Quantum K-theory &Integrability

Peter Koroteev

Talk at workshop GLSM@30 5/22/2023

Enumerative AG and Integrability

String theory have been suggesting for a long time that there is a strong connection between **geometry** and **integrability**

Study of **Gromov-Witten** invariants was influenced by progress in string theory. For a symplectic manifold **X** GW invariants appear in the expansion of quantum multiplication in **quantum cohomology** of **X.**

A particular attention is given to genus zero GW invariants.

In this talk, we study **equivariant quantum K-theory** of a large family of varieties and its connection to **integrable systems** as well as some applications to representation theory and number theory

Classical Integrability

Equations of motion

$$\frac{df}{dt} = \{H_1, f\}$$

Integrability — family of n conserved quantities which Poisson commute with each other

$$\{H_i, H_j\} = 0 \quad i, j = 1, \ldots, n$$

Liouville-Arnold Theorem

Compact Lagrangians $\mathscr{L}$: $\{H_i = E_i\}$ are isomorphic to tori

Evolution in the neighborhood of $\mathscr{L}$ is linearized in action/angle variables $\{I_i, \varphi_i\}_{i=1}^{n}$

$$\frac{d\varphi_i}{dt} = \omega_i, \qquad \frac{dI_i}{dt} = 0$$

Action/angle variables are hard to find

Examples

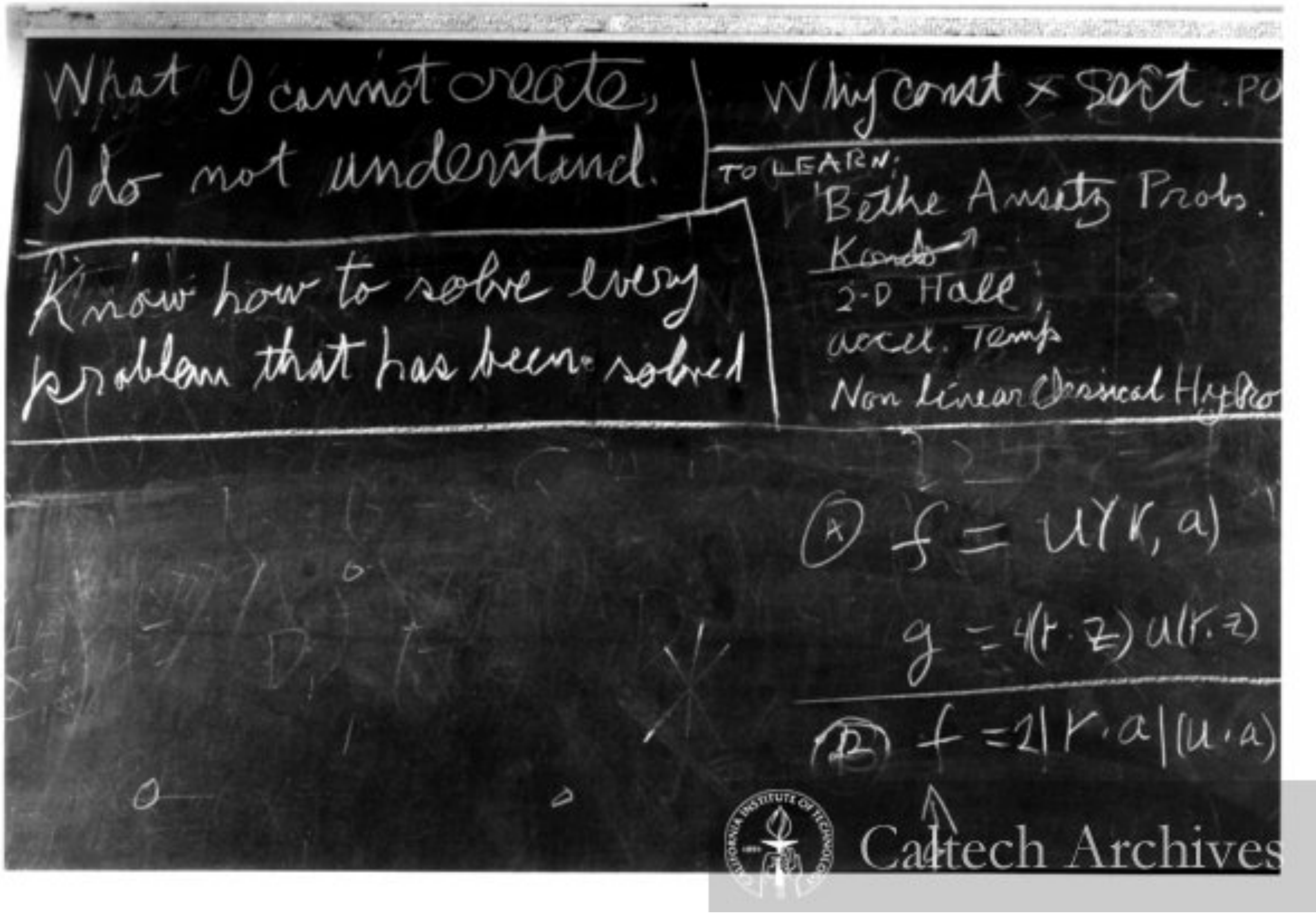
Many-body integrable systems — Calogero, Toda, Ruijsenaars (more on this later)

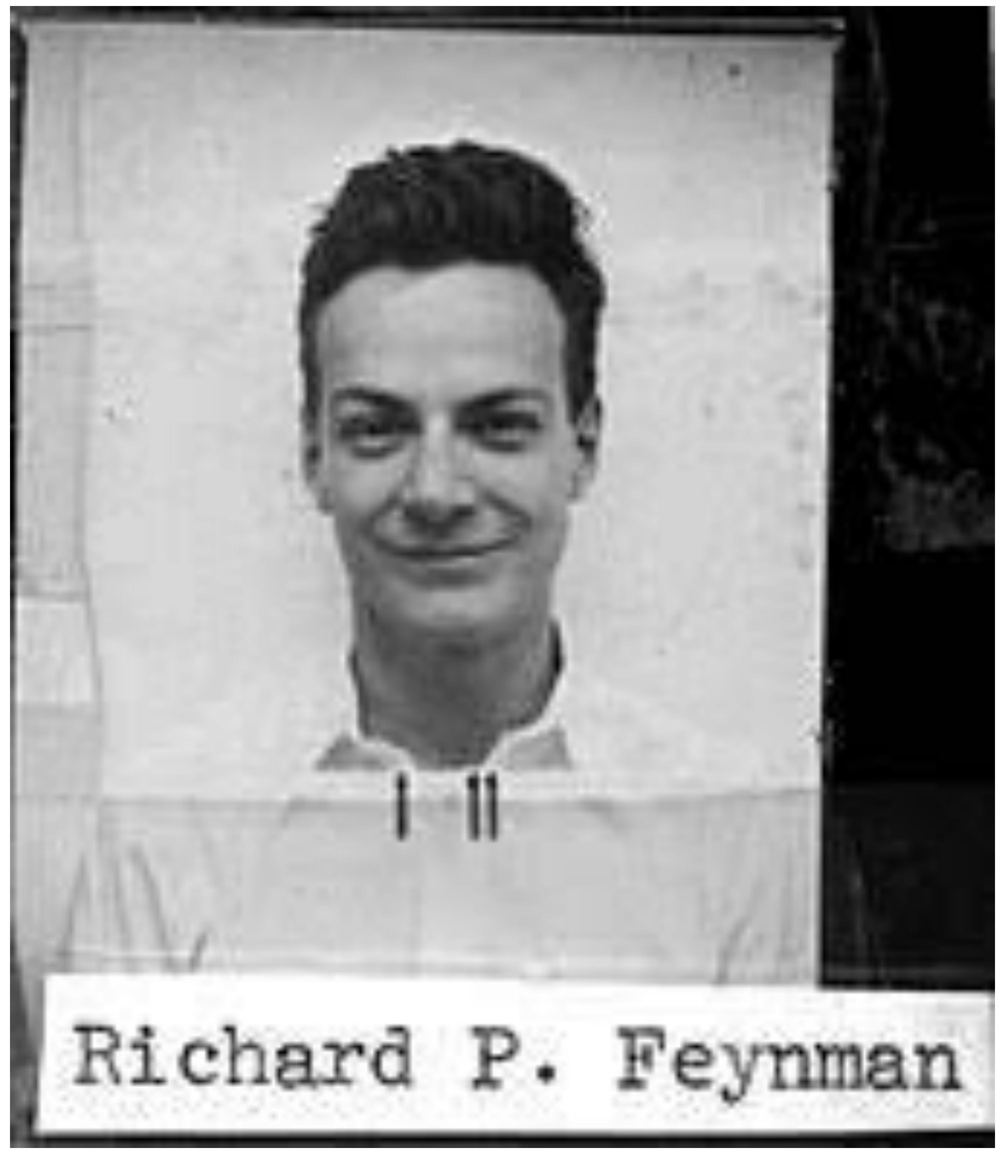
Continuous integrable models in (1+1)-dimensions: Korteweg-de-Vries, Intermediate Long-Wave, etc.

$$u_t = 6uu_x - u_{xxx}$$

They admit soliton solutions. Sectors with N solitons are described by finite N-body integrable systems

Inverse scattering method — Lax pair data → action/angle variables





I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better.

Literature

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qKZ/tRS Duality via Quantum K-Theoretic Counts
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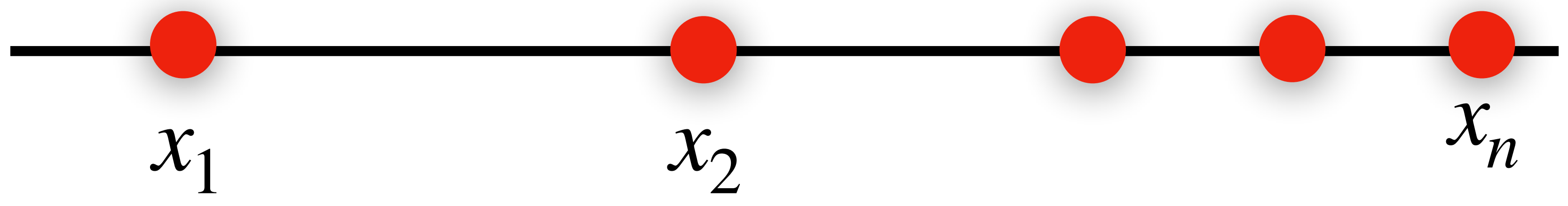
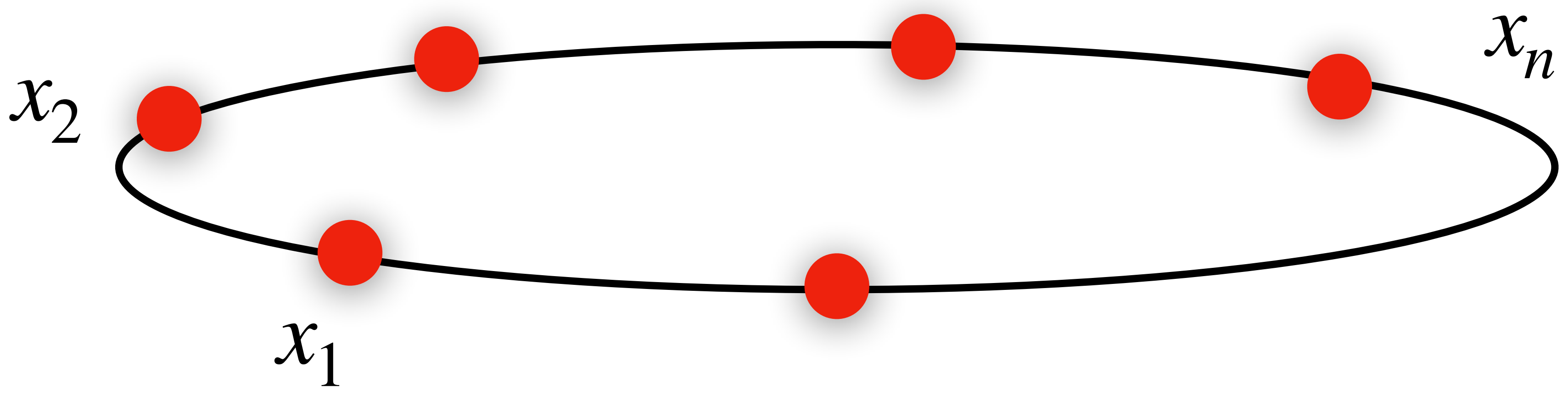
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

I. Many-Body Systems

Calogero in 1971 introduced a new integrable system. Moser in 1975 proved its integrability using Lax pair





$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$

The **Calogero-Moser (CM)** system has several generalizations: rational CM $\rightarrow$ trigonometric CM $\rightarrow$ elliptic CM

$$V(x) \simeq \sum \frac{1}{(x_i - x_j)^2} \qquad V(x) \simeq \sum \frac{1}{\sinh(x_i - x_j)^2} \qquad V(x) \simeq \wp(x_j - x_i)$$

Another relativistic generalization called **Ruijsenaars-Schneider (RS)** family

rRS $\rightarrow$ tRS $\rightarrow$ eRS

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

Example: tRS Model with 2 Particles

Hamiltonians

$$T_1 = \frac{\xi_1 - t\xi_2}{\xi_1 - \xi_2} p_1 + \frac{\xi_2 - t\xi_1}{\xi_2 - \xi_1} p_2$$

$$T_2 = p_1 p_2$$

Coordinates ξ_i, momenta p_i

coupling constant t, energies E_i

Symplectic form

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{d\xi_i}{\xi_i}$$

Integrals of motion

$$T_i = E_i$$

Quantization

$$p_i \xi_j = \xi_j p_i q^{\delta_{ij}} \qquad q \in \mathbb{C}^\times$$

tRS Momenta are shift operators

$$p_i f(\xi_i) = f(q\xi_i)$$

Eigenvalue Equations

$$T_i V = E_i V$$

Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

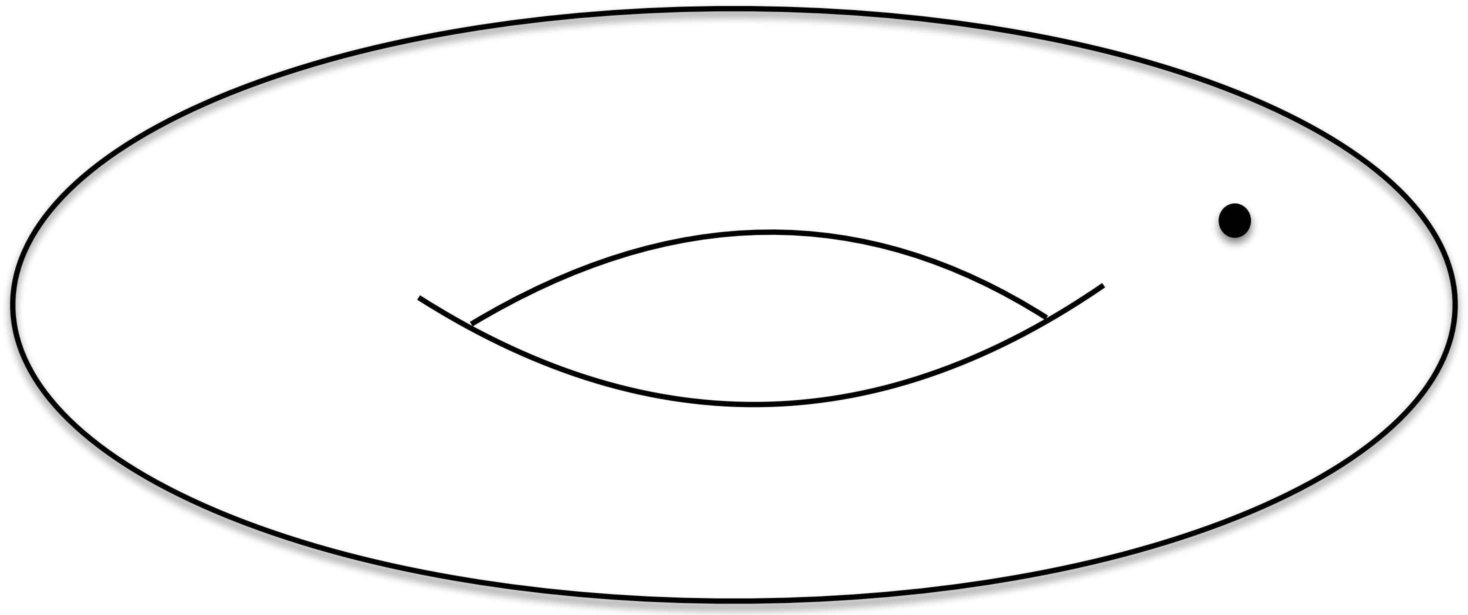
$$qMT - TM = u \otimes v^T$$

The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space** $\mathscr{M}$

Flat connections on punctured torus



$$\mathcal{M}_n = \{A, B, C\}/GL(n; \mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

Integrable Hamiltonians are ~TrT^k

T-Lax matrix

[my DAHA paper with Gukov, Nawata, Pei, Saberi
[arXiv:2206.03565] **SpringerBriefs** (2023)]

II. Quantum Integrability

Let $\mathfrak{g}$ Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g}(t)$ loop algebra (Laurent poly valued in g)

Evaluation modules form a tensor category of $\hat{\mathfrak{g}}$

$$V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

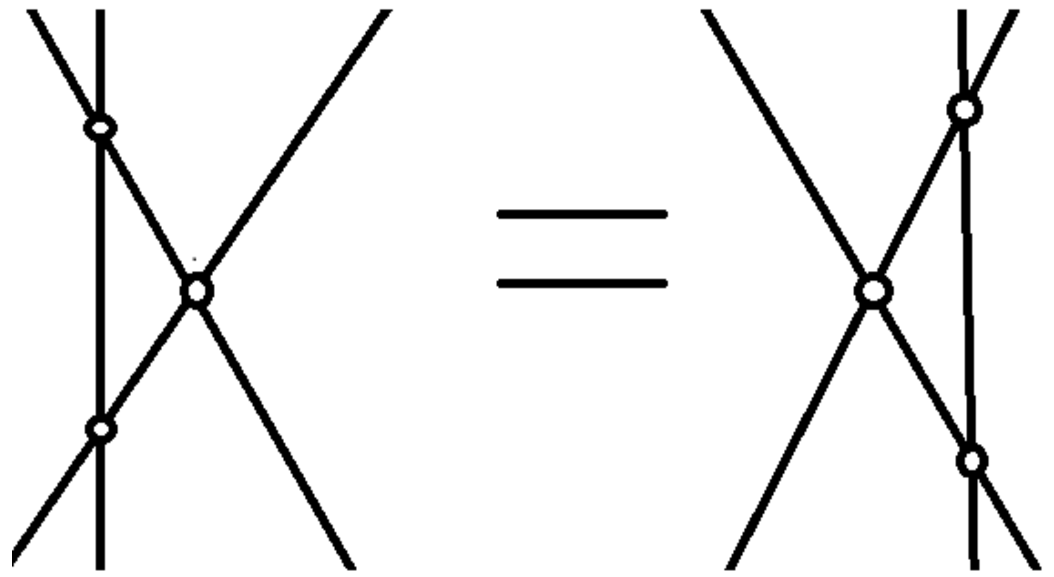
V_i are representations of $\mathfrak{g}$ a_i are special values of spectral parameter t

Quantum group is a noncommutative deformation $U_\hbar(\hat{\mathfrak{g}})$

with a nontrivial intertwiner — R-matrix

$$R_{V_1,V_2}(a_1/a_2) : V_1(a_1) \otimes V_2(a_2) \to V_2(a_2) \otimes V_1(a_1)$$

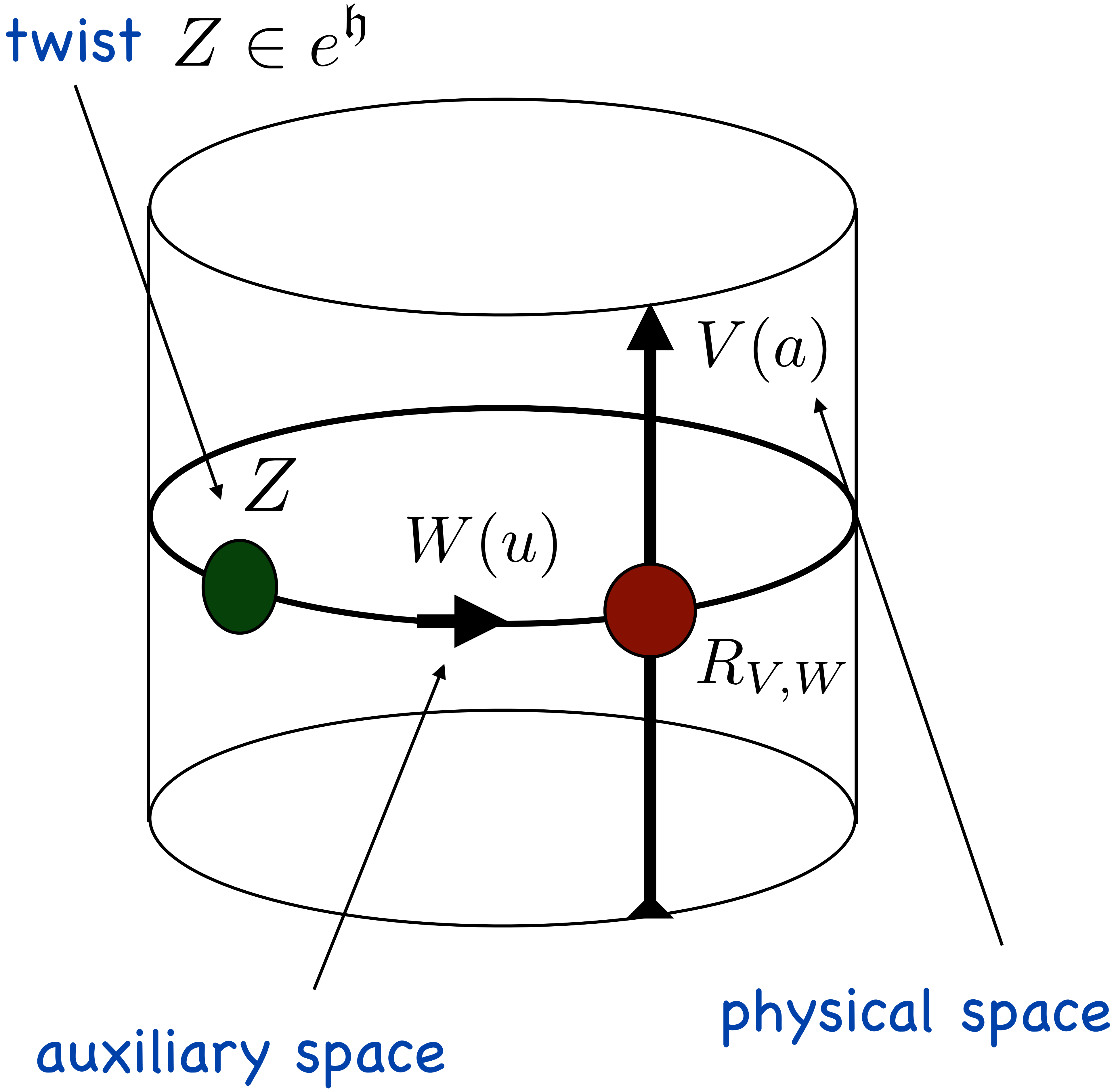
satisfying Yang-Baxter equation



Transfer Matrix

[Faddeev Reshetikhin Tachtajan]

The intertwiner represents an interaction vertex in integrable models. The quantum group is generated by matrix elements of R



Integrability comes from transfer matrices which generates Bethe algebra

$$T_W(u) = Tr_{W(u)}((Z \otimes 1)R_{V,W})$$

$$[T_W(u), T_W(u')] = 0$$

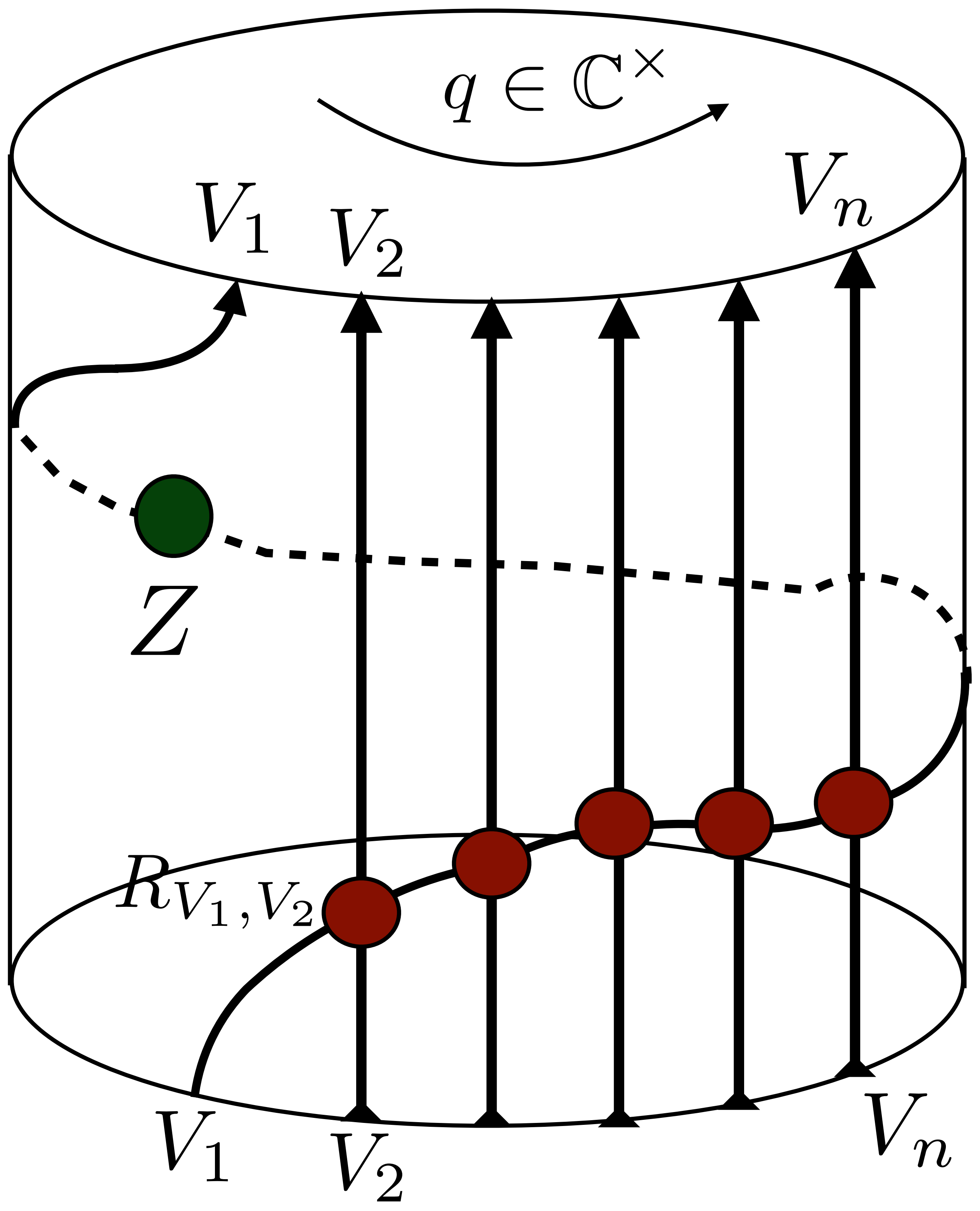
Transfer matrices are usually polynomials in u whose coefficients are the **integrals of motion**

The XXZ Spin Chain

$\mathfrak{g} = \mathfrak{sl}_2$ spin-1/2 chain on n sites $V = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$

Consider Knizhnik-Zamolodchikov (qKZ) difference equation [I. Frenkel Reshetikhin]

$$\Psi(qa_1, \ldots a_n) = (Z \otimes 1 \otimes \cdots \otimes 1) R_{V_1,V_n} \cdots R_{V_1,V_2} \Psi(a_1, \ldots a_n)$$



where

$$\Psi(a_1, \ldots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

In the limit $q \to 1$

qKZ becomes an eigenvalue problem

Solutions of qKZ

[Aganagic Okounkov]

Schematic solution

$$\Psi_\alpha = \int \frac{d\mathbf{x}}{\mathbf{x}} \, f_\alpha(\mathbf{x}, a) \, \mathcal{K}(\mathbf{x}, z, a, q)$$

indexed by physical space

representation

universal kernel

$$\log \mathcal{K}(\mathbf{x}, z, a, q) \underset{q \to 1}{\sim} \frac{S(\mathbf{x}, z, a)}{\log q}$$

$$\frac{\partial S}{\partial x_i} = 0$$ Bethe equations for Bethe roots $\mathbf{x}$

$$a_i \frac{\partial S}{\partial a_i} = \Lambda_i$$ Eigenvalues of qKZ operators

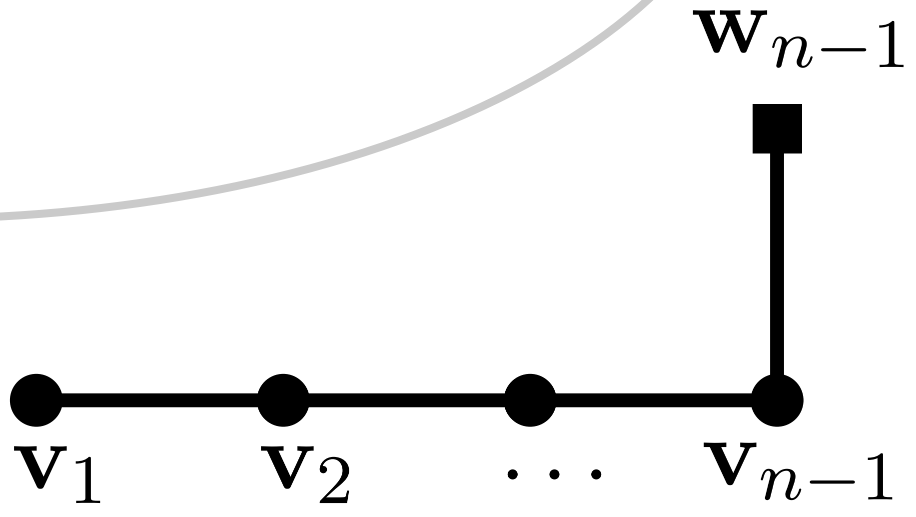
The map $\alpha \mapsto f_\alpha(\mathbf{x}^*)$ provides diagonalization

So we need to find `off shell' Bethe eigenfunctions $f_\alpha(\mathbf{x}, a)$

The Nekrasov-Shatashvili Correspondence

The answer will come from enumerative algebraic geometry inspired by physics

Hilbert space of states of quantum integrable system $\longleftrightarrow$ Equivariant K-theory of Nakajima quiver varitey



gauge group $G = \prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} U(v_i)$ $(v_1, v_2, \ldots)$ encode weight of rep α

Bethe roots **x** live in the maximal torus of G, by integrating over **x** we project on Weyl invariant functions of Bethe roots

Flavor group $G_F = \prod_i U(w_i)$ whose maximal torus gives parameters **a**

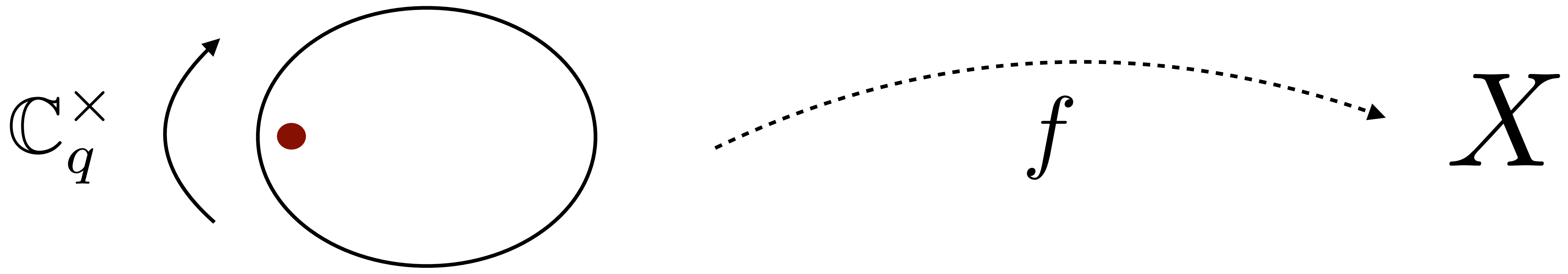
Bifundamental matter $\mathrm{Hom}(V_i, V_j)$

Quantum K-theory of X

The quiver variety X = {Matter fields}/gauge group

X is a module of a quantum group in the Nakajima correspondence construction

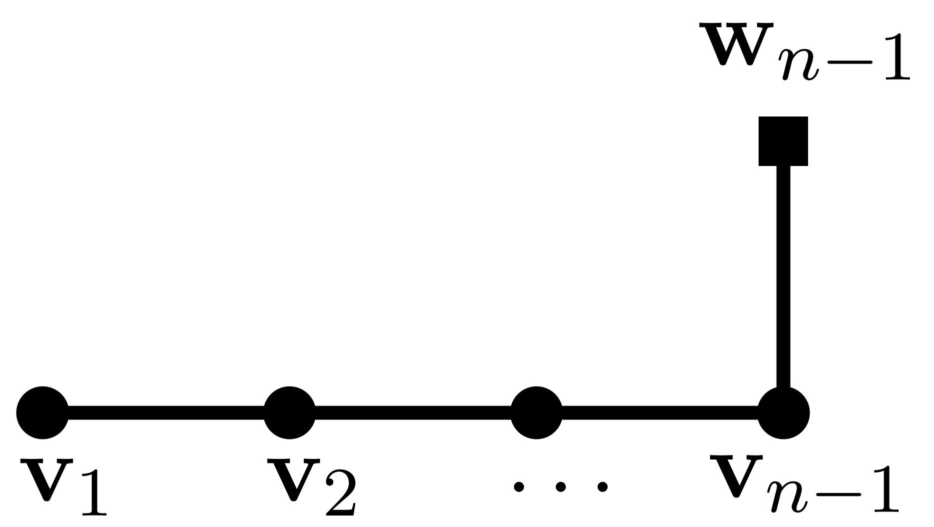
We will be computing integrals in K-theory of the space of quasimaps $f : \mathcal{C} --- > X$ weighted by degree $\mathbf{z}^{\deg f}$ subject to equivariant action on the base nodal curve $\mathbb{C}_q^\times$



(cf Gromov-Witten invariants)

Nakajima Quiver Varieties

Rep(**v**,**w**) — linear space of quiver reps



Moment map $\mu : T^*\mathrm{Rep}(\mathbf{v},\mathbf{w}) \to \mathrm{Lie}(G)^*$

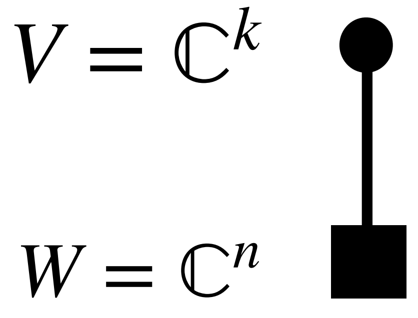
Quiver variety $X = \mu^{-1}(0)//_\theta G = \mu^{-1}(0)_{ss}/G$ $\quad G = \prod GL(\mathbf{v}_i)$

Automorphism group $\mathrm{Aut}(X) = \prod GL(Q_{ij}) \times \prod GL(W_i) \times \mathbb{C}^\times_\hbar$

Maximal torus (a) $T = \mathbb{T}(\mathrm{Aut}(X))$

Tensorial polynomials of tautological bundles V_i, W_i and their duals generate classical T-equivariant K-theory ring of X

Ex: $T^*Gr_{k,n}$ $\quad \tau(V) = V^{\otimes 2} - \Lambda^3 V^*$



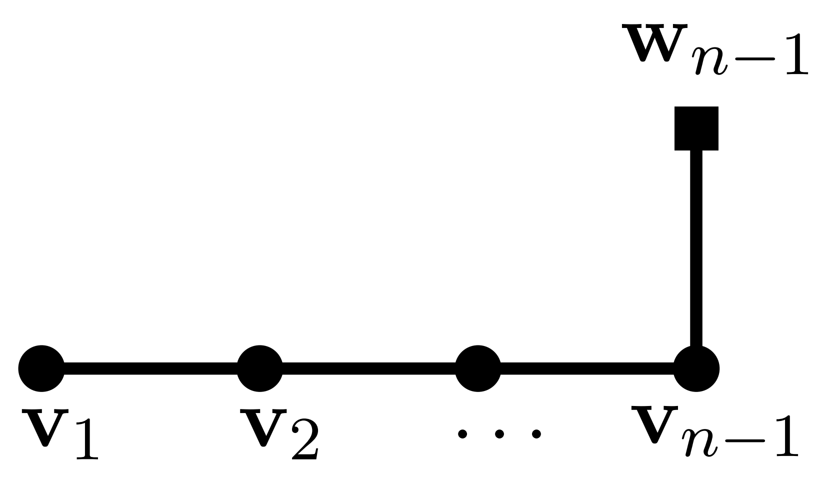
$\mathbf{v}_1 = k,\ \mathbf{w}_1 = n$ $\quad \tau(s_1,\cdots,s_k) = (s_1+\cdots+s_k)^2 - \sum_{1\le i_1<i_2<i_3\le k} s_{i_1}^{-1}s_{i_2}^{-1}s_{i_3}^{-1}$

Quasimaps

[Ciocan-Fontanine, Kim, Maulik]
[Okounkov]

A **quasimap** $f : \mathcal{C} - - \to X$ is described by

- vector bundles $\mathscr{V}_i$ on $\mathcal{C}$ of ranks $\mathbf{v}_i$, trivial bundles $\mathscr{W}_i$ of ranks $\mathbf{w}_i$
- section $f \in H^0(\mathcal{C}, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$ satisfying $\mu = 0$



$$\mathscr{M} = \sum_{i \in I} Hom(\mathscr{W}_i, \mathscr{V}_i) \oplus \sum_{i,j \in I} Q_{ij} \otimes Hom(\mathscr{V}_i, \mathscr{V}_j)$$

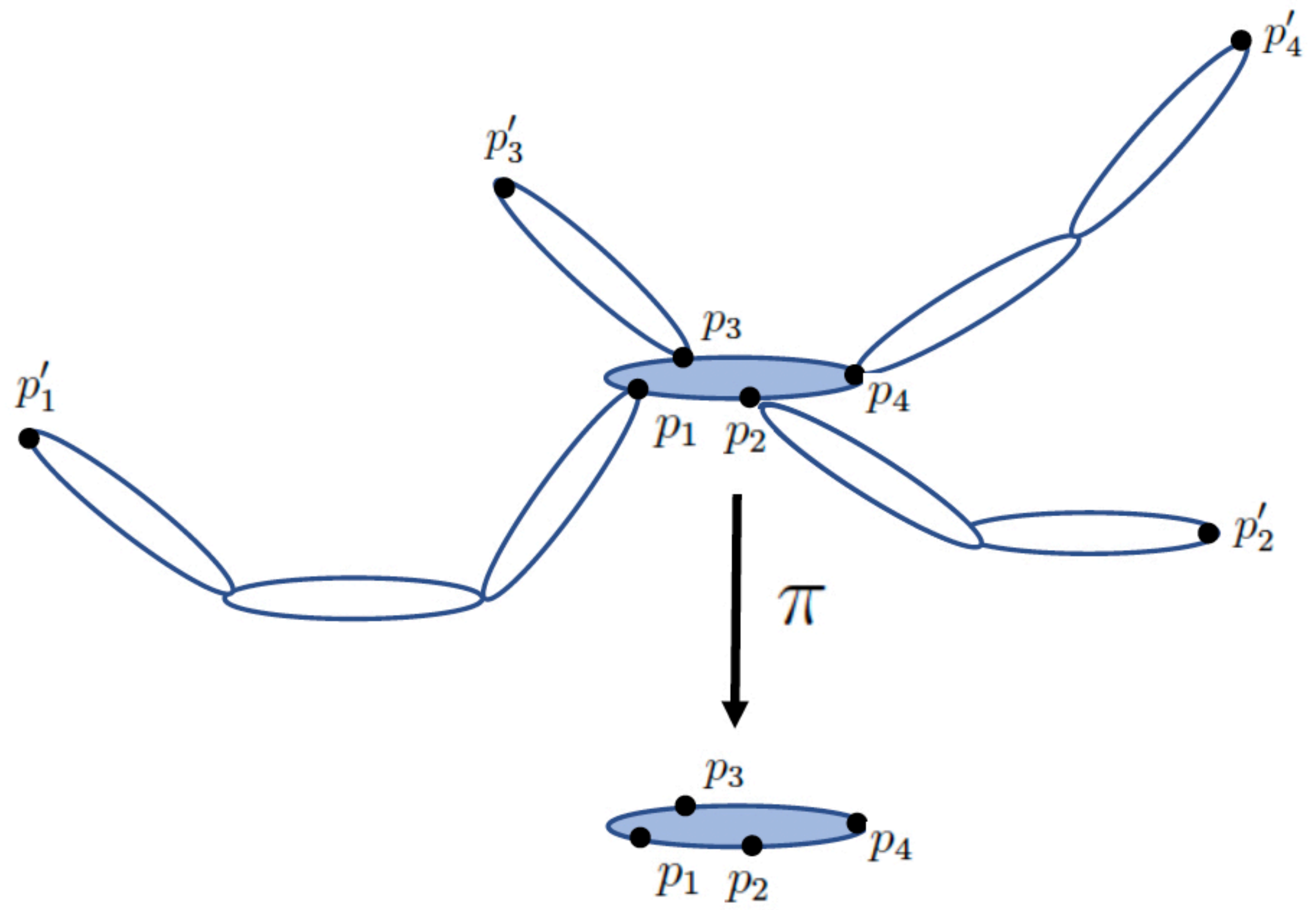
Evaluation map to quotient stack

$$\mathrm{ev}_p(f) = f(p) \in [\mu^{-1}(0)/G] \supset X$$

Quasimap is **stable** if $f(p) \in X$ for all but finitely many points — singularities

The moduli space of stable quasimaps $\mathbf{QM}^d(X)$

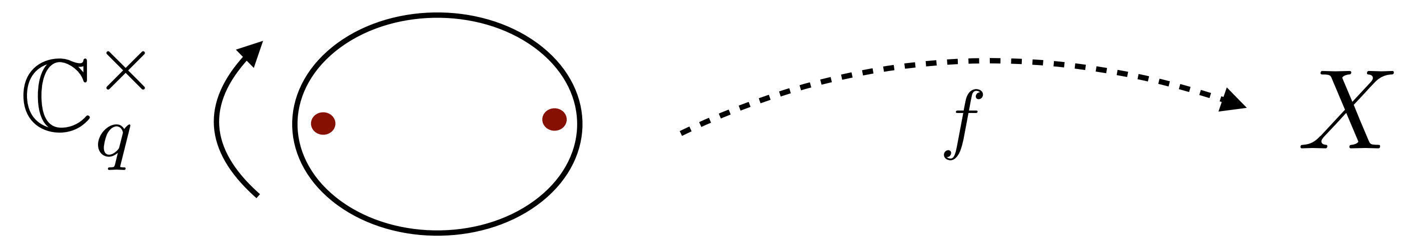
$\mathscr{V}_i$ and f vary



Quantum K-theory

[Okounkov]
[Pushkar Smirnov Zeitlin]

Quasimaps spaces admit action of $\mathbb{C}_q^\times$ on base $\mathbb{P}^1$ with two fixed points $p_1 = 0,\ \ p_2 = \infty$



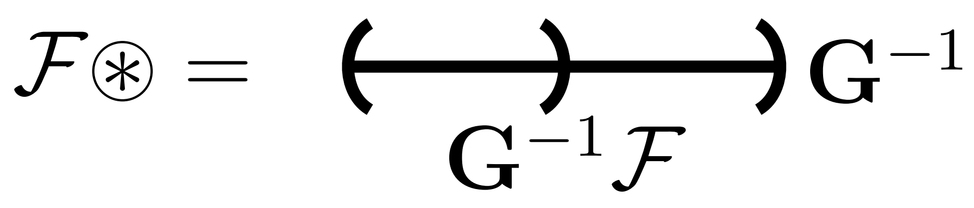
Define **vertex function** for τ with quantum (Novikov) parameters **z**

$$\mathbf{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}(\widehat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \tau|_{p_1}, \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{nonsing}\, p_2})\boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}_q^\times}(X)_{loc}[[\boldsymbol{z}]]$$

fixed pts

$$K_T(X)_{loc} = K_T(X) \otimes_{\mathbb{Z}[a,\hbar]} \mathbb{Q}(a,\hbar)$$

Define **quantum K-theory** as a ring with multiplication



$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

Theorem: $QK(X)$ is a commutative associative unital algebra

$$\hat{\mathbf{1}}(z) = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\left(\mathsf{QM}^d_{\mathrm{relative}\, p_2}, \widehat{\mathcal{O}}^d_{\mathrm{vir}}\right)$$

Vertex computation for T^*G/P

Localization $$V^{(\tau)}_{\mathbf{p}}(z) = \sum_{\mathbf{d}\in\mathbb{Z}^n_{\geq 0}} \sum_{(\mathscr{V},\mathscr{W})\in(\mathsf{QM}^{\mathbf{d}}_{\text{nonsing}\,p_2})^{\mathsf{T}}} \hat{s}(\chi(\mathbf{d}))\, z^{\mathbf{d}} q^{\deg(\mathscr{P})/2} \tau(\mathscr{V}|_{p_1}).$$

$$\chi(\mathbf{d}) = \mathrm{char}_{\mathsf{T}}\Big(T^{vir}_{\{(\mathscr{V}_i\},\ \mathscr{W}_{n-1})}\mathsf{QM}^{\mathbf{d}}\Big)$$

$$\hat{s}(x) = \frac{1}{x^{1/2} - x^{-1/2}}$$

At a fixed point

$$\mathcal{M} = (\mathcal{O}(d)\otimes q^{-d}) \oplus \left(\mathcal{O}(d)\otimes q^{-d}\otimes \frac{a_i}{a_j}\right)$$

character

$$char_{\mathsf{T}}\Big(H^{\bullet}(a_i q^{-d_i}\mathcal{O}(d_i))\Big) = a_i \frac{q^{-d_i-1}-1}{q^{-1}-1}$$

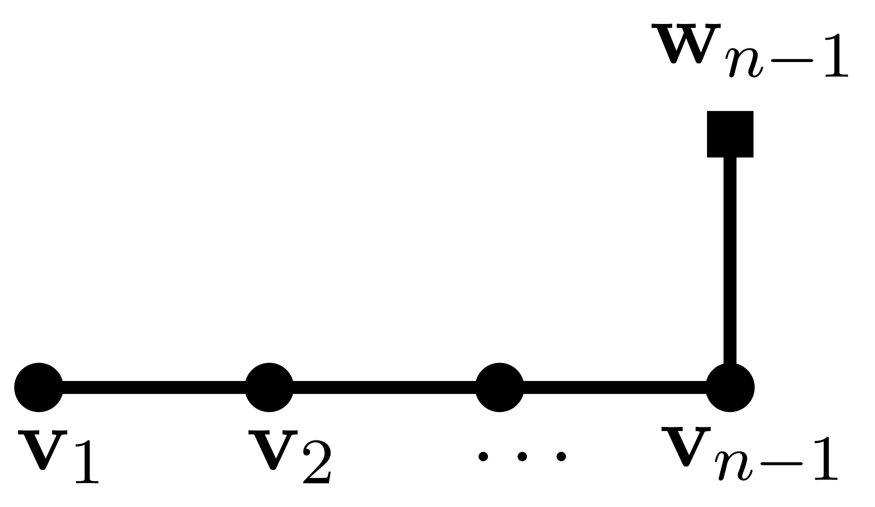
Contribution of $xq^{-d}\mathcal{O}(d)$ to the character is

$$\{x\}_d = \frac{(\hbar/x,q)_d}{(q/x,q)_d}(-q^{1/2}\hbar^{-1/2})^d, \quad \text{where} \quad (x,q)_d = \frac{\varphi(x)}{\varphi(q^d x)}$$

$$\varphi(x) = \prod_{i=0}^{\infty}(1-q^i x)$$

Vertex coefficient function $$V^{(\tau)}_{\boldsymbol{p}}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j}q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1}\prod_{j,k=1}^{\mathbf{v}_i}\{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}}$$



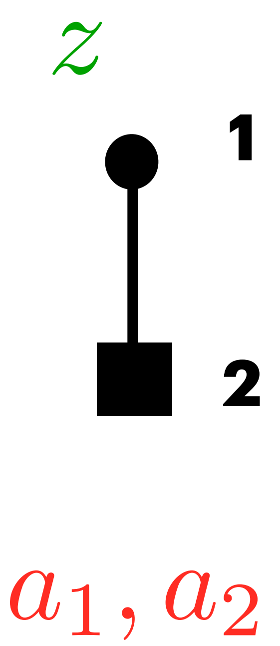
Vertex for $T^*\mathbb{P}^1$

Vertex function coefficient with trivial insertion

$$V^{(1)}_{\mathbf{p}} = \sum_{d>0} z^d \prod_{i=1}^{2} \frac{\left(\frac{q}{\hbar}\frac{a_{\mathbf{p}}}{a_i};q\right)_d}{\left(\frac{a_{\mathbf{p}}}{a_i};q\right)_d} = {}_2\phi_1\left(\hbar, \hbar\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, q\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; \frac{q}{\hbar}z\right) .$$

two fixed points

$\mathbf{p} = \{a_1\}$ and $\mathbf{p} = \{a_2\}$.



Truncation on V - Macdonald Polynomials!!

As a contour integral

$$V = \frac{e^{-\frac{\log z \cdot \log a_1 a_2}{\log q}}}{2\pi i} \int_C \frac{ds}{s}\, e^{\frac{\log z \cdot \log s}{\log q}} \frac{\varphi\left(\hbar\frac{s}{a_1}\right)}{\varphi\left(\frac{s}{a_1}\right)} \frac{\varphi\left(\hbar\frac{s}{a_2}\right)}{\varphi\left(\frac{s}{a_2}\right)}$$

[PK [arXiv:1805.00986]
Comm.Math.Phys. (2021)]

Vertex functions are eigenfunctions of quantum tRS difference operators!

$$T_i(a)V(z,a) = e_i(z)V(z,a) \qquad T_i(z)V(z,a) = e_i(a)V(z,a)$$

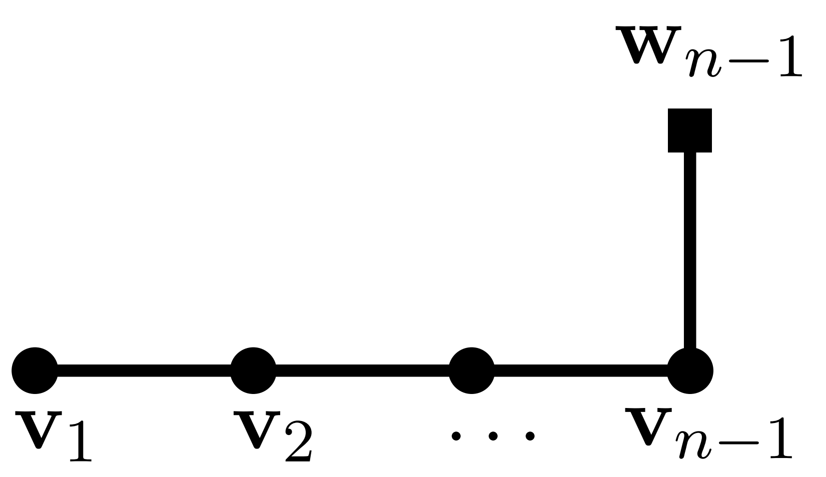
$$\hbar \to \hbar^{-1}$$

[PK Zeitlin [arXiv:1802.04463]
Math.Res.Lett. **28** (2021) 435]

Quantum K-theory Ring

[PK Pushkar Smirnov Zeitlin]

Classical limit $q \to 1$ yields



$$QK_T(T^*\mathbb{F}l_n) = \frac{\mathbb{C}[\zeta_1^{\pm 1}, \dots, \zeta_n^{\pm 1}; a_1^{\pm 1}, \dots, a_n^{\pm 1}, \hbar^{\pm 1}; p_1^{\pm 1}, \dots, p_n^{\pm 1}]}{(H_r(\zeta_i, p_i, \hbar) - e_r(a_1, \dots, a_n))}$$

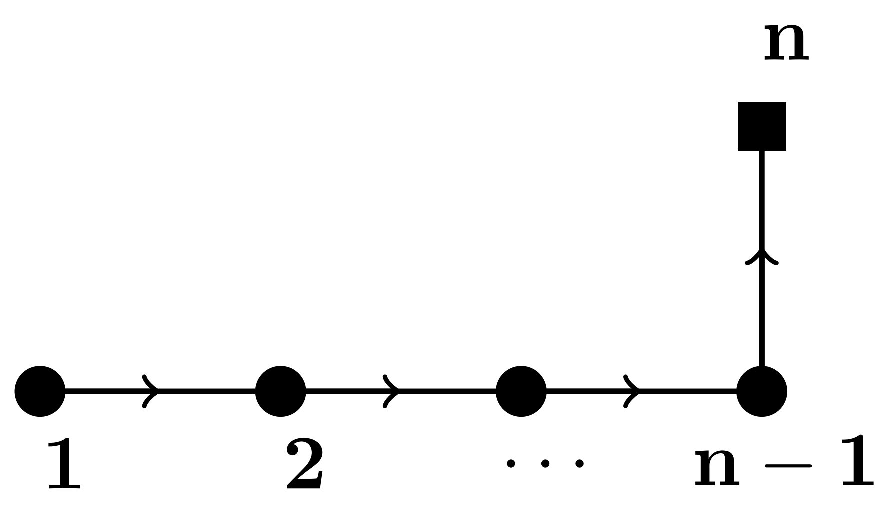
tRuijsenaars-Schneider integrals of motion

Contributions from the base and the fiber in T*G/B split $\quad (\omega, \omega^{-1}\hbar)$

[cf Givental Lee]

$$\frac{1}{\omega^{1/2} - \omega^{-1/2}} \frac{1}{(\hbar\omega^{-1})^{1/2} - (\hbar\omega^{-1})^{-1/2}} = \frac{1}{1-\omega^{-1}} \frac{-\hbar^{1/2}}{1-\hbar^{-1}\omega^{-1}}$$

$$\hbar \to \infty \qquad \hat{s}(\omega, \omega^{-1}\hbar) \to \frac{1}{1-\omega^{-1}}$$



Vertex functions satisfy q-Toda difference relations

$$V_{\mathbf{p}}^{(1)} \to {}_2\phi_1\left(0, 0, \frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; z^\sharp\right) =: {}_1\phi_0\left(\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; z^\sharp\right) = \sum_{k=0}^{\infty} \frac{(z^\sharp)^k}{\left(\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, q\right)_k (q,q)_k}$$

$$H_r^{\text{q-Toda}} = \sum_{\substack{\mathcal{I}=\{i_1<\cdots<i_r\} \\ \mathcal{I}\subset\{1,\dots,n\}}} \prod_{\ell=1}^{r} \left(1 - \frac{\mathfrak{z}_{i_\ell - 1}}{\mathfrak{z}_{i_\ell}}\right)^{1-\delta_{i_\ell - i_{\ell-1},1}} \prod_{k\in\mathcal{I}} \mathfrak{p}_k$$

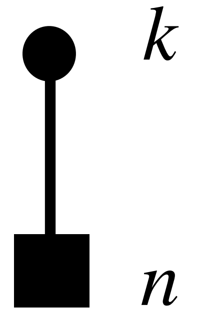
$$q \to 1 \qquad QK_{T'}(\mathbb{F}l_n) = \frac{\mathbb{C}[\mathfrak{z}_1^{\pm 1}, \dots, \mathfrak{z}_n^{\pm 1}; \mathfrak{a}_1^{\pm 1}, \dots, \mathfrak{a}_n^{\pm 1}; \mathfrak{p}_1^{\pm 1}, \dots, \mathfrak{p}_n^{\pm 1}]}{\left(H_r^{q\text{-}Toda}(\mathfrak{z}_i, \mathfrak{p}_i) = e_r(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\right)}$$

Bethe Equations for $T^*Gr_{k,n}$

[Pushkar Smirnov Zeitlin]

Operator of quantum multiplication

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V_{\boldsymbol{p}}^{(\tau)}(z)}{V_{\boldsymbol{p}}^{(1)}(z)}$$



Theorem *The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(s_1, \cdots, s_k)$ evaluated at the solutions of the following equations:*

$$\prod_{j=1}^{n} \frac{s_i - a_j}{\hbar a_j - s_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{s_i \hbar - s_j}{s_i - s_j \hbar}, \quad i = 1 \cdots k.$$

Equivariant parameters a_i,

twist z,

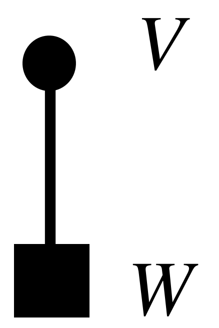
Planck constant $\hbar$

Baxter Q-operator $$Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$$ has eigenvalue $$Q(u) = \prod_{i=1}^{k} (u - s_i)$$

The QQ-System for A_1

Short exact sequence of bundles

$$0 \to V \to W \to V^{\vee} \to 0$$



Eigenvalues of Q-operators

$$Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$$

$$\widetilde{Q}(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V^{\vee})(z) \circledast$$

Satisfy the QQ-relation

$$z\, \widetilde{Q}(\hbar u) Q(u) - \widetilde{Q}(u) Q(\hbar u) = \prod_{i=1}^{n} (u - a_i)$$

equivalent to the XXZ Bethe equations

QQ-System in General

Consider complex simple Lie algebra $\mathfrak{g}$ of rank r

Cartan matrix $a_{ij} = \langle \check{\alpha}_i, \alpha_j \rangle$

$$\widetilde{\xi}_i\, Q^i_-(u) Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u) Q^i_+(u) = \Lambda_i(u) \prod_{j>i} \left[Q^j_+(\hbar u) \right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(u) \right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Polynomials $Q_+(u)$ contain Bethe roots, $\Lambda(u)$ contain equivariant parameters

Polynomials $Q_-(u)$ are auxiliary

The Ubiquitous QQ-System

Bethe Ansatz equations for XXX, XXZ models — eigenvalues of Baxter operators

[Mukhin, Varchenko] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\hat{g})$

[Frenkel, Hernandez] ....

Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Nekrasov-Shatashvili] [Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

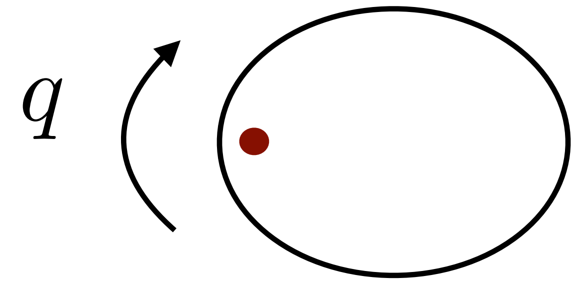
Spectral determinants in the QDE/IM Correspondence

[Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

(G,q)-Opers

III. (G,q)-Connection

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$u \mapsto qu$$



G-simple simply-connected complex Lie group

Consider vector bundle $\mathscr{F}_G$ over $\mathbb{P}^1$

(G, q)-connection A is a meromorphic section of $Hom_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{F}_G, \mathscr{F}_G^q)$

Locally q-gauge transformation of the connection

$$A(u) \mapsto g(qu)A(u)g(u)^{-1} \qquad g(u) \in G(\mathbb{C}(u))$$

Compare with (standard) gauge transformations

$$\partial_u + A(u) \mapsto g(u)(\partial_u + A(u))g(u)^{-1} \qquad g(u) \in \mathfrak{g}(u)$$

(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathfrak{F}_G, A, \mathfrak{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathfrak{F}_{B_-}$ is a reduction of $\mathfrak{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathfrak{F}_G \longrightarrow \mathfrak{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)]) c B_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally $$A(u) = n'(u) \prod_i (\phi_i(u)_i^{\check{\alpha}} s_i)\, n(u)$$

$$\phi_i(u) \in \mathbb{C}(u),\ n(u), n'(u) \in N_-(u) = [B_-(u), B_-(u)]$$

(SL(2),q)-Opers

Let $G = SL(2)$ The q-oper definition can be formulated as

Triple $(E, A, \mathscr{L})$

(E, A) is the $(SL(2), q)$ connection

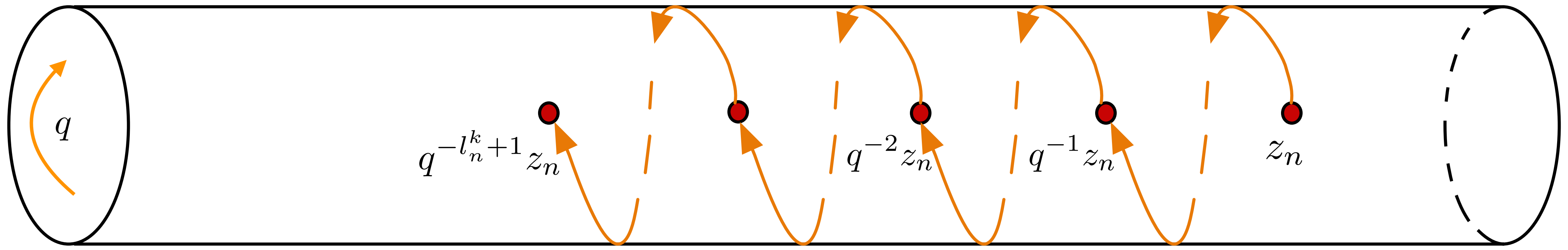
$\mathscr{L} \subset E$ is a line subbundle

The induced map $\bar{A} : \mathscr{L} \to (E/\mathscr{L})^q$ is an isomorphism

in a trivialization $\mathscr{L} =$ Span(s)

$$s(qu) \wedge A(u)s(u) \neq 0$$

Allow singularities $\quad s(qu) \wedge A(u)s(u) = \Lambda(u)$



$$\Lambda(u) = \prod_{l, j_l}(u - q^{j_l} a_l)$$

Add Twists $\quad Z = g(qu)A(u)g(u)^{-1}$

$$Z \in H \subset H(u) \subset G(u)$$

q-Opers, QQ-System & Bethe Ansatz

Chose trivialization of $\mathcal{L}$ $\quad s(u) = \begin{pmatrix} Q_+(u) \\ Q_-(u) \end{pmatrix}$

Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition — SL(2) **QQ-system**

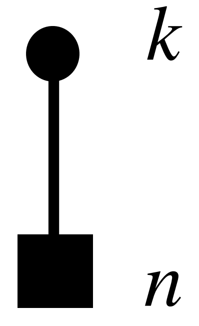
$$s(qu) \wedge A(u) s(u) = \Lambda(u) \longrightarrow \zeta\, Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(qu) Q_+(u) = \Lambda(u)$$

QQ-system to XXZ Bethe equations

$$Q_+(u) = \prod_{k=1}^{m} (u - s_k)$$

$$\prod_{l=1}^{n} \frac{s_i - q^{r_l} a_l}{s_i - a_l} = \zeta^2 q^k \prod_{j=1}^{k} \frac{q s_i - s_j}{s_i - q s_j}$$

$$i = 1, \ldots, k$$



$\hbar = q$

q-Miura Transformation

Miura q-oper: $(E, A, \mathscr{L}, \hat{\mathscr{L}})$, where $(E, A, \mathscr{L})$ is a q-oper and $\hat{\mathscr{L}}$ is preserved by q-connection A

$$A(u) = \begin{pmatrix} g(u) & \Lambda(u) \\ 0 & g(u)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition $A(u) = v(qu) Z v(u)^{-1}$ $Z = \mathrm{diag}(\zeta, \zeta^{-1})$

$$g(u) = \zeta \frac{Q_+(qu)}{Q_+(u)}$$

$$v(u) = \begin{pmatrix} Q_+(u) & \zeta Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(u) Q_+(qu) \\ 0 & Q_+(u) \end{pmatrix} \in B_+(u)$$

The q-oper condition becomes the **SL(2) QQ-system** $\zeta\, Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(qu) Q_+(u) = \Lambda(u)$

Difference Equation $D_q(s) = As$

Scalar difference operator $\left(D_q^2 - T(qu) D_q - \dfrac{\Lambda(qu)}{\Lambda(u)} \right) s_1 = 0$

tRS Hamiltonians

Recover 2-body tRS Hamiltonian from an (SL(2),q)-Oper

$$\det\begin{pmatrix} Q_+(u) & \zeta Q_+(qu) \\ Q_-(u) & \zeta^{-1}Q_-(qu) \end{pmatrix} = \Lambda(u)$$

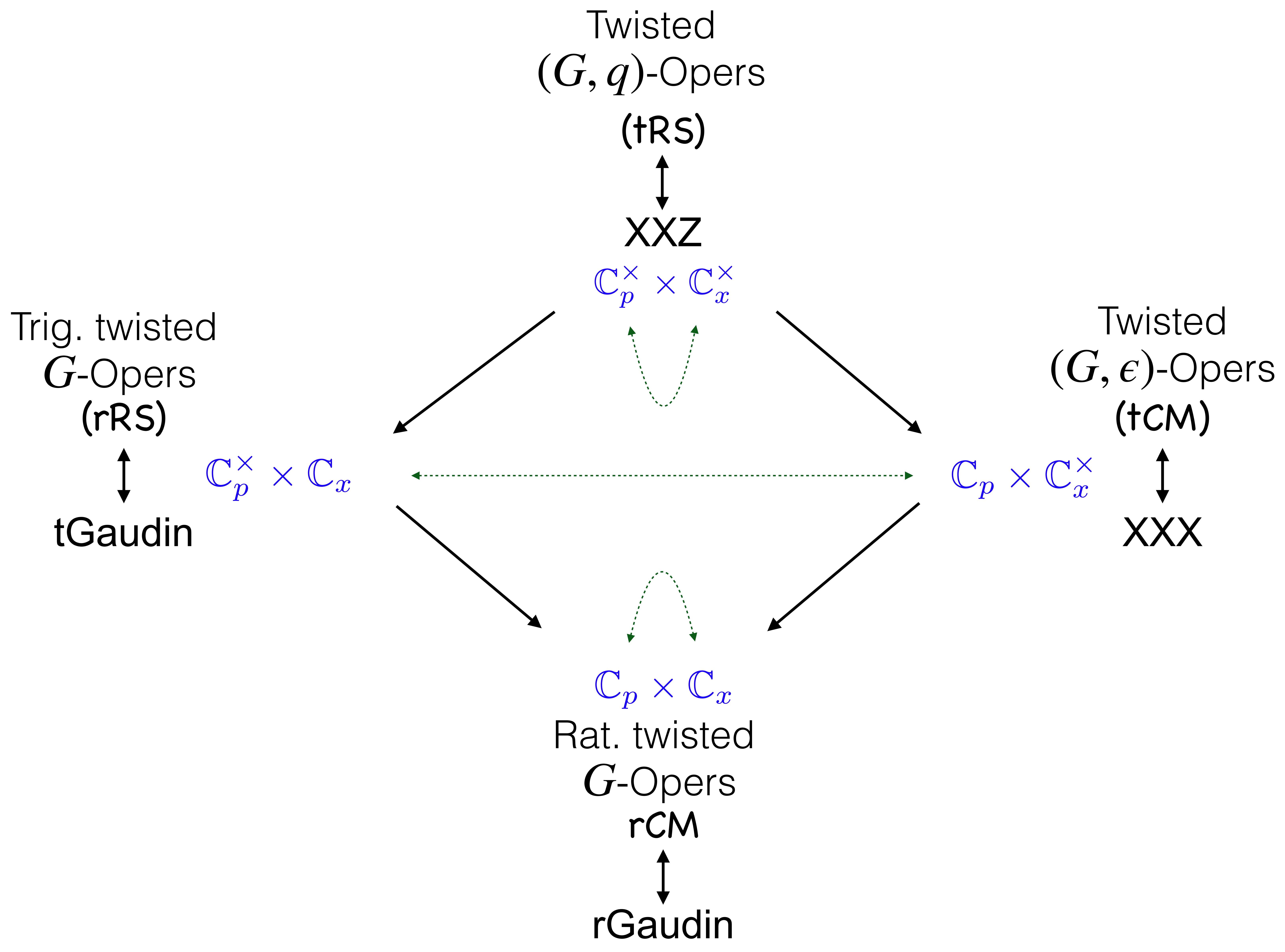
Let $\quad Q_+(u) = u - p_+ \qquad Q_-(u) = u - p_-$

$$u^2 - u\underbrace{\left[\frac{\zeta - q\zeta^{-1}}{\zeta-\zeta^{-1}}p_+ + \frac{q\zeta - q\zeta^{-1}}{\zeta^{-1}-\zeta}p_-\right]}_{T_1} + \underbrace{p_+p_-}_{T_2} = (u-a_+)(u-a_-)$$

qOper condition yields
tRS Hamiltonians!

$$\det(u - T) = (u-a_+)(u-a_-)$$

Network of Dualities



q-Opers and q-Langlands

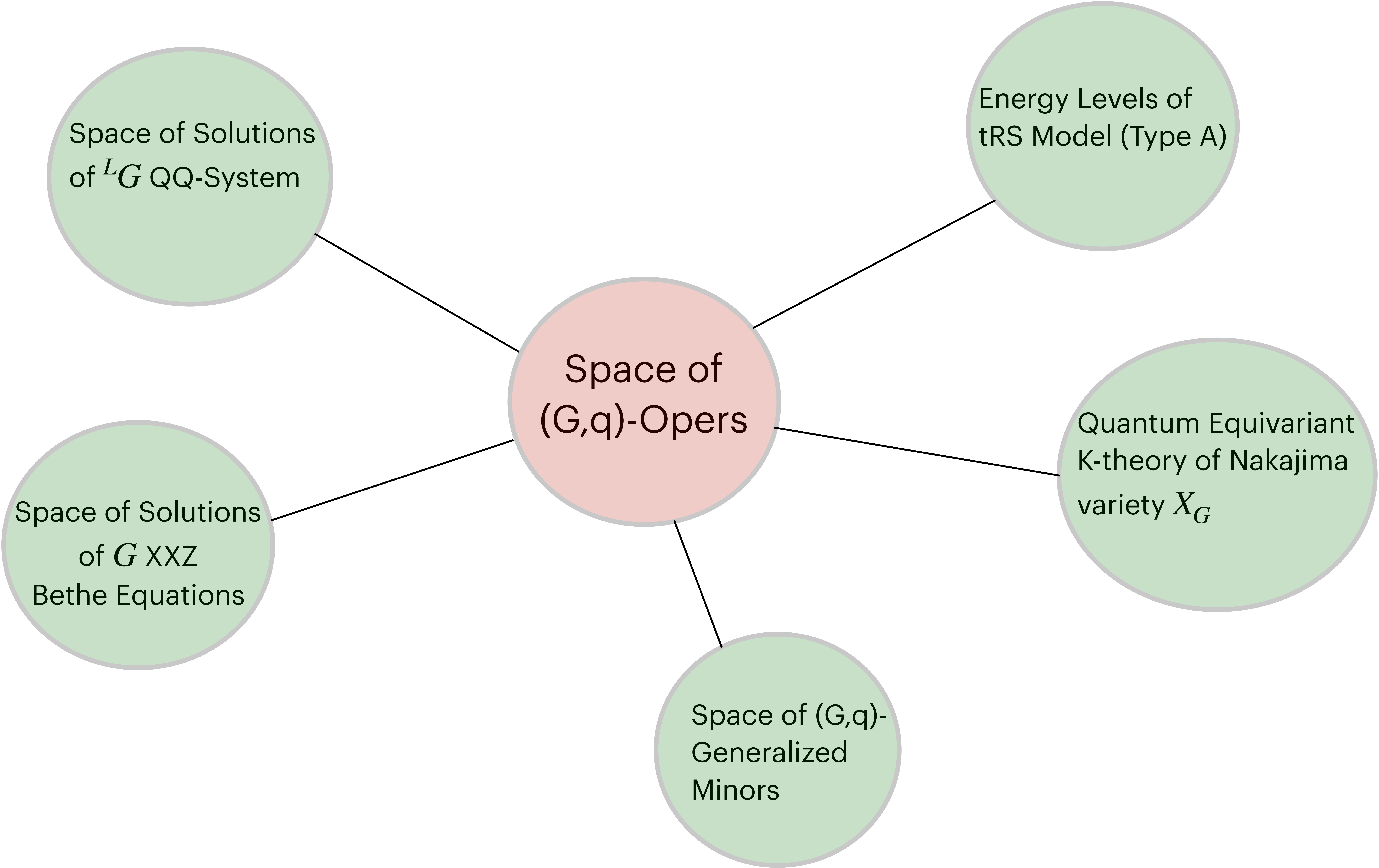
[Frenkel, PK, Zeitlin, Sage, **JEMS 2023**]

Miura (G,q)-oper with singularities
$$A(u) = \prod_i \left(\zeta_i \frac{Q_+^i(qu)}{Q_+^i(u)}\right)^{\check{\alpha}_i} \exp \frac{\Lambda_i(u)}{g_i(u)} e_i$$

Theorem: There is a 1-to-1 correspondence between the set of nondegenerate Z-twisted (G,q)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

$$\widetilde{\xi}_i\, Q_-^i(u)Q_+^i(\hbar u) - \xi_i Q_-^i(\hbar u)Q_+^i(u) = \Lambda_i(u)\prod_{j>i}\left[Q_+^j(\hbar u)\right]^{-a_{ji}}\prod_{j<i}\left[Q_+^j(u)\right]^{-a_{ji}}, \qquad i=1,\dots,r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i}\zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1}\prod_{j<i}\zeta_j^{-a_{ji}}$$

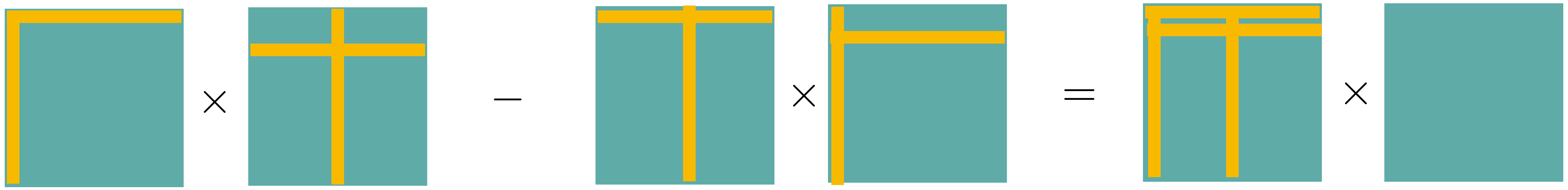

Space of Solutions
of ^{L}G QQ-System
Energy Levels of
tRS Model (Type A)
Space of
(G,q)-Opers
Quantum Equivariant
K-theory of Nakajima
variety X_G
Space of Solutions
of G XXZ
Bethe Equations
Space of (G,q)-
Generalized
Minors

IV. Cluster Algerbras

[PK, Zeitlin, **Crelle (2023)**]

The QQ-system $\quad \xi_{i+1}\, Q^i_-(u)Q^i_+(u+\epsilon) - \xi_i Q^i_-(u+\epsilon)Q^i_+(u) = \Lambda_i(u)Q^{i+1}_+(u+\epsilon)Q^{i+1}_+(u)$

For $G = SL(n)$ obtain Lewis Carrol (Desnanot-Jacobi-Trudi) identity $\quad M^1_1 M^2_i - M^1_i M^2_1 = M^{12}_{1i} M$



For general G obtain relation on generalized minors $\quad \Delta^{\omega_i}(v(u)) = Q^i_+(u)$ [Fomin Zelevinsky]

$$\Delta_{u\cdot\omega_i,v\cdot\omega_i}\Delta_{uw_i\cdot\omega_i,vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i,v\cdot\omega_i}\Delta_{u\cdot\omega_i,vw_i\cdot\omega_i} = \prod_{j\neq i}\Delta^{-a_{ji}}_{u\cdot\omega_j,v\cdot\omega_j}\,,$$

$u, v \in W_G$

q-Langlands Correspondence

[Aganagic Frenkel Okounkov]

Two types of solutions of the qKZ equation:

Analytic in chamber of equivariant parameters $\{a_i\}$— conformal blocks of $U_\hbar(\widehat{\mathfrak{g}})$

Analytic in chamber of quantum parameters (twists) $\{\zeta_i\}$— conformal blocks for deformed W-algebra $W_{q,\hbar}({}^L\widehat{\mathfrak{g}})$

The q-Langlands correspondence

$$\begin{tikzcd} U_\hbar(\widehat{\mathfrak{g}}) \arrow[d] \arrow[r, leftrightarrow] & W_{q,\hbar}({}^L\widehat{\mathfrak{g}}) \arrow[d, "q \to 1"] \\ \text{XXZ Bethe} & (G,\hbar)\text{-oper} \end{tikzcd}$$

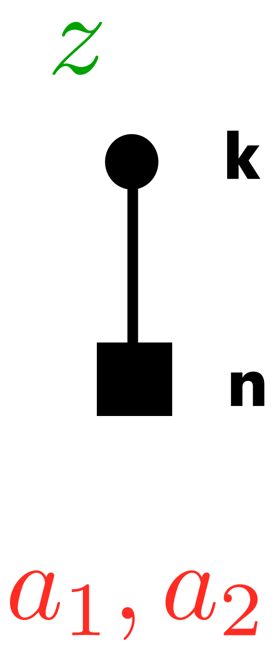
Equivalence of categories

$$\mathrm{Dmod}_\kappa(Bun_G) \longleftrightarrow \mathrm{Dmod}_{-\frac{1}{m\kappa}}(Bun_{{}^LG})$$

Number Theory

[Smirnov Varchenko]

Consider cohomological vertex (J-function)



$$\mathsf{V}(z) = \sum_{d=0}^{\infty} c_d\, z^d \in \mathbb{Q}[[z]] \qquad c_d := \int_{\mathsf{QM}_d(X,\infty)} \omega^{vir}$$

For a prime p construct a sequence of polynomials $T_s(z) \in \mathbb{Z}[z]$ from the superpotnential which converges to the vertex in the **p-adic** norm

$$\lim_{s\to\infty} \mathsf{T}_s(z) = \mathsf{V}(z) \qquad \mathsf{T}_s(z) = \mathrm{coeff}_{x^{dp^s-1}}\Big(\Phi_s(x,z)\Big)$$

Some properties

$$\mathsf{V}(z) = \prod_{i=0}^{\infty} \frac{\mathsf{T}_m(z^{p^i})}{\mathsf{T}_{m-1}(z^{p^{i+1}})} \mod p^m, \quad m = 1, 2 \ldots$$

Dwork identity

$$\frac{\mathsf{T}_{s+1}(z)}{\mathsf{T}_s(z^p)} = \frac{\mathsf{T}_s(z)}{\mathsf{T}_{s-1}(z^p)} \mod p^s$$