Geometric action-angle coordinates for the spin RS system

Elliptic integrable systems workshop

(arXiv: 1909.08107)

Outline:

- CM system according to D. Ben-Zvi, T. Nevins
- RS spectral sheaves, main result
- Sketch of the proof
- Advertisement for this perspective:
  - KP-CM / 2D Toda - RS correspondence
  - RS $\longrightarrow$ CM limit
  - Action-angle duality
  - Extension to DELL system (?)

The Calogero–Moser system:

Fix an irreducible cubic curve E and $b \in E^{sm}$.

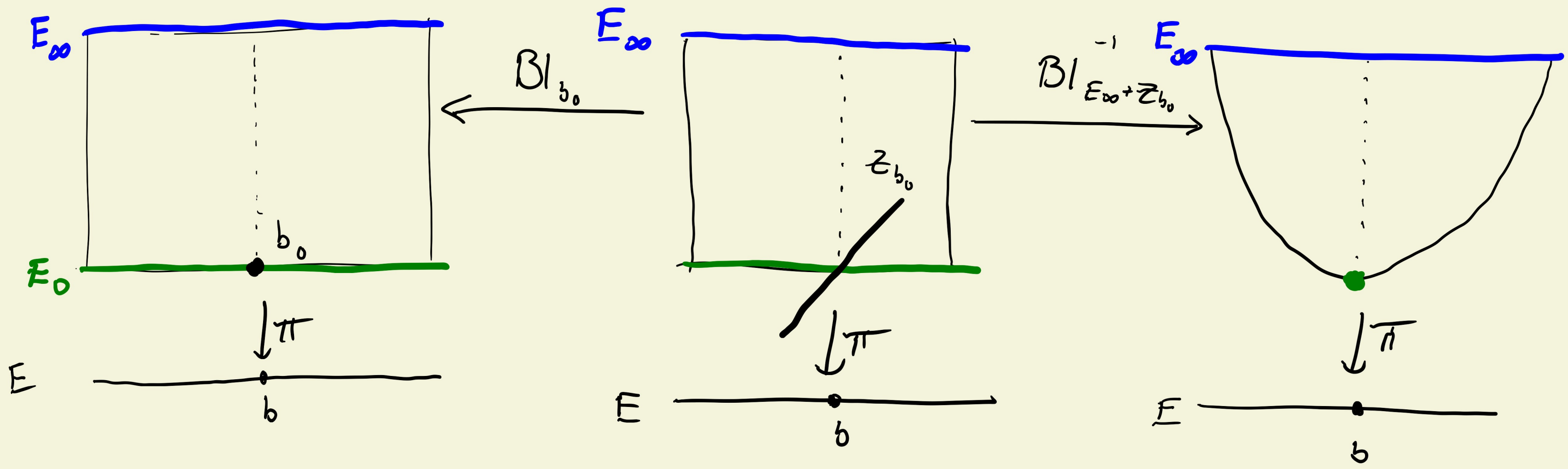
- $E^{sm} = E$, then (E, b) is an elliptic curve
- E has a nodal singularity, $\mathbb{P}^1 \longrightarrow E$ is a normalization, and $E^{sm} = \mathbb{P}^1 \setminus \{q_1, q_2\} = \mathbb{C}^*$
- E has a cuspidal singularity, $E^{sm} = \mathbb{P}^1 \setminus \{q\} = \mathbb{C}$.

Let A be the Atiyah bundle on E:

$$A \in \mathrm{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) \cong \mathbb{C} \rightsquigarrow \quad 0 \longrightarrow \mathcal{O}_E \longrightarrow A \longrightarrow \mathcal{O}_E \longrightarrow 0$$

Let $\overline{E}^{\natural} = \mathbb{P}(A)$, a ruled surface over E with section $E_\infty \subseteq \overline{E}^{\natural}$ coming from $A \twoheadrightarrow \mathcal{O}_E$.

Note: $\overline{E}^{\natural}$ is birational to $\overline{T^*E} \cong E \times \mathbb{P}^1$.



This identifies sections of A with vector fields on E with simple pole at b.

$E^{\natural} = \overline{E}^{\natural} \setminus E_\infty$ appears in the work of Treibich–Verdier on elliptic solitons / tangential covers.

Def: Fix T be a torsion sheaf supported on $E_\infty \subseteq \overline{E}^{\natural}$.

A T-*framed* *CM* *spectral* *sheaf* is a pure 1-dim'l sheaf $\mathcal{F}$ on $\overline{E}^{\natural}$, together with an isomorphism $\phi: \mathcal{F}|_{E_\infty} \xrightarrow{\sim} T$ subject to:

$$\left.\begin{array}{l} \text{i) } W = \pi_* \mathcal{F}(-E_\infty) \text{ is s.stable of degree } 0 \\ \text{ii) } \deg \pi_* \mathcal{F}(kE_\infty) = (k+1)\deg T \text{ for } k \gg 0 \end{array}\right] \begin{array}{l}\text{Technical}\\ \text{cohom +}\\ \text{stability}\\ \text{conditions}\end{array}$$

Th'm (Ben-Zvi–Nevins): Let $CM_n(E, \mathcal{O}_b^k)$ be the moduli space of $\mathcal{O}_b^k$-framed CM spectral sheaves with $\operatorname{rk} W = n$. $CM_n(E, \mathcal{O}_b^k)$ can be identified with a completion of the k-spin, n-particle CM phase space with flows coming from *tweaking* of CM spectral sheaves.

Tweaking flows: Let $\Sigma = \mathrm{supp}(\mathcal{F})$, and $p \in \Sigma \cap E_\infty$.

Let D be the formal disk at p, then w can consider $\mathcal{F}|_{\Sigma \setminus p}$ and $\mathcal{F}|_D$ with transition map φ along $D^\times = \mathrm{Spec}\, \mathbb{C}((z))$.

Given $f \in \mathbb{C}((z))$, we get a tangent vector at $\mathcal{F}$ by changing

$$\varphi \longmapsto \varphi \cdot (1 + \varepsilon f)$$

over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$.

The corresponding $f = z^k$ flow yields the k^{th} CM flow.

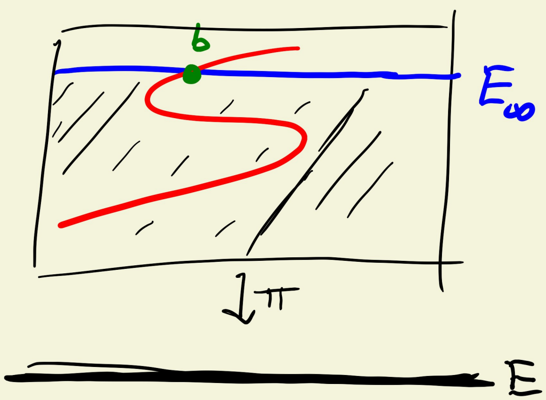
A compelling calculation:

• There is an open locus in $\mathcal{CM}_n(E, \mathcal{O}_b^k)$ where

$$W \cong \bigoplus_{i=1}^{n} \mathcal{O}(q_i - b) \quad \text{for points} \quad q_i \in E^{sm}$$

• The action of $\mathcal{O}_{E^k}$ on $\mathcal{F}$ descends to a Higgs field

$$\begin{array}{ccc} \phi : W & \longrightarrow & W(b) \\ \wr\wr & & \\ \bigoplus \mathcal{O}(q_i - b) & \longrightarrow & \bigoplus \mathcal{O}(q_i) \end{array}$$



• The (i,j) component is given by $\mathrm{Hom}(\mathcal{O}(q_j - b), \mathcal{O}(q_i))$

$$= H^0(E, \mathcal{O}(-(q_j - q_i) + b))$$

$$\frac{\sigma(z - q_j - q_i)}{\sigma(z)\sigma(q_j - q_i)}$$

(rational picture) • Pole at zero, zero at $(q_j - q_i)$: $\frac{1}{z} - \frac{1}{q_j - q_i} \rightsquigarrow \phi = \begin{pmatrix} p_1 & & \boxed{ij} \\ & \ddots & \\ & & p_n \end{pmatrix}$

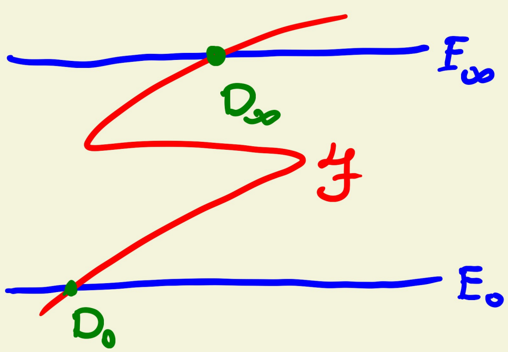
Ruijsenaars – Schneider description:

Take $\alpha \in \overline{\operatorname{Jac} E} \cong E$, with corresponding line bundle L_α.

Consider the surface $\overline{S}_\alpha = \mathbb{P}(L_\alpha \oplus \mathcal{O}_E)$.

The two projections $L_\alpha \oplus \mathcal{O}_E \longrightarrow \mathcal{O}_E$, $L_\alpha \oplus \mathcal{O}_E \twoheadrightarrow L_\alpha$ yield

sections $E_0, E_\infty \subseteq \overline{S}_\alpha$



<u>Def:</u> Fix T a torsion sheaf on $E_0 \cup E_\infty$.
An <u>RS spectral sheaf</u> is a pure 1-dim'l sheaf on S_α with fixed framing $\phi : \mathcal{F}|_{E_0 \cup E_\infty} \xrightarrow{\sim} T$ + stability & cohom. conditions.

Let $RS_{n,\alpha}(E,T)$ be the moduli space of T-framed RS spectral sheaves with $\pi_* \mathcal{F}$ a rank n bundle on E.

- The geometry of linear systems of on the ruled surface $\overline{S_\alpha}$ $\Rightarrow$
  If $\Sigma = \operatorname{supp} \mathcal{F}$, then $D_0 = \Sigma \cap E_0$, $D_\infty = \Sigma \cap E_\infty$ are related by: $\boxed{D_\infty = D_0 + n(b-\sigma)}$ ☆

- There is an open locus of $RS_{n,\alpha}(E,T)$ consisting of pairs $(\Sigma, \mathcal{L})$ for $\mathcal{L} \in \operatorname{Pic} \Sigma$. But there are more general spectral sheaves.

- $RS_{n,\alpha}(E,T)$ inherits a Poisson structure as a moduli space of sheaves on the Poisson surface $\overline{S_\alpha}$.
  In fact the framing defines it as a symplectic leaf.

- Restricting to the case when $T \cong \mathcal{O}_{p_0}^k \oplus \mathcal{O}_{p_\infty}^k$ with p_0, p_∞ related by ☆, denote $RS_{n,\alpha}^k(E) := RS_{n,\alpha}(E, \mathcal{O}_{p_0}^k \oplus \mathcal{O}_{p_\infty}^k)$.

- Can define a collection of tweaking flows by changing the transition function at p_0 $\left(\Sigma = \mathrm{Supp}\,\mathcal{F},\quad D_0 = \mathrm{Spec}\,\widehat{\mathcal{O}}_{\Sigma, p_0},\quad \mathcal{U}_0 = \Sigma \setminus p_0\right)$

  or p_∞ $\left(\quad \text{''} \quad,\quad D_\infty = \mathrm{Spec}\,\widehat{\mathcal{O}}_{\Sigma, p_\infty},\quad \mathcal{U}_\infty = \Sigma \setminus p_\infty\right)$

- Both of these flows span the same Lagrangian subspace in

$$T_{\mathcal{F}} RS_{n,\alpha}^k(E)$$

Th'm (P.): $RS^k_{n,\alpha}(E)$ can be identified with a completion of the k-spin, n-particle RS phase space with RS flows coming from tweaking of RS spectral sheaves.

Sketch of proof:

1) Describe $RS^k_{n,\alpha}(E)$ as a Hitchin system:

Let $W = \pi_* \mathcal{F}$

On $\overline{S}_\alpha \setminus E_\infty$ have a Higgs field $\eta_0 : W \longrightarrow W \otimes L_\alpha^{-1}(p_\infty)$

" $\overline{S}_\alpha \setminus E_0$ " " " " $\eta_\infty : W \longrightarrow W \otimes L_\alpha(p_0)$

- η_0, and η_∞ are invertible away from p_0, p_∞.

- Relation $\bigstar$ implies $\eta_0(z) = \eta_\infty(z + n(\alpha - b))^{-1}$

- The composite $W \xrightarrow{\eta_\infty} W \otimes L_\alpha(p_0) \xrightarrow{\eta_0 \otimes id} W(p_0 + p_\infty)$

  Realizes $RS^k_{n,\alpha}(E)$ as a space of <u>multiplicative</u> <u>Higgs</u> <u>bundle</u> with values in $\mathcal{O}_E(p_0 + p_\infty)$.

  (example of an abstract Higgs bundle defined by Donagi-Gaitsgory)

2) Repeat the compelling calculation as in the CM case:

Restrict to the open set $\mathcal{U} \subseteq RS^{k}_{n,\alpha}(E)$ where W decomposes as $W \cong \bigoplus_{i=1}^{n} \mathcal{O}(q_i - b)$.

In coordinates η_0, η_∞ become elliptic Cauchy matrices, and we recover the RS Lax matrix as

$$\eta_\infty \cdot \eta_0 = \eta_0^{-1}(z) \cdot \eta_0(z + n\hbar)$$

Note: We get the **factorized** Lax matrix of K. Hasegawa introduced earlier in Andrei Zotov's talk.

$$\eta_0(z) = g(z), \text{ the intertwiner in IRF-Vertex}$$

3) Define a Hitchin system on $RS^k_{n,\alpha}(E)$

$$\vec{H} : RS^k_{n,\alpha}(E) \longrightarrow B_{RS} \subseteq \bigoplus_{i=0}^{n} H^0(E, \mathcal{O}_E(p_0+p_\infty)^{\otimes i})$$

$$H_i(W, \eta_0, \eta_\infty) \longmapsto \frac{1}{i+1} \operatorname{Tr}(\eta_0 \cdot \eta_\infty)^i$$

and show tweaking flows on $\mathcal{Y} \rightsquigarrow$ Hitchin flows on (W, η_0, η_∞)

Role of α: Define a limit $\alpha \longrightarrow 0$ in which $L_\alpha \oplus \mathcal{O}_E \longrightarrow A$

$$\overline{S_\alpha} \longrightarrow E^\natural$$

RS spectral sheaves limit to CM spectral sheaves!

(work in progress w/ D. B-Z, T.N.)

$\boxed{\text{Krichever, Zabrodin: poles of 2D Toda solutions flow as RS particles}}$

Can try to understand the 2D Toda - RS correspondence in this perspective:

Fourier Mukai: Let $\mathcal{P}$ be the Poincaré bundle on $E \times \check{E}$

$$\begin{tikzcd} & E \times \check{E} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E & & \check{E} \simeq E \end{tikzcd} \qquad FM(\mathcal{E}) = p_{2*}(\mathcal{P} \otimes p_1^* \mathcal{E})$$

- FM exchanges rank n v.bundles with length n torsion sheaves

- $FM\left(W \xrightarrow{\eta_0} W \otimes L_\alpha(p_\infty)\right) \simeq \bigoplus_{q_i} T_{q_i} \xrightarrow{\sigma^+} \bigoplus T_{q_i+\alpha}$

  $FM\left(W \xrightarrow{\eta_\infty} W \otimes L_\alpha^{-1}(p_0)\right) = \bigoplus T_{q_i} \xrightarrow{\sigma^-} \bigoplus T_{q_i-\alpha}$

  \+ commutativity of difference operators.

(Warning: Reckless speculation ahead)

- Start with $Y = E_\tau \times E_{\tau'}$, find the appropriate birational (isogeny?) transformation to an abelian surface X. ($(1,n)$ – polarized abelian surface?)

- Define a space of Dell spectral sheaves on X
(completion of $(\Sigma, \mathcal{L})$ w/ Σ a Seiberg-Witten curve from $6D \to 4D$)

- Using the birational transformation $Y \dashrightarrow X$, find a (parabolic) Hitchin description of the space of Dell spectral sheaves

- If all the choices were made appropriately, recover a P–Q self dual Hamiltonian system!

Thank you for listening!