Hello and welcome to class!

Last time

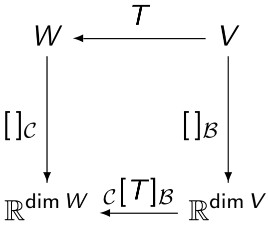
We discussed change of basis.

This time

We will introduce the notions of eigenvalues and eigenvectors.
These some of the most powerful ideas you will see in this class.

Review: the matrix of a linear transformation

If $\mathcal{B}$ is a basis in V and $\mathcal{C}$ is a basis in W, a linear transformation $T : V \to W$ is written in coordinates by the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ which completes the square

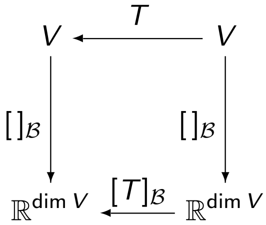


For a new choice of basis $\mathcal{B}'$ of V and $\mathcal{C}'$ of W, we have

$$ {}_{\mathcal{C}'}[T]_{\mathcal{B}'} = \underset{\mathcal{C}'\leftarrow\mathcal{C}}{P}\ {}_{\mathcal{C}}[T]_{\mathcal{B}}\ \underset{\mathcal{B}\leftarrow\mathcal{B}'}{P} $$

Review: the matrix of a linear transformation

In the special case when $V = W$ and $\mathcal{B} = \mathcal{C}$, we write just $[T]_{\mathcal{B}}$.

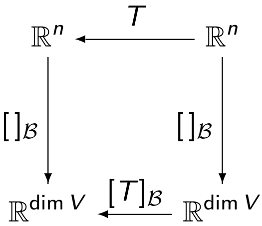


For a new choice of basis $\mathcal{B}'$ of V, we have

$$[T]_{\mathcal{B}'} = \underset{\mathcal{B}'\leftarrow\mathcal{B}}{P}\,[T]_{\mathcal{B}}\,\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}\right)^{-1}[T]_{\mathcal{B}}\,\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}$$

Review: change of basis and conjugation

Finally, if $V = \mathbb{R}^n$,



we recall that $[\mathbf{v}]_{\mathcal{B}} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]^{-1} \cdot \mathbf{v}$ so

$$[T]_{\mathcal{B}} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]^{-1} \cdot T \cdot [\mathbf{b}_1, \ldots, \mathbf{b}_n]$$

In examplestan

No-one knows how to make pants the correct size, so all the people have to wear suspenders or belts.

Every year, 1 % of belt-wearers decide they'd prefer suspenders, and 2 % of suspender-wearers decide they'd rather have a belt.

Today, belts and suspenders are about equally popular. What will people be wearing next year? In five years? In 100 years?

In examplestan

Symbolically:

$$\begin{bmatrix} \text{belts in year } n+1 \\ \text{suspenders in year } n+1 \end{bmatrix} = \begin{bmatrix} .99 & .02 \\ .01 & .98 \end{bmatrix} \begin{bmatrix} \text{belts in year } n \\ \text{suspenders in year } n \end{bmatrix}$$

Iterating this process:

$$\begin{bmatrix} \text{belts after } n \text{ years} \\ \text{suspenders after } n \text{ years} \end{bmatrix} = \begin{bmatrix} .99 & .02 \\ .01 & .98 \end{bmatrix}^n \begin{bmatrix} \text{belts now} \\ \text{suspenders now} \end{bmatrix}$$

How can we compute $\begin{bmatrix} .99 & .02 \\ .01 & .98 \end{bmatrix}^n$?

Discrete dynamical systems

There are many processes whose:

- Possible states are elements of a vector space V
- State at time $n = 0, 1, 2, 3, \cdots$ is written $\mathbf{v}(n)$
- Time evolution is

$$\mathbf{v}(n+1) = A \cdot \mathbf{v}(n)$$

For some linear transformation $A : V \to V$

Such systems are called (time independent) linear discrete dynamical systems. We just saw one.

Discrete dynamical systems

For the linear discrete dynamical system

$$\mathbf{v}(n+1) = A \cdot \mathbf{v}(n)$$

The state at time n is

$$\mathbf{v}(n) = A^n \mathbf{v}(0)$$

So to understand the behavior of such a system is to understand how to take powers of a linear transformation (or matrix).

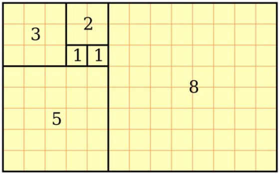
The Fibonacci numbers

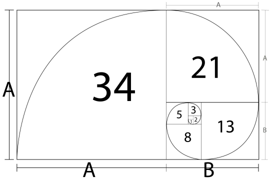
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Each is the sum of the previous two:

$$F_{n+2} = F_{n+1} + F_n$$

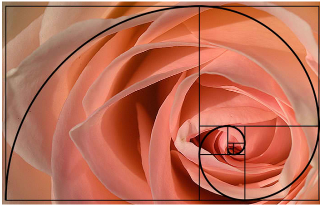
Squares of these side-lengths fit together nicely in a spiral

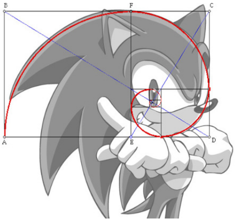


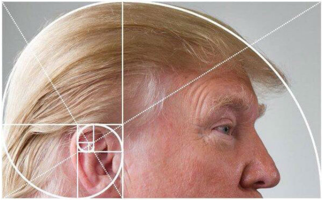


The Fibonacci numbers

This spiral can be seen in nature...







The Fibonacci numbers

The recursion $F_{n+2} = F_{n+1} + F_n$ can be described by a matrix

$$\left[\begin{array}{c} F_{n+2} \\ F_{n+1} \end{array} \right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} F_{n+1} \\ F_n \end{array} \right]$$

Since $F_1 = 1$ and $F_0 = 0$,

$$\left[\begin{array}{c} F_{n+1} \\ F_n \end{array} \right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right]^n \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$$

The Fibonacci numbers

Consider a population of creatures. Every month, each creature older than one month reproduces, creating one new creature.

How does the population grow?

$$\begin{bmatrix} \text{pop. at } n+1 \\ \geq \text{one month at } n+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \text{pop. at } n \\ \geq \text{one month at } n \end{bmatrix}$$

This was why Fibonacci introduced his numbers. The appearance of them in nature is sometimes explained by the above mechanism.

Powers of matrices

To understand a linear discrete dynamical system given by

$$A : V \to V$$

we should compute A^n.

If $A : \mathbb{R}^n \to \mathbb{R}^n$ is given by a diagonal matrix, this is easy:

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}^n = \begin{bmatrix} a_1^n & 0 & 0 \\ 0 & a_2^n & 0 \\ 0 & 0 & a_3^n \end{bmatrix}$$

Diagonal matrices

A matrix A is diagonal if and only if:

For each $\mathbf{e}_i$, there is a scalar λ_i so that

$$A \cdot \mathbf{e}_i = \lambda_i \cdot \mathbf{e}_i$$

E.g. when $n = 3$, this would mean

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Diagonal matrices

It's almost as good for A to be diagonal *in some basis*

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$$

Since in this case, we can change basis to $\mathcal{B}$, compute powers of the diagonal matrix $[A]_{\mathcal{B}}$, and then change back.

Note that A is diagonal in the basis $\mathcal{B}$ exactly when

$$A \cdot \mathbf{b}_i = \lambda_i \mathbf{b}_i$$

Powers in other bases

If $A : \mathbb{R}^n \to \mathbb{R}^n$ is given by a matrix (also called A),

If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis, set $B = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$, so that

$$[\mathbf{v}]_{\mathcal{B}} = B^{-1} \cdot \mathbf{v}$$

$$[A]_{\mathcal{B}} = B^{-1} A B$$

hence

$$A = B [A]_{\mathcal{B}} B^{-1}$$

Powers in other bases

Since

$$A = B[A]_{\mathcal{B}} B^{-1}$$

We can compute

$$A^2 = B[A]_{\mathcal{B}} B^{-1} B[A]_{\mathcal{B}} B^{-1} = B[A]_{\mathcal{B}} [A]_{\mathcal{B}} B^{-1} = B[A]_{\mathcal{B}}^2 B^{-1}$$

More generally,

$$A^n = B[A]_{\mathcal{B}}^n B^{-1}$$

So if we can find a basis $\mathcal{B}$ in which $[A]_{\mathcal{B}}$ is diagonal, we can compute $[A]_{\mathcal{B}}^n$, hence A^n.

Eigenvalues and eigenvectors

As we saw, A is diagonal in the basis $\mathcal{B}$ exactly when

$$A \cdot \mathbf{b}_i = \lambda_i \mathbf{b}_i$$

Any vector $\mathbf{b}$ with $A \cdot \mathbf{b} = \lambda \mathbf{b}$ is called an eigenvector of A.

In this case λ is called an eigenvalue of A.

Eigenvalues and eigenvectors

Example

Consider the identity matrix I. Every vector is an eigenvector, since $I \cdot v = v$ They all have eigenvalue 1.

Eigenvalues and eigenvectors

Example

Consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Since $A \cdot \mathbf{e}_1 = 2\mathbf{e}_1$ and $A \cdot \mathbf{e}_2 = 3\mathbf{e}_2$, the vectors $\mathbf{e}_1, \mathbf{e}_2$ are eigenvectors.

The vectors $a\mathbf{e}_1$ and $b\mathbf{e}_2$ are also eigenvectors, for any scalars a, b.

Are there any other eigenvectors?

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 3b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

This can only happen if $a = 0$ or $b = 0$.

Thus the only eigenvectors are $a\mathbf{e}_1$ and $b\mathbf{e}_2$. The only eigenvalues are 2 and 3.

Eigenvalues and eigenvectors

Example

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. Since $A \cdot \mathbf{e}_1 = 2\mathbf{e}_1$, the vector $\mathbf{e}_1$ is an eigenvector. So are its multiples.

Are there any other eigenvectors?

Finding eigenvalues

The equation $A \cdot \mathbf{v} = \lambda \mathbf{v}$ is equivalent to $(A - \lambda I)\mathbf{v} = 0$.

There's a nonzero solution if and only if

$$det(A - \lambda I) = 0$$

So λ is an eigenvalue if and only if it solves $det(A - \lambda I) = 0$.

This is called the characteristic equation.

Try it yourself

Write the characteristic equation $det(A - \lambda I) = 0$ for:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (1 - \lambda)^2 = 0$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad (2 - \lambda)(3 - \lambda) = 0$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad (2 - \lambda)(3 - \lambda) = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \lambda^2 - \lambda - 1 = 0$$

Eigenvalues and eigenvectors

Example

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. Since $A \cdot \mathbf{e}_1 = 2\mathbf{e}_1$, the vector $\mathbf{e}_1$ is an eigenvector. So are its multiples.

Are there any other eigenvectors? Yes: the characteristic equation is $(2 - \lambda)(3 - \lambda) = 0$, so there's an eigenvector of eigenvalue 3.

Finding eigenvectors

To find the eigenvectors of A of eigenvalue λ

means solving the equation $A\mathbf{v} = \lambda\mathbf{v}$

i.e., finding the kernel of $(A - \lambda I)$.

Finding eigenvectors

Example

Let's find eigenvectors of $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ of eigenvalue 3.

That means finding the kernel of $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$.

By inspection, it's the linear subspace spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Checking: $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Try it yourself!

Find the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

The characteristic equation is $\lambda^2 - \lambda - 1$. The eigenvalues are given by the roots of this:

$$\lambda_+ = \frac{1+\sqrt{5}}{2}$$

$$\lambda_- = \frac{1-\sqrt{5}}{2}$$

Try it yourself!

Find the eigenvectors of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

The eigenvalues are $\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}$. We want to find the kernel of

$$\begin{bmatrix} 1 - \frac{1 \pm \sqrt{5}}{2} & 1 \\ 1 & -\frac{1 \pm \sqrt{5}}{2} \end{bmatrix}$$

By inspection, the kernel is spanned by $\begin{bmatrix} \frac{1 \pm \sqrt{5}}{2} \\ 1 \end{bmatrix}$.

$\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$, (and their multiples) are the eigenvectors.

Back to Fibonacci

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Back to Fibonacci

$\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$ are eigenvectors for $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

with eigenvalues $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

In other words, in the basis $\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$, the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ becomes diagonal with entries $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Back to Fibonacci

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$