Hello and welcome to class!

Last time

We discussed when a matrix has a basis of real eigenvectors.

This time

We talk a bit about the complex numbers and discuss their uses in finding eigenvalues and eigenvectors. Then we'll move on to the next chapter, about orthogonality.

The complex numbers

The negative real numbers have no square-roots.

We just invent them: introduce a symbol i, whose square is -1.

Now all real numbers have square-roots: $\sqrt{-7} = i\sqrt{7}$.

The complex numbers

Definition

The complex numbers are the collection of expressions $a + bi$, where a, b are real numbers, and i squares to -1.

We write $\mathbb{C}$ for the set of these numbers.

They add and multiply just like you think they do:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i$$

The complex numbers

A useful fact: observe

$$(a + bi)(a - bi) = a^2 + b^2$$

This is nonzero so long as $a + bi \neq 0$.

Thus any nonzero complex number has an inverse:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2} i$$

Is that really ok?

For a long time, even many mathematicians didn't think so.

The following may reassure you.

At some point, you learned to count. Then, addition, multiplication, subtraction, division.

But, in terms of counting numbers, the questions "what is $1 - 2$" and "what is $1/2$" didn't have answers. So we just invented some.

Now you have fractions and negative numbers, but still questions like "what number squares to two" or "what is the ratio of the circumference of the circle to its diameter" do not have answers. So we just invented some.

The complex numbers are the next step in the progression.

The fundamental theorem of algebra

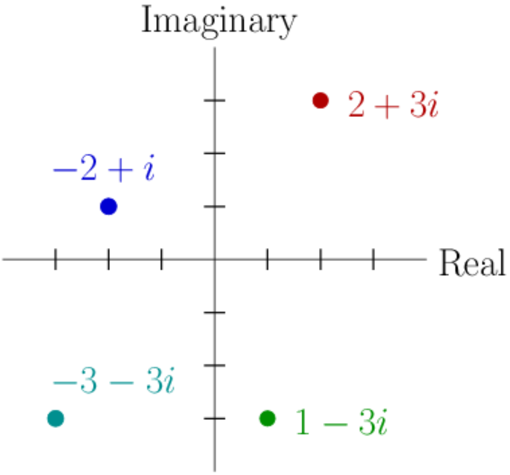
Theorem

Any polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$, possibly with complex coefficients, factors into linear factors over the complex numbers. That is, there exist complex numbers $r_1, \ldots, r_n$ such that

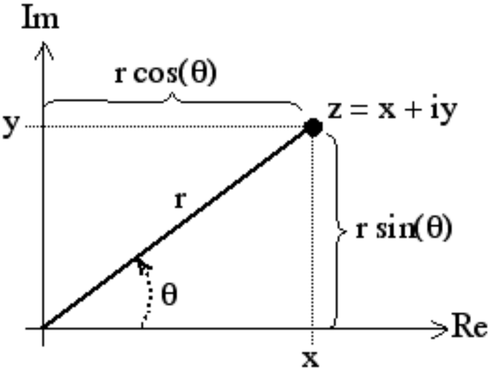
$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = (x - r_1)(x - r_2)\cdots(x - r_n)$$

For example, $x^2 + 1 = (x + i)(x - i)$.

The complex plane



Polar coordinates



Euler's identity

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

So we can write any complex number $x + iy$ first via polar coordinates as $r\cos\theta + ir\sin\theta$ and then as $re^{i\theta}$.

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}(y/x)$$

$$e^{i\pi} + 1 = 0$$

The geometric meaning of complex multiplication

Multiplying by a complex number $a + bi$ is a linear transformation on the 2-dimensional real vector space underlying the complex numbers (the complex plane)

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ bc + ad \end{bmatrix}$$

In polar form: $re^{i\theta}$ acts by the matrix

$$r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

So it scales by r and rotates by θ.

Back to linear algebra

We developed linear algebra with real coefficients.

But everything we did since the beginning of class actually makes sense with complex coefficients as well.

Good review exercise: go back through and check this! Hint: it will be important that nonzero complex numbers have inverses.

Why is this useful?

We saw that an $n \times n$ matrix can be diagonalized when its characteristic polynomial has n distinct real roots.

If we're willing to use complex numbers, then we can diagonalize a matrix whenever its characteristic polynomial has n distinct complex roots. (By the same argument.)

This is a lot more likely:

the fundamental theorem of algebra tells us that every polynomial factors into linear factors over the complex numbers.

Rotation

Consider the matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Its characteristic polynomial is

$$(\cos(\theta) - \lambda)^2 + \sin(\theta)^2 = \lambda^2 - 2\cos(\theta) + 1$$

This has roots:

$$\lambda = \frac{2\cos(\theta) \pm \sqrt{4\cos(\theta)^2 - 4}}{2} = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta}$$

So: no real eigenvalues — geometrically: rotation preserves no line — but it can still be diagonalized over $\mathbb{C}$.

Length

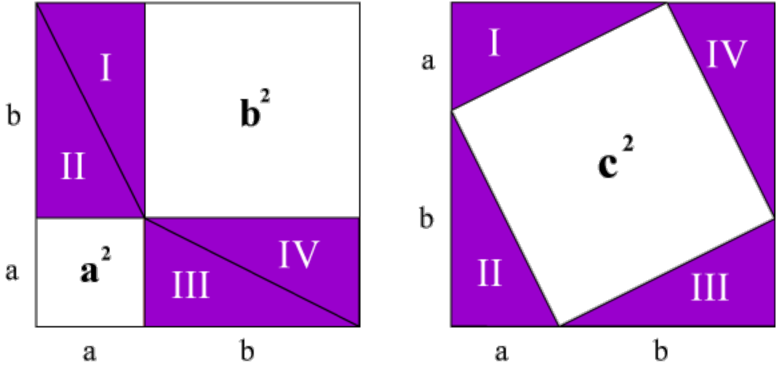
In the real world, we are quite interested in distance; length.

In our abstract world of vector spaces, we need to define the corresponding notion.

For $\mathbb{R}^1$ this is easy: we just use the absolute value.

For $\mathbb{R}^n$, we are guided by the Pythagorean theorem.

The Pythagorean theorem



$$a^2 + b^2 = c^2$$

Length

Definition

The length of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Note that for a positive scalar λ, we have $||\lambda\mathbf{v}|| = \lambda||\mathbf{v}||$.

Example

The vector $(5, 3, 1, 1)$ has length

$$\sqrt{5^2 + 3^2 + 1^2 + 1^2} = \sqrt{25 + 9 + 1 + 1} = \sqrt{36} = 6$$

Try it yourself!

Find the lengths:

$$||(0,0,0,0)|| = \sqrt{0^2+0^2+0^2+0^2} = 0$$

$$||(1,1)|| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$||(-1,2,-3,4)|| =$$

$$\sqrt{(-1)^2+2^2+(-3)^2+4^2} = \sqrt{1+4+9+16} = \sqrt{30}$$

Unit vectors

Given a vector $\mathbf{v} \in \mathbb{R}^n$, there is a unique vector of length 1 pointing in the same direction: $\frac{\mathbf{v}}{||\mathbf{v}||}$. Indeed, since $||\mathbf{v}||$ is a positive scalar:

$$\left|\left|\frac{\mathbf{v}}{||\mathbf{v}||}\right|\right| = \frac{1}{||\mathbf{v}||}||\mathbf{v}|| = 1$$

We call this the unit vector in the direction of $\mathbf{v}$.

Example

The vector $(5, 3, 1, 1)$ has length

$$\sqrt{5^2 + 3^2 + 1^2 + 1^2} = \sqrt{25 + 9 + 1 + 1} = \sqrt{36} = 6$$

The unit vector in the same direction is $(\frac{5}{6}, \frac{3}{6}, \frac{1}{6}, \frac{1}{6})$.

Try it yourself!

Find a unit vector in the given direction:

$(3, 4)$ The length is $\sqrt{3^2 + 4^2} = 5$. So the unit vector is $(3/5, 4/5)$.

$(3, 4, 5)$ The length is

$$||(3,4,5)|| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$$

So the unit vector in the same direction is $\frac{1}{5\sqrt{2}}(3, 4, 5)$.

Distance

A notion of length of vectors determines a notion of distance in $\mathbb{R}^n$:

The distance between $\mathbf{a}$ and $\mathbf{b}$ is $||\mathbf{b} - \mathbf{a}||$.

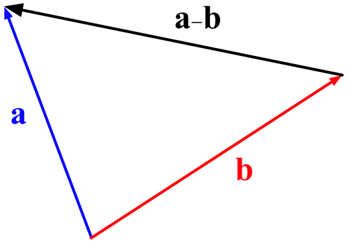
Example

The distance between $(1, 2, 3)$ and $(3, 2, 1)$ is

$$||(1, 2, 3) - (3, 2, 1)|| = ||(-2, 0, 2)|| = \sqrt{(-2)^2 + 0^2 + 2^2} = 2\sqrt{2}$$

Orthogonality

Recall: for a triangle with sides of lengths a, b, c, we have the relation $a^2 + b^2 = c^2$ if and only if the angle between the sides of lengths a and b is a right angle.



So in $\mathbb{R}^2$, the vectors **a** and **b** are at a right angle if and only if

$$||\mathbf{a}||^2 + ||\mathbf{b}||^2 = ||\mathbf{a} - \mathbf{b}||^2$$

Orthogonality

In higher dimensions, we turn this fact into a...

Definition

In $\mathbb{R}^n$, the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if

$$||\mathbf{a}||^2 + ||\mathbf{b}||^2 = ||\mathbf{a} - \mathbf{b}||^2$$

Orthogonality

Expanding these out, for $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$:

$$||\mathbf{a}||^2 + ||\mathbf{b}||^2 = \sum_i a_i^2 + b_i^2$$

$$||\mathbf{a} - \mathbf{b}||^2 = \sum_i a_i^2 + b_i^2 - 2a_i b_i$$

We find

$$||\mathbf{a}||^2 + ||\mathbf{b}||^2 - ||\mathbf{a} - \mathbf{b}||^2 = 2\sum_i a_i b_i$$

The dot product

So $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\sum_i a_i b_i = 0$.

Definition

For vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, we write

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n$$

This is called the dot product.

The vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Dot product properties

Observe:

$$\mathbf{a} \cdot \mathbf{a} = (a_1, \ldots, a_n) \cdot (a_1, \ldots, a_n) = a_1^2 + \cdots + a_n^2 = ||\mathbf{a}||^2$$

$$||\mathbf{a}+\mathbf{b}|| = \sum_i (a_i + b_i)^2 = \sum_i a_i^2 + 2a_i b_i + b_i^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 + 2(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{a} \cdot (c\mathbf{v} + d\mathbf{w}) = c(\mathbf{a} \cdot \mathbf{v}) + d(\mathbf{a} \cdot \mathbf{w})$$

Try it yourself!

Compute the dot products:

$(1, 2) \cdot (3, 4) = 1 \cdot 3 + 2 \cdot 4 = 11$

$(2, -1, 3) \cdot (1, 2, 4) = 2 \cdot 1 + -1 \cdot 2 + 3 \cdot 4 = 12$

Try it yourself!

Are they orthogonal?

$(1,1,0),(0,0,1)$? The dot product is

$$(1,1,0)\cdot(0,0,1) = 1\cdot 0 + 1\cdot 0 + 0\cdot 1 = 0$$

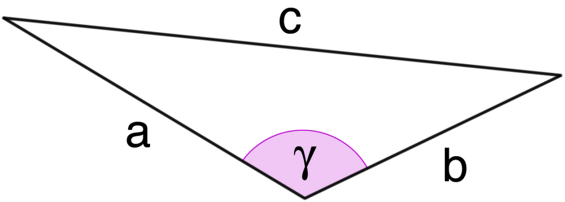
so yes.

$(3,1,4),(-2,1,1)$? The dot product is

$$3\cdot(-2) + 1\cdot 1 + 4\cdot 1 = -1$$

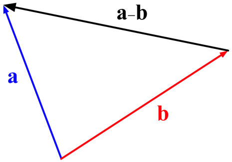
so no.

The law of cosines



$$c^2 = a^2 + b^2 - 2ab(\cos(\gamma))$$

Angles



Thus if θ is the angle between **a** and **b**,

$$||\mathbf{b} - \mathbf{a}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2||\mathbf{a}||||\mathbf{b}|| \cos(\theta)$$

On the other hand, we saw

$$||\mathbf{b} - \mathbf{a}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2(\mathbf{a} \cdot \mathbf{b})$$

Thus

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos(\theta)$$

Angles

Example

The angle θ between the vectors $(1,2)$ and $(3,4)$ can be determined by

$$(1,2)\cdot(3,4) = ||(1,2)||\,||(3,4)||\cos(\theta)$$

We compute

$$(1,2)\cdot(3,4) = 11 \qquad ||(1,2)|| = \sqrt{5} \qquad ||(3,4)|| = 5$$

$$\theta = \cos^{-1}\left(\frac{11}{5\sqrt{5}}\right)$$