^2 - u\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}} p_+ + \frac{q\zeta - q\zeta^{-1}}{\zeta^{-1} - \zeta} p_-\right] + p_+ p_- = (u - a_+)(u - a_-)$$

$T_1 \qquad T_2$

qOper condition yields
tRS Hamiltonians!

$$\det(u - T) = (u - a_+)(u - a_-)$$

Network of Dualities



q-Opers and q-Langlands

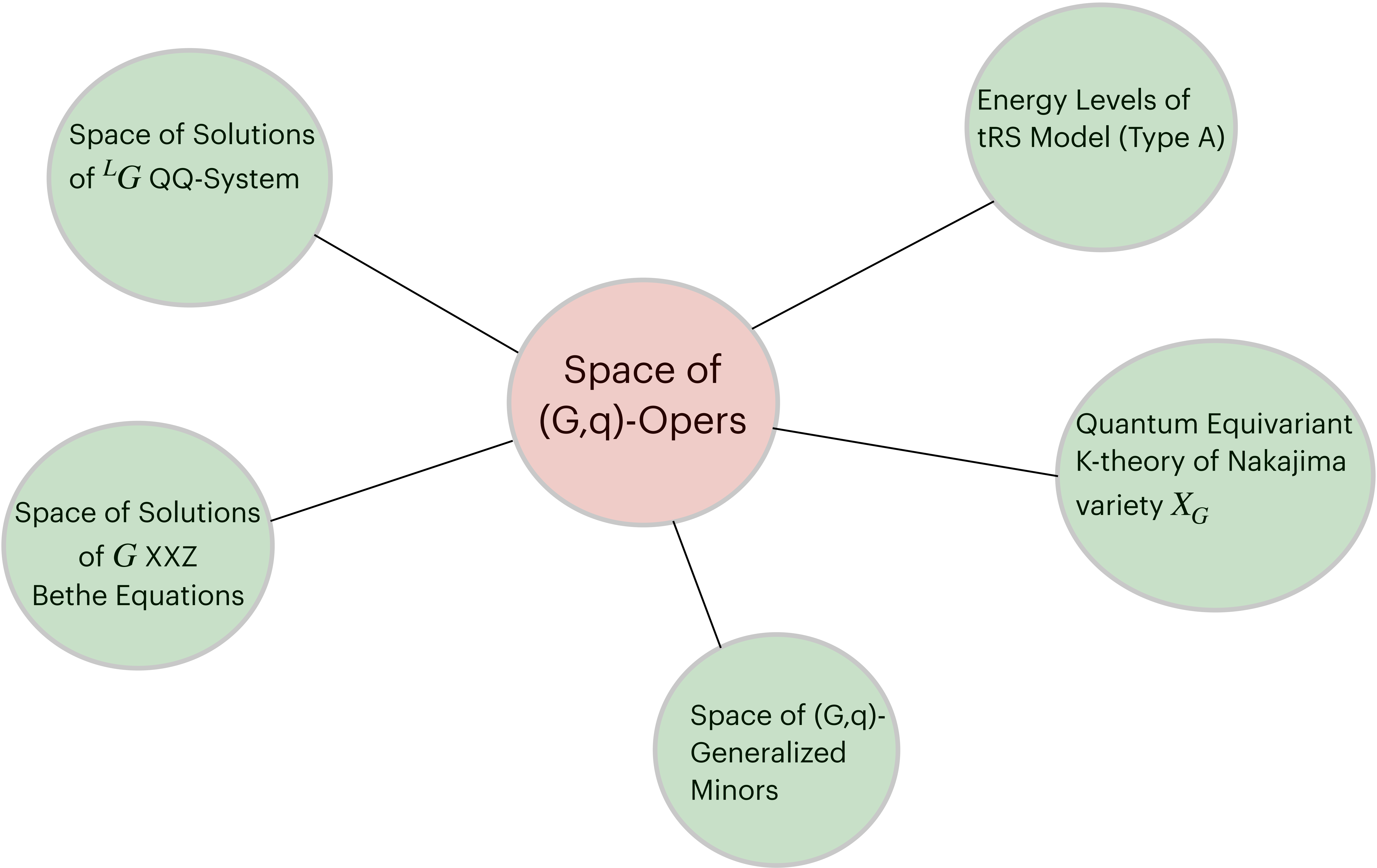
[Frenkel, PK, Zeitlin, Sage, **JEMS 2023**]

Miura (G,q)-oper with singularities $$A(u) = \prod_i \left(\zeta_i \frac{Q^i_+(qu)}{Q^i_+(u)}\right)^{\check{\alpha}_i} \exp \frac{\Lambda_i(u)}{g_i(u)} e_i$$

Theorem: There is a 1-to-1 correspondence between the set of nondegenerate Z-twisted (G,q)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

$$\widetilde{\xi}_i\, Q^i_-(u) Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u) Q^i_+(u) = \Lambda_i(u) \prod_{j>i} \left[Q^j_+(\hbar u)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(u)\right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$



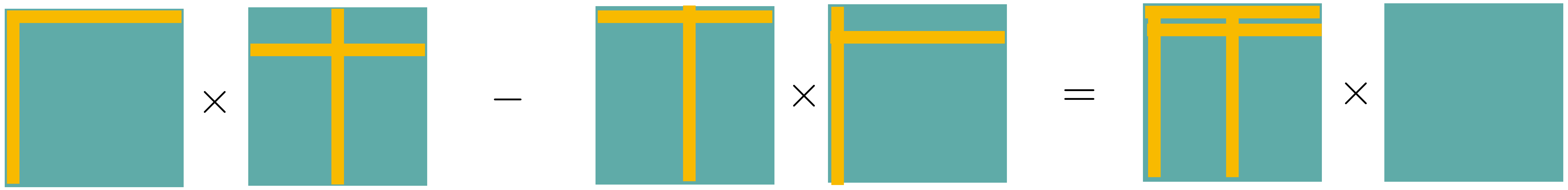
Space of Solutions of ^LG QQ-System
Energy Levels of tRS Model (Type A)
Space of (G,q)-Opers
Quantum Equivariant K-theory of Nakajima variety X_G
Space of Solutions of G XXZ Bethe Equations
Space of (G,q)-Generalized Minors

IV. Cluster Algerbras

[PK, Zeitlin, **Crelle (2023)**]

The QQ-system $\xi_{i+1}\, Q^i_-(u)Q^i_+(u+\epsilon) - \xi_i Q^i_-(u+\epsilon)Q^i_+(u) = \Lambda_i(u) Q^{i+1}_+(u+\epsilon)Q^{i+1}_+(u)$

For $G = SL(n)$ obtain Lewis Carrol (Desnanot-Jacobi-Trudi) identity $M^1_1 M^2_i - M^1_i M^2_1 = M^{12}_{1i} M$



For general G obtain relation on generalized minors $\Delta^{\omega_i}(v(u)) = Q^i_+(u)$ [Fomin Zelevinsky]

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}\Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i}\Delta_{u\cdot\omega_i, vw_i\cdot\omega_i} = \prod_{j\neq i} \Delta^{-a_{ji}}_{u\cdot\omega_j, v\cdot\omega_j},$$

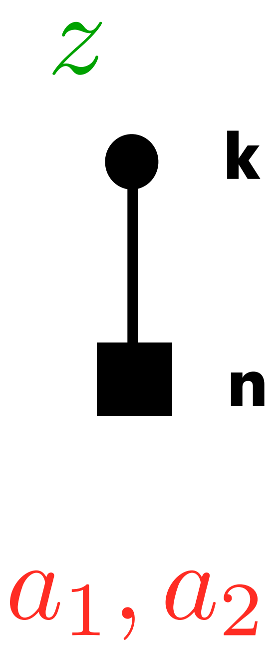
$u, v \in W_G$

Number Theory Applications

[Smirnov Varchenko]

Consider cohomological vertex (J-function)

$$\mathsf{V}(z) = \sum_{d=0}^{\infty} c_d\, z^d \in \mathbb{Q}[[z]] \qquad c_d := \int_{\mathsf{QM}_d(X,\infty)} \omega^{vir}$$



For a prime p construct a sequence of polynomials $T_s(z) \in \mathbb{Z}[z]$ from the superpotnential which converges to the vertex in the **p-adic** norm

$$\lim_{s\to\infty} \mathsf{T}_s(z) = \mathsf{V}(z) \qquad \mathsf{T}_s(z) = \mathrm{coeff}_{x^{dp^s-1}}\Big(\Phi_s(x,z)\Big)$$

Some properties

$$\mathsf{V}(z) = \prod_{i=0}^{\infty} \frac{\mathsf{T}_m(z^{p^i})}{\mathsf{T}_{m-1}(z^{p^{i+1}})} \mod p^m, \quad m = 1, 2 \ldots$$

Dwork identity

$$\frac{\mathsf{T}_{s+1}(z)}{\mathsf{T}_s(z^p)} = \frac{\mathsf{T}_s(z)}{\mathsf{T}_{s-1}(z^p)} \mod p^s$$