Hello and welcome to class!

Last time

We began discussing eigenvalues and eigenvectors.

This time

We'll see more examples, and study the question of when there is a basis of eigenvectors.

Review: Eigenvectors and eigenvalues

If $T : V \to V$ is a linear transformation — perhaps $V = \mathbb{R}^n$ and T is given by a matrix — and

$$T\mathbf{v} = \lambda\mathbf{v}$$

then $\mathbf{v}$ is said to be an eigenvector of T with eigenvalue λ.

The eigenvalues can be determined by solving the characteristic equation $det(T - \lambda I) = 0$.

The eigenvectors for a given eigenvalue λ can be determined by finding the kernel of $T - \lambda I$, by row reduction.

Review: diagonalization

If $\mathcal{B}$ is a basis for V consisting of eigenvectors of $T : V \to V$, then the matrix $[T]_{\mathcal{B}}$ is diagonal.

If $V = \mathbb{R}^n$, $[T]$ is the matrix of T in the standard basis, and

$$B = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$$

is the matrix whose columns are the basis vectors then

$$[T]_{\mathcal{B}} = B^{-1}[T]B$$

Review: diagonalization

A square matrix M can be written as

$$M = X \cdot (\text{diagonal matrix}) \cdot X^{-1}$$

exactly when there's a basis of eigenvectors $\mathcal{B}$ for M in which case we take

$$X = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$$

This is good for computing the powers of M:

$$M^n = X \cdot (\text{diagonal matrix})^n \cdot X^{-1}$$

Basis independence

Note that the notions of eigenvector and eigenvalue

$$T\mathbf{v} = \lambda \mathbf{v}$$

depend only on the linear transformation T and not on the basis you write T in. Of course, the way that you write down the eigenvectors does depend on the basis.

The characteristic polynomial also does not depend on the basis. Indeed, changing basis always amounts to a transformation $M \mapsto B^{-1}MB$, and

$$\det(M - \lambda I) = \det(B^{-1}(M - \lambda I)B) = \det(B^{-1}MB - \lambda)$$

Example

Consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Its characteristic equation is: $0 = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1$.

So its eigenvalues are $1, -1$.

Eigenvectors of eigenvalue 1 form the kernel of $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. It's spanned by $(1, 1)$.

Eigenvectors of eigenvalue -1 form the kernel of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It's spanned by $(1, -1)$. There is a basis of real eigenvectors.

Example

Consider the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Its characteristic equation is: $0 = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} = 1 - \lambda^3$

This has one real solution: $\lambda = 1$.

Eigenvectors of eigenvalue 1 form the kernel of $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

This is spanned by $(1, 1, 1)$.

There is no basis of real eigenvectors.

Example

Consider the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Its characteristic equation is: $0 = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$. This has no real solutions, so there are no real eigenvalues or nonzero eigenvectors.

There is no basis of real eigenvectors.

Example

Consider the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Characteristic equation: $0 = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1$.

So the eigenvalues are $\lambda = 1$

The eigenvectors of eigenvalue 1 are the kernel of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This is spanned by $(1, 0)$.

There is no basis of real eigenvectors.

Example

Consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Characteristic equation: $0 = \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1$.

So the eigenvalues are $\lambda = 1$

The eigenvectors of eigenvalue 1 are the kernel of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This is all vectors in $\mathbb{R}^2$.

There is a basis of real eigenvectors.

Example

Consider the linear transformation $\frac{d}{dx} : \mathbb{P}_n \to \mathbb{P}_n$.

Constant polynomials are eigenvectors of eigenvalue zero. There are no others: the derivative lowers the degree of a polynomial so any eigenvector must have eigenvalue zero hence be a constant.

Example

Or, we pick a basis, say $1, x, x^2, \ldots$ and write the matrix of $\frac{d}{dx}$.

I'll write the case $\mathbb{P}_4$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic equation of this matrix is $-\lambda^5 = 0$. So 0 is the only eigenvalue.

The corresponding eigenspace is in this basis spanned by $(1, 0, 0, 0, 0)$. These are the constant polynomials.

When is there a basis of eigenvectors?

Theorem

Let $T : V \to V$ be a linear transformation, and suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero eigenvectors,

$$T\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

If the eigenvalues λ_i are distinct, then the eigenvectors $\mathbf{v}_i$ are linearly independent.

When is there a basis of eigenvectors?

Proof.

We check $n = 2$. Assume $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent, i.e. $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$ where c_1, c_2 are constants not both zero. Applying T, we have $\lambda_1 c_1 \mathbf{v}_1 + \lambda_2 c_2 \mathbf{v}_2 = 0$ as well.

If $c_1 = 0$, then we can conclude $\mathbf{v}_2 = 0$, which is a contradiction since we assumed the $\mathbf{v}_i \neq 0$. So $c_1 \neq 0$.

If $\lambda_2 = 0$, then $\lambda_1 c_1 \mathbf{v}_1 = 0$ but $\lambda_1 \neq 0$ since $\lambda_1 \neq \lambda_2$. Then since $c_1 \neq 0$, we have $\mathbf{v}_1 = 0$. This is a contradiction.

Otherwise, subtract $\frac{1}{\lambda_2}$ times the second equation from the first. We find $c_1(1 - \lambda_1/\lambda_2)\mathbf{v}_1 = 0$. Since $\lambda_1 \neq \lambda_2$, this implies $\mathbf{v}_1 = 0$, which is a contradiction. □

When is there a basis of eigenvectors?

More generally, consider a minimal dependent subset of the $\mathbf{v}_i$. There must be at least two, since each $\mathbf{v}_i$ was nonzero. Then $0 = \sum c_i \mathbf{v}_i$ where all the $c_i \neq 0$ (by minimality). For notational convenience assume $\mathbf{v}_1$ is among them.

Applying T, we also have $0 = \sum c_i \lambda_i \mathbf{v}_i$. By minimality, none of the λ_i can be zero. But then we subtract λ_1 times the first equation from the second, getting

$$0 = \sum_{i>1} c_i(\lambda_i - \lambda_1)\mathbf{v}_i$$

This is a nontrivial — since $\lambda_i \neq \lambda_1$ — linear combination of fewer of the basis vectors. This contradicts minimality.

When is there a basis of eigenvectors?

Theorem

Let $T : V \to V$ be a linear transformation, and suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ are nonzero eigenvectors,

$$T\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

If the eigenvalues λ_i are distinct, then the eigenvectors $\mathbf{v}_i$ are linearly independent.

Corollary

If the characteristic equation of an $n \times n$ matrix has n distinct real solutions, then there is a basis of real eigenvectors.

When is there a basis of eigenvectors?

This is not a necessary condition: the identity matrix has characteristic equation $(1 - \lambda)^n$, which has only one real solution but any basis is a basis of eigenvectors for the identity matrix.

If $T : V \to V$ is a linear transformation and λ is an eigenvalue, we say that the kernel of $T - \lambda I$, i.e., the space spanned by the eigenvectors of eigenvalue λ, is the eigenspace of eigenvalue λ.

Example

The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

has eigenvalues $1, 2$.

The eigenspace of eigenvalue 1 is spanned by $\mathbf{e}_1$.

The eigenspace of eigenvalue 2 is spanned by $\mathbf{e}_2, \mathbf{e}_3$.

When is there a basis of eigenvectors?

One can show (although I won't here):

Theorem

Let $T : V \to V$ be a linear transformation. The following are equivalent.

- *There's a basis for V consisting of eigenvectors for T.*
- *The eigenspaces of T span V.*
- *The eigenspace of eigenvalue λ has dimension equal to the multiplicity of λ as a root of the characteristic polynomial.*

This is harder to check than the condition that the characteristic equation has distinct solutions, since you have to actually determine the eigenspaces (or at least their dimensions).