Painlevé-Calogero correspondence: The elliptic 8-coupling level

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Outline

1. Introduction

- Recall there are 6 Painlevé ODEs, the most general one being P(VI). The latter comes in several guises, but in the 'elliptic' setting of this seminar there is a preferred one. It reads

$$\partial_\tau^2 \lambda = \sum_{s=0}^{3} c_s \, \partial_\lambda \wp(\lambda + \omega_s; 1/2, \tau/2),$$

  where $\wp$ is the Weierstrass function, $\Im(\tau) > 0$, and

$$\omega_0 = 0, \ \ \omega_1 = 1/2, \ \ \omega_2 = \tau/2, \ \ \omega_3 = 1/2 + \tau/2,$$

  with 1 and τ being the primitive $\wp$-periods.

- It admits an isomonodromy interpretation, which can also be formulated in terms of a zero-curvature Lax pair.

- The pertinent linear Lax ODE is of the Schrödinger form

$$(H(\lambda; x) - E)\psi(x) = 0,$$

with $H(\lambda; x)$ given by

$$-\frac{d^2}{dx^2} + \frac{\partial_x \wp(x)}{\wp(x) - \wp(\lambda)}\frac{d}{dx} + \frac{C}{\wp(x) - \wp(\lambda)} + \sum_{s=0}^{3} g_s(g_s - 1)\wp(x + \omega_s).$$

- The limit $\lambda \to 0$ yields the BC_1 Calogero-Moser Hamiltonian

$$H(0; x) = -\frac{d^2}{dx^2} + \sum_{s=0}^{3} g_s(g_s - 1)\wp(x + \omega_s).$$

- This Hamiltonian is also known as the 1-particle specialization of the N-particle Inozemtsev integrable system. Furthermore, its time-independent Schrödinger equation is the Heun equation in elliptic form.

- Van Diejen [3] introduced an 8-coupling 'relativistic' elliptic Hamiltonian whose associated Schrödinger equation can be viewed as a difference equation counterpart of the 4-coupling Heun ODE (in the above elliptic form). This Hamiltonian has a great many specializations and confluence limits.
- Sakai [2] introduced an 8-parameter elliptic difference equation, which can be viewed as the most general equation in an extensive hierarchy of (generalized) Painlevé equations.
- In the recent joint paper Ref. [1] with Noumi and Yamada, we have shown that one of the two linear Lax q-difference equations associated with the nonlinear Sakai equation can be specialized to the Schrödinger equation for van Diejen's BC_1 Hamiltonian.
- In previous work [4], we proved that the spectrum of the latter has $W(E_8)$-invariance. Sakai's equation has a $W(E_8^{(1)})$-symmetry.
- In the sequel we sketch the findings of Ref. [1].

2. Van Diejen's 8-coupling elliptic Hamiltonian

- We start from the elliptic gamma function $G(r, a_+, a_-; x)$ given by

$$G(x) \equiv \prod_{m,n=0}^{\infty} \frac{1 - \exp\big(-(2m+1)ra_+ - (2n+1)ra_- - 2irx \big)}{1 - \exp\big(-(2m+1)ra_+ - (2n+1)ra_- + 2irx \big)},$$

with $r, \Re(a_\delta) > 0$, $\delta = +, -$. Obviously, G is a π/r-periodic meromorphic function, which satisfies $G(-x) = 1/G(x)$ and is modular invariant (symmetric under swapping a_+ and a_-).

- The building block for van Diejen's Hamiltonian is one of the two right-hand side functions arising from the analytic difference equations (henceforth AΔEs)

$$\frac{G(x + ia_\delta/2)}{G(x - ia_\delta/2)} = R_{-\delta}(x), \quad \delta = +, -,$$

namely,

$$R_+(x) = \prod_{m=0}^{\infty} [1 - \exp(2irx - (2m+1)ra_+)][x \to -x].$$

- Clearly R_+ is an entire, even, π/r-periodic function without real zeros; moreover, it satisfies the AΔE

$$R_+(x + ia_+/2) = -\exp(-2irx)R_+(x - ia_+/2).$$

(In essence, R_+ is one of Jacobi's theta functions.)

- Van Diejen's operator is of the form

$$A_+(\gamma; x) = V(\gamma; x)\exp(-ia_- d/dx) + (x \to -x) + V_b(\gamma; x),$$

where

$$V(\gamma; x) \equiv \frac{\prod_{\mu=0}^{7} R_+(x - i\gamma_\mu - ia_-/2)}{R_+(2x + ia_+/2)R_+(2x + ia_+/2 - ia_-)},$$

and $V_b(\gamma; x)$ is an even elliptic function with periods $\pi/r, ia_+$.

- As such, V_b is uniquely determined up to a constant by specifying the residues at its (generically) simple poles, which occur at the zeros of the factors $R_+(\pm 2x + ia_+/2 - ia_-)$ of the shift coefficients $V(\gamma; \pm x)$.

- Hence, the poles occur at $x \equiv \pm x_n$, $n = 0, 1, 2, 3$ (modulo the elliptic lattice Λ), where

$$\{x_0, x_1, x_2, x_3\} := -ia_-/2 + \{0, \pi/2r, ia_+/2, ia_+/2 + \pi/2r\}.$$

- The residues at these poles are given by

$$\mathrm{Res}(x_0) = \eta \prod_{\mu} R_+(i\gamma_\mu),$$

$$\mathrm{Res}(x_1) = \eta \prod_{\mu} R_+(i\gamma_\mu + \pi/2r),$$

$$\mathrm{Res}(x_2) = \eta \exp\Big(- 2ra_+ - r\sum_{\mu} \gamma_\mu\Big) \prod_{\mu} R_+(i\gamma_\mu + ia_+/2),$$

$$\mathrm{Res}(x_3) = \eta \exp\Big(- 2ra_+ - r\sum_{\mu} \gamma_\mu\Big) \prod_{\mu} R_+(i\gamma_\mu + ia_+/2 + \pi/2r),$$

with

$$\eta := 1 \Big/ 4irR_+(ia_- + ia_+/2) \prod_{k=1}^{\infty} (1 - \exp(-2kra_+))^2.$$

- These residues are invariant under $W(D_8)$ (permutations and even sign flips), and so is the (unspecified) additive constant.
- Using a gauge transformation involving a product of G-functions, the operator $A_+(\gamma; x)$ can be transformed to

$$\mathcal{A}_+(\gamma; x) = e^{-ia_- d/dx} + V(\gamma; x + ia_-)V(\gamma; -x)e^{ia_- d/dx} + V_b(\gamma; x).$$

  This operator is $W(D_8)$-invariant; for $a_+, a_- > 0$ it is also (formally) self-adjoint.
- Likewise, we can change the sign of γ_μ in $V(\gamma; x)$ by using a gauge factor

$$G_\mu(x) \equiv G(x + i\gamma_\mu)/G(x - i\gamma_\mu).$$

  In particular, we can use $G_0 G_1 G_2 G_3$ to transform the shift coefficient

$$\tilde{V}(\gamma; x) \equiv \frac{\prod_{\mu=0}^{3} R_+(x + i\gamma_\mu - ia_-/2) \prod_{\mu=4}^{7} R_+(x - i\gamma_\mu - ia_-/2)}{R_+(2x + ia_+/2)R_+(2x + ia_+/2 - ia_-)},$$

  (needed below) back to $V(\gamma; x)$.

3. The 8-parameter elliptic Painlevé Lax equation

- Nowadays, another version of the elliptic gamma function is more widely used, namely

$$\Gamma_{p,q}(z) \equiv \prod_{m,n=0}^{\infty} \frac{1 - z^{-1}p^{m+1}q^{n+1}}{1 - zp^m q^n}.$$

- It leads to the q-difference equation

$$\Gamma_{p,q}(qz) = [z]\Gamma_{p,q}(z),$$

with

$$[z] = \prod_{m=0}^{\infty}(1 - zp^m)(1 - z^{-1}p^{m+1})$$

yielding the building block for the Lax equation associated to Sakai's elliptic Painlevé equation.

- The relation to $G(x)$ is given by the substitutions

$$p = \exp(-2ra_+), \;\; q = \exp(-2ra_-), \;\; z = \exp(2irx - ra_+ - ra_-).$$

- Moreover, these substitutions yield the connection

$$[z] = R_+((\ln z)/2ir - ia_+/2)$$

between the two building blocks.

- Our aim is now to compare the time-independent Schrödinger ('relativistic Heun') equation

$$(A_+(\gamma; x) - E)\psi(x) = 0,$$

to the Lax q-difference equation, which is of the form

$$W_-(z)y(z/q) + W_+(z)y(qz) - R(z)y(z) = 0.$$

- Clearly, we need the correspondence

$$\psi(x) = y(2irx - ra_+ - ra_-),$$

so we are reduced to comparing the coefficients of the two equations. We proceed to discuss the Lax equation coefficients.

- The shift coefficients are given by

$$W_-(z) \equiv A(k/z)B(z)\mathcal{F}(qz)[k/q^2z^2],$$
$$W_+(z) \equiv A(qz)B(k/qz)\mathcal{F}(z)[k/z^2],$$

with

$$A(z) \equiv \prod_{j=1}^{4}[z/a_j], \quad B(z) \equiv \prod_{j=1}^{4}[z/b_j], \quad \mathcal{F}(z) \equiv Cz[z/\lambda][k/z\lambda].$$

Here, C can be viewed as a gauge parameter and k arises from the extended affine E_8 Weyl group picture associated with the elliptic Sakai equation. Also, λ encodes one of the two initial values for this equation.
- In the additive coefficient $R(z)$, the dependence on the 8 parameters a_j, b_j, is given by a factor

$$U(z) \equiv A(z)B(z).$$

It is expedient to postpone the remaining details for $R(z)$.

4. Comparing the shift parts

- The first order of business is to connect the 8 parameters γ_μ and a_j, b_j. We have gauge freedom in the shift coefficients, but not in the additive parts. In particular, we can change signs of the γ_μ in the former, but not in the latter. Therefore, the optimal choice is

$$a_j = q\exp(-2r\gamma_{j-1}), \quad b_j = q\exp(-2r\gamma_{j+3}), \quad j = 1, 2, 3, 4,$$

since then we get the permutation-invariant function

$$U(z) = \prod_{j=1}^{4}[z/a_j][z/b_j] = \prod_{\mu=0}^{7} R_+(x - i\gamma_\mu - ia_-/2).$$

- For the factors $[k/za_j]$ in $W_-(z)$ and the factors $[k/qzb_j]$ in $W_+(z)$ this choice yields

$$[k/za_j] = R_+(x + i\gamma_{j-1} - (\ln k)/2ir + 3ia_-/2 + ia_+),$$

$$[k/qzb_j] = R_+(x + i\gamma_{j+3} - (\ln k)/2ir + 5ia_-/2 + ia_+).$$

- When we now choose

$$k = pq^2 = \exp(-2ra_+)\exp(-4ra_-),$$

then the 8-parameter factors in $W_\pm(z)$ and $\tilde{V}(\gamma;\pm x)$ become equal. (Recall the shift coefficients $\tilde{V}(\gamma;\pm x)$ arise from $V(\gamma;\pm x)$ by a gauge transformation.)

- Using from now on the convention

$$f(x \pm y) := f(x+y)f(x-y),$$

we also get

$$[k/z^2][k/qz^2][k/q^2z^2] = R_+(2x + ia_+/2 \pm ia_-)R_+(2x + ia_+/2).$$

- The three factors on the rhs occur in the denominators of $\tilde{V}(\gamma;\pm x)$, so the next step is to divide the Lax equation by this product.

- The result is that the shift part of the Lax equation is of the form

$$\mathcal{F}(qz)\tilde{V}(\gamma;x)\psi(x-ia_-)+q\mathcal{F}(z)e^{8irx}\tilde{V}(\gamma;-x)\psi(x+ia_-),$$

with

$$\mathcal{F}(z)=Cz[z/\lambda][pq^2/z\lambda].$$

- When we now set $\lambda = q\nu$ and divide the shift part by

$$Cp^{-1}z^3[z/q\nu][p/z\nu],$$

then we obtain

$$e^{-4irx}\frac{[z/\nu]}{[z/q\nu]}\tilde{V}(\gamma;x)\psi(x-ia_-)+e^{4irx}\frac{[pq/z\nu]}{[p/z\nu]}\tilde{V}(\gamma;-x)\psi(x+ia_-).$$

- Finally, setting $\nu = \exp(-2r\gamma_8)$, we get

$$\frac{[z/\nu]}{[z/q\nu]} = \frac{R_+(x - i\gamma_8 + ia_-/2)}{R_+(x - i\gamma_8 - ia_-/2)}, \quad \frac{[pq/z\nu]}{[p/z\nu]} = \frac{R_+(x + i\gamma_8 - ia_-/2)}{R_+(x + i\gamma_8 + ia_-/2)}.$$

  Thus we need only specialize γ_8 to γ_7 (say) to arrive at coefficients $\tilde{V}(\tilde{\gamma}; \pm x)$, with

$$\tilde{\gamma}_\mu \equiv \gamma_\mu, \quad \mu = 0, \ldots, 6, \quad \tilde{\gamma}_7 \equiv \gamma_7 - a_-.$$

- Up to the plane wave factors, we therefore get the shift part of a (gauge-transformed) van Diejen operator with $\gamma_7 \to \gamma_7 - a_-$.
- To remove the plane waves, we can gauge transform with $g(x) \equiv R_-(x \pm ia_-/2)$, since we have

$$\frac{1}{g(x)} \exp(\delta i a_- d/dx) g(x) = q^{-1} e^{-4\delta irx} \exp(\delta i a_- d/dx), \quad \delta = +, -.$$

- Now it remains to divide by q^{-1} to 'get what we want' as concerns the shift part.

5. The acid test

- The upshot is that we have made several reparametrizations and have divided the Lax equation by various factors, so as to tie in the shift parts with those of van Diejen's Hamiltonian. The key question is therefore whether the resulting additive part in the Lax equation can be tied in with $V_b(x) - E$.
- Thus we now need to detail the additive coefficient of the Lax equation. It is of the form

$$R(z) = \sum_{n=1}^{3} S_n(z).$$

The summands S_1, S_2 are of the form

$$S_1(z) \equiv U(z)\mathcal{F}(qz)\mathcal{G}(k/z)[k/q^2z^2]/\mathcal{G}(z),$$

$$S_2(z) \equiv U(k/qz)\mathcal{F}(z)\mathcal{G}(qz)[k/z^2]/\mathcal{G}(k/qz),$$

with the new factor $\mathcal{G}(z)$ given by

$$\mathcal{G}(z) \equiv z[z/\xi_1][z/\xi_2].$$

Here, we have

$$\xi_1 \xi_2 = \ell,$$

with

$$k^2 \ell^2 = q \prod_{j=1}^{4} a_j b_j.$$

(Like k, the parameter ℓ stems from the extended affine E_8 Weyl group; moreover, ξ_1 may be viewed as the second initial value for the Sakai equation.)

- The third summand reads

$$S_3(z) \equiv -\mathcal{F}(z)\mathcal{F}(qz)\overline{\mathcal{F}}(z)[k/z^2][k/qz^2][k/q^2z^2]/\mathcal{G}(z)\mathcal{G}(k/qz),$$

  where $\overline{\mathcal{F}}(z)$ equals $\mathcal{F}(z)$ with 'evolved' parameters $\overline{C}, \overline{\lambda}$, $\overline{k} \equiv k/q$. Thus,

$$\overline{\mathcal{F}}(z) = \overline{C} z[z/\overline{\lambda}][k/qz\overline{\lambda}].$$

- The evolution of C and λ is fixed by requiring

$$\mathcal{F}(\xi_j)\overline{\mathcal{F}}(\xi_j)[k/\xi_j^2][k/q\xi_j^2] = \mathcal{G}(k/\xi_j)\mathcal{G}(k/q\xi_j)U(\xi_j), \quad j = 1, 2.$$

(This amounts to demanding that $R(z)$ have no poles at the zeros $z = \xi_1, \xi_2$, of $\mathcal{G}(z)$.)

- With $R(z)$ now determined, we reparametrize it as before, using in addition

$$\xi_i = q\exp(-2r\phi_i), \quad i = 1, 2, \qquad \overline{\lambda} = q\exp(-2r\overline{\gamma_8}).$$

- Finally, we should divide $R(z)$ by the various (reparametrized) factors we needed to tie in the shift part of the Lax equation with the one of $A_+(\gamma; x)$. After a straightforward (but long!) calculation, the resulting additive coefficient $Z(x)$ is of the form

$$Z(x) = E(x) + E(-x) + V_e(x),$$

with E and V_e detailed next.

- The 'extra' summand V_e is given by

$$V_e(x) \equiv C\overline{C}e^{-2ra_-} \frac{R_+(x \pm i(\overline{\gamma_8} - a_-/2))R_+(x \pm i(\gamma_8 - a_-/2))}{\prod_{j=1}^{2} R_+(x \pm i(\phi_j + a_-/2))},$$

and is manifestly an even elliptic function.

- The function E reads

$$E(x) \equiv -e^{-8irx}e^{-4ra_-} \frac{R_+(x - i\gamma_8 + ia_-/2)}{R_+(x - i\gamma_8 - ia_-/2)}$$

$$\times \frac{\prod_{\mu=0}^{7} R_+(x - i\gamma_\mu - ia_-/2)}{R_+(2x + ia_+/2)R_+(2x + ia_+/2 - ia_-)} \prod_{j=1}^{2} \frac{R_+(x + i\phi_j - ia_-/2)}{R_+(x - i\phi_j - ia_-/2)}.$$

It is clearly π/r-periodic, but at face value it seems not to be ia_+-periodic. In fact, however, it is, due to the above relations.

- The upshot is that $Z(x)$ is an even elliptic function. By evennness, it has no poles when the factors $R_+(\pm 2x + ia_+/2)$ vanish.

- Thanks to the evolution requirement on $\overline{C}$ and $\overline{\gamma}_8$, we get vanishing pole residue sums for $\pm x \equiv i\phi_1 + ia_-/2 - ia_+/2$ and $\pm x \equiv i\phi_2 + ia_-/2 - ia_+/2$ (modulo the elliptic lattice Λ). But $Z(x)$ does have poles for $\pm x \equiv x_n$, $n = 0, 1, 2, 3$, and also for $\pm x \equiv i\gamma_8 + ia_-/2 - ia_+/2$ (mod Λ).
- Taking γ_8 equal to γ_7, however, the latter poles drop out. Thus we are left with the same pole locations as those in $V_b(x)$. When we now calculate the residues at these poles, they turn out to coincide with those of $V_b(\tilde{\gamma}; x)$.
- As a consequence, we have

$$V_b(\tilde{\gamma}; x) - Z(x) =: E,$$

with E depending on the parameters, but not on x: The acid test is passed!

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