

Hello and welcome to class!

Last time

We studied the formal properties of **determinants**, and how to compute them by **row reduction**.

Today

We'll see some more formulas involving the determinant — **minor expansion** and **Cramer's rule** — and discuss the interpretation of the determinant as a **signed volume**.

Review: computing determinants by row reduction

To compute the determinant of a matrix, row reduce it, and keep track of any row switches or rescalings of rows.

At the end, multiply together:

- ▶ the inverses of the row rescaling factors
- ▶ the diagonal entries of the final echelon matrix
- ▶ $(-1)^{\# \text{rowswaps}}$

That's the determinant of the original matrix.

This method is much much faster than summing all the terms.

Example

Let's compute the determinant of this matrix

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{bmatrix}$$

First, we row reduce, **keeping track of rescalings and row switches**

Example

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -3 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & -4 & -3 & 3 \\ 0 & 1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -7 & 7 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -7 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1/7} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the determinant is $(-1)^2 \cdot (-7) \cdot (1 \cdot 1 \cdot 1 \cdot 1) = -7$.

Try it yourself!

Compute the determinant of this matrix:

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Row reduce, **keeping track of rescalings and row switches:**

Try it yourself!

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 4 & 2 & -5 \\ 0 & 5 & 1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 5 & 1 & -2 \\ 0 & 4 & 2 & -5 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -9 & -17 \\ 0 & 0 & -6 & -17 \end{bmatrix}$$

The determinant is 51.

Review: terms in the determinant

In the 2x2 case:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$+ad$$

$$-bc$$

Review: terms in the determinant

In the 3x3 case:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$+aei$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$+bfg$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$+cdh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$-afh$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$-bdi$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$-ceg$$

Another perspective

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$+aei - afh$$

$$+a \begin{vmatrix} e & f \\ h & i \end{vmatrix}$$

no orange-green
inversions

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$-bdi + bfg$$

$$-b \begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

one orange-green
inversions

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$+cdh - ceg$$

$$+c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

two orange-green
inversions

Minor expansion

For a matrix A , I'll write A_{ij} for the matrix formed by **omitting** row i and column j . **For example**, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We have:

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11} |A_{11}| - a_{12} |A_{12}| + a_{13} |A_{13}| \end{aligned}$$

Minor expansion

More generally, by the same argument, for a square $n \times n$ matrix A with entry $a_{i,j}$ in row i and column j ,

for any k in $1, \dots, n$, there is a **minor expansion along the k 'th row**

$$|A| = \sum_{j=1}^n (-1)^{j+k} a_{kj} |A_{\cancel{k}\cancel{j}}|$$

and a **minor expansion along the k 'th column**

$$|A| = \sum_{j=1}^n (-1)^{j+k} a_{jk} |A_{j\cancel{k}}|$$

The sign $(-1)^{\text{row}+\text{column}}$

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix}$$

Example

Compute by minor expansion along the second row:

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{vmatrix}$$

Example

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{vmatrix} = -2 \begin{vmatrix} \text{[row 1 crossed out]} & 2 & 3 & -1 \\ \text{[col 1 crossed out]} & 1 & -1 & 2 \\ \text{[col 1 crossed out]} & 7 & 8 & -2 \end{vmatrix} + 0 \begin{vmatrix} 1 & \text{[col 2 crossed out]} & 3 & -1 \\ \text{[row 2 crossed out]} & \text{[col 2 crossed out]} & -1 & 2 \\ 3 & \text{[col 2 crossed out]} & 8 & -2 \end{vmatrix} \\ -3 \begin{vmatrix} 1 & 2 & \text{[col 3 crossed out]} & -1 \\ \text{[row 3 crossed out]} & \text{[col 3 crossed out]} & \text{[col 3 crossed out]} & 2 \\ 3 & 7 & \text{[col 3 crossed out]} & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 & 3 & \text{[col 4 crossed out]} \\ 0 & 1 & -1 & \text{[col 4 crossed out]} \\ 3 & 7 & 8 & \text{[col 4 crossed out]} \end{vmatrix}$$

Example

$$-2 \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 7 & 8 & -2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 & -1 \\ 0 & -1 & 2 \\ 3 & 8 & -2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 7 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 7 & 8 \end{vmatrix}$$

Now we minor-expand each of these 3×3 determinants.

We'll use the second row for each (to catch the zero).

Example

$$\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 7 & 8 & -2 \end{vmatrix} = -1 \begin{vmatrix} 3 & -1 \\ 8 & -2 \end{vmatrix} + (-1) \begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix} \\ = -2 - 3 + 10 = 5$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 7 & -2 \end{vmatrix} = -0 \begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} \\ = 0 + 1 - 2 = -1$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 7 & 8 \end{vmatrix} = -0 \begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} \\ = 0 - 1 + 1 = 0$$

Example

$$-2 \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 7 & 8 & -2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 & -1 \\ 0 & -1 & 2 \\ 3 & 8 & -2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 7 & -2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 7 & 8 \end{vmatrix}$$

$$(-2 \times 5) + (0 \times ?) + (-3 \times -1) + (1 \times 0) = -7$$

That's the same as we got doing this the other way.

Which was easier?

Try it yourself!

Compute by minor expansion the determinant of the matrix.

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

A formula for the inverse

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{Adj}(A) = \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix}$$

$$A \cdot \text{Adj}(A) = \begin{bmatrix} a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} & -a_{11}A_{21} + a_{12}A_{22} - a_{13}A_{23} & a_{11}A_{31} - a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} - a_{22}A_{12} + a_{23}A_{13} & -a_{21}A_{21} + a_{22}A_{22} - a_{23}A_{23} & a_{21}A_{31} - a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} - a_{32}A_{12} + a_{33}A_{13} & -a_{31}A_{21} + a_{32}A_{22} - a_{33}A_{23} & a_{31}A_{31} - a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

The diagonal terms, e.g., $a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$, are **minor expansions** of $\det(A)$.

A formula for the inverse

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{Adj}(A) = \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix}$$

Let's look at an off-diagonal term of $A \cdot \text{Adj}(A)$, say

$$a_{21}A_{11} - a_{22}A_{12} + a_{23}A_{13}$$

Expanding this out from the definition,

$$a_{21} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

A formula for the inverse

The quantity

$$a_{21} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

is the **minor expansion** of the determinant

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The matrix has a repeated row, so the determinant is zero! The same is true for all the off diagonal terms.

A formula for the inverse

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{Adj}(A) = \begin{bmatrix} A_{11} & -A_{21} & A_{31} \\ -A_{12} & A_{22} & -A_{32} \\ A_{13} & -A_{23} & A_{33} \end{bmatrix}$$

$$A \cdot \text{Adj}(A) = \det(A) \cdot I = \text{Adj}(A) \cdot A$$

This holds for any square matrix A , where

$$\text{Adj}(A)_{ij} = (-1)^{i+j} |A_{j\cancel{i}}|$$

The entry in **row** i , **column** j of $\text{Adj}(A)$ is the determinant of the matrix formed by removing **column** i and **row** j of A , times $(-1)^{i+j}$.

Try it yourself!

For the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, determine $\text{Adj}(A)$, and verify

$$A \cdot \text{Adj}(A) = \det(A) \cdot I = \text{Adj}(A) \cdot A$$

$$\text{Adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Cramer's rule

Consider a matrix equation $A\mathbf{x} = \mathbf{b}$ where A is square. Then

$$\det(A) \cdot \mathbf{x} = (\text{Adj}(A) \cdot A)\mathbf{x} = \text{Adj}(A) \cdot \mathbf{b}$$

Take the i 'th row of the column vector on both sides:

$$\det(A) \cdot x_i = \sum_j \text{Adj}(A)_{ij} b_j = \sum_j (-1)^{i+j} |A_{j\neq i}| b_j$$

I.e., the **minor expansion** along the i 'th column of the determinant of the matrix formed by replacing the i 'th column of A by $\mathbf{b}$.

Cramer's rule

Consider a matrix equation $A\mathbf{x} = \mathbf{b}$ where A is square.

Then if $\det(A) \neq 0$,

$$x_i = \frac{\det(\text{replace column } i \text{ of } A \text{ by } \mathbf{b})}{\det(A)}$$

Never use these formulas to compute

As we saw, taking the determinant of a 4×4 matrix by **minor expansion** was more difficult than by **row reduction**.

It only gets worse as the size of the matrix grows.

Likewise, **row reduction** beats **computing Adj** for inverting matrices, and beats **Cramer's rule** for solving systems.

Why learn these formulas at all?

It's **conceptually satisfying** to know that, not only is there a **procedure** for solving systems or inverting matrices,

there's in fact a **closed form formula**.

The properties of the formula reveal facts about the solutions.

Integer inverses and solutions

Say you have an invertible matrix M with integer entries.

Does its inverse also have integer entries?

It does, **if and only** $\det(M) = \pm 1$.

Observe $\det(M) \det(M^{-1}) = \det(MM^{-1}) = 1$. The determinant of an integer matrix is always an integer — **it's made by additions and multiplications**. If M^{-1} has integer entries, then $\det(M)$ and $\det(M^{-1})$ are **two integers which multiply to 1**, hence both ± 1 .

Similarly, the Adj of an integer matrix is an integer: **it's made by additions and multiplications**. So, if $\det M = \pm 1$, then $M^{-1} = \text{Adj}(M) / \det M$ is an integer matrix as well.

Integer inverses and solutions

Similarly, consider the equation $A\mathbf{x} = \mathbf{b}$.

Assume

- ▶ A is square and has integer entries
- ▶ $\mathbf{b}$ has integer entries
- ▶ $\det(A) = \pm 1$

We saw that A^{-1} has integer entries, so the (unique) solution $\mathbf{x} = A^{-1}\mathbf{b}$ also has integer entries.

Volumes

You probably have an intuitive notion of what **volume** means: the **amount of stuff** that can fit inside something. For our purposes, the **stuff** is going to be **cubes of a fixed side length**:

$$\text{Volume}(S) \sim \text{number of cubes that fit inside } S$$

To be more precise,

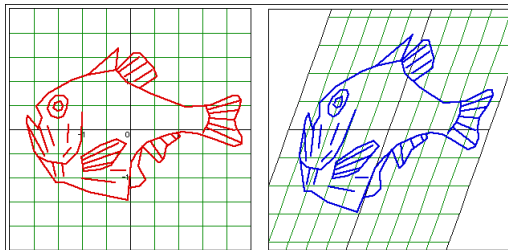
$$\text{Volume}(S) = \lim_{\epsilon \rightarrow 0} \epsilon^{-\dim} \cdot \left(\begin{array}{l} \text{number of cubes} \\ \text{of side length } \epsilon \\ \text{that fit inside } S \end{array} \right)$$

By this we mean: for $S \subset \mathbb{R}^n$, we **overlay the ϵ -mesh grid on S** , and count the **number of cubes** which fall completely inside.

Linear transformations and volumes

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Given a set X , we want to think about how the volumes of X and $T(X)$ compare.

The key observation is that the number of cubes in X is the same as the number of T -transformed such cubes in $T(X)$.



Linear transformations and volumes

Filling the transformed cubes with yet smaller regular cubes, and observing that **the failure of these to pack correctly at the boundary is washed out as $\epsilon \rightarrow 0$** , we conclude:

$$\frac{\text{Volume}(X)}{\text{Volume}(\text{cube})} = \frac{\text{Volume}(T(X))}{\text{Volume}(T(\text{cube}))}$$

Rearranging,

$$\text{Volume}(T(X)) = \text{Volume}(T(\text{cube})) \cdot \text{Volume}(X)$$

But what's $\text{Volume}(T(\text{cube}))$?

Linear transformations and volumes

Suppose now given **two linear transformations**, $T, S : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

We apply the **formula**

$$\text{Volume}(T(X)) = \text{Volume}(T(\text{cube})) \cdot \text{Volume}(X)$$

to the set $X = S(\text{cube})$:

$$\text{Volume}(T(S(\text{cube}))) = \text{Volume}(T(\text{cube})) \cdot \text{Volume}(S(\text{cube}))$$

Linear transformations and volumes

In other words, the function

$$\begin{aligned} V : \text{linear transformations} &\rightarrow \mathbb{R} \\ T &\mapsto \text{Volume}(T(\text{unit cube})) \end{aligned}$$

respects multiplication in the sense that

$$V(T \circ S) = V(T)V(S)$$

Note also that $V(\text{Identity}) = 1$, and $V(\text{non invertible matrix}) = 0$.

Linear transformations and volumes

Now consider any linear transformation T .

If it's **not invertible**, $V(T) = 0$.

If it is invertible, then **by row reduction** we can expand it as a product of elementary matrices,

$$T = E_n \cdots E_1$$

Since **volume scaling is multiplicative**,

$$V(T) = V(E_n) \cdots V(E_1)$$

Linear transformations and volumes

It remains to understand volume scaling of elementary matrices.

Rescaling a row — stretching a coordinate — rescales volume by the same factor: **the volume of a box is the product of its side lengths**, and we rescaled one of them.

Switching two rows doesn't change volume at all — **we're just renaming the sides of the box**.

Adding a multiple of one row to another **takes a box to a parallelepiped with the same base and the same height**, so again doesn't change volume.

Determinants and volumes

So for an elementary linear transformation,

$$\text{Volume}(T(\text{unit cube})) = |\det(T)|$$

Volume scaling is multiplicative, so this holds for any linear transformation.

Collecting these observations, for any linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and any set $X \subset \mathbb{R}^n$,

$$\text{Volume}(T(X)) = |\det(T)| \cdot \text{Volume}(X)$$