Hello and welcome to class!

Last time

We introduced linear subspaces and bases.

Today

We talk a bit more about bases, then move on to study the determinant of a matrix. I am going to do what corresponds to chapter 3.2 in the book today and then chapters 3.1 and 3.3 next time.

Bases of $\mathbb{R}^n$

A linearly independent subset of $\mathbb{R}^n$ must have at most n elements, and a spanning set of $\mathbb{R}^n$ must have at least n elements, so

Every basis of $\mathbb{R}^n$ has exactly n elements.

Bases of $\mathbb{R}^n$

For an *n element subset* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of $\mathbb{R}^n$,

The following are equivalent:

- The $\mathbf{v}_i$ span $\mathbb{R}^n$
- The $\mathbf{v}_i$ are linearly independent
- The $\mathbf{v}_i$ form a basis
- The square matrix whose columns are the $\mathbf{v}_i$ is invertible.

Bases and linear transformations

From a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of a linear subspace V of $\mathbb{R}^n$, we can define a linear map

$$\begin{aligned} T : \mathbb{R}^k &\to \mathbb{R}^n \\ \mathbf{x} &\mapsto \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix} \cdot \mathbf{x} \end{aligned}$$

This linear map has range V, since the $\mathbf{v}_i$ span V, and is one-to-one because the $\mathbf{v}_i$ are linearly independent.

A basis for V is essentially the same as a one-to-one linear map with range V. This is also called an isomorphism from $\mathbb{R}^k$ to V.

The size of a basis

Suppose given two bases, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ of the same linear subspace V of $\mathbb{R}^n$.

Let $T : \mathbb{R}^k \to \mathbb{R}^n$ be the linear map $\mathbf{x} \mapsto \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix} \cdot \mathbf{x}$.

Since T is one-to-one with range V, there is a unique element of $\mathbb{R}^k$ which maps to $\mathbf{w}_i$. We write it as $T^{-1}(\mathbf{w}_i)$.

Note that $T(T^{-1}(\mathbf{w}_i)) = \mathbf{w}_i$.

The size of a basis

Given a linear dependency $0 = \sum c_i T^{-1}(\mathbf{w}_i)$, we obtain

$$0 = T\left(\sum c_i T^{-1}(\mathbf{w}_i)\right) = \sum c_i T(T^{-1}(\mathbf{w}_i)) = \sum c_i \mathbf{w}_i$$

The $\mathbf{w}_i$ are linearly independent, hence the c_i must all be zero.
Thus the $T^{-1}(\mathbf{w}_i)$ are linearly independent.

The size of a basis

Since the $\mathbf{w}_i$ span V, each $\mathbf{v}_j$ can be written as

$$\mathbf{v}_j = \sum_i c_{ij} \mathbf{w}_i$$

By definition, $\mathbf{v}_j = T(\mathbf{e}_j)$. So

$$T(\mathbf{e}_j) = \sum_i c_{ij} \mathbf{w}_i = \sum_i c_{ij} T(T^{-1}(\mathbf{w}_i)) = T\left(\sum_i c_{ij} T^{-1}(\mathbf{w}_i)\right)$$

Since T is one-to-one,

$$\mathbf{e}_j = \sum_i c_{ij} T^{-1}(\mathbf{w}_i)$$

Thus the span of the $T^{-1}(\mathbf{w}_i)$ contains all $\mathbf{e}_j$, hence all $\mathbb{R}^k$.

The size of a basis

Thus there are as many $\mathbf{w}_i$ as elements of a basis for $\mathbb{R}^k$ — i.e., k of them — which is the same as the number of $\mathbf{v}_j$.

We have learned that every basis for a linear subspace has the same number of elements. This common number is called the dimension of the linear subspace.

Rank-Nullity

Theorem

If A is a matrix with r rows and c columns, i.e., determines a linear transformation $A : \mathbb{R}^c \to \mathbb{R}^r$, then the dimensions of the column space and the null space add up to c.

Proof.

The column space has dimension equal to the number of pivot columns, and the null space has dimension equal to the number of non-pivot columns. $\square$

Row rank and column rank

Theorem

The dimension of the space spanned by the rows of a matrix is equal to the dimension of the space spanned by the columns.

Proof.

Both are equal to the number of pivots. $\square$

Invertibility

Recall that for a square matrix, the following are equivalent:

- The matrix is invertible
- The rows are linearly independent
- The rows span
- The rows form a basis
- The columns are linearly independent
- The columns span
- The columns form a basis
- The linear transformation is one-to-one
- The linear transformation is onto

Review!

Is the matrix invertible?

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ no

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ yes

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$ no

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$ yes

2×2 determinants

Is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible?

If $a \neq 0$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$ so the matrix is invertible if and only if $ad - bc \neq 0$.

If $a = 0$, then by inspection the matrix is invertible if and only if b, c are not both zero.

In short, the matrix is invertible if and only if

$$ad - bc \neq 0$$

The quantity $ad - bc$ is called the determinant of the matrix.

Try it yourself!

Calculate the determinant.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad 0 \times 0 - 0 \times 0 = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 1 \times 1 - 0 \times 0 = 1$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad 1 \times 6 - 2 \times 3 = 0$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \qquad 1 \times 5 - 2 \times 3 = -1$$

Review!

Is the matrix invertible?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ yes

$$\begin{bmatrix} 3 & 5 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$ yes

$$\begin{bmatrix} 8 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 5 & 4 \end{bmatrix}$$ yes

Review!

Is the matrix invertible?

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$ no

$$\begin{bmatrix} 1 & 3 & 4 \\ 1 & 3 & 2 \\ 2 & 6 & 8 \end{bmatrix}$$ no

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$ no

3×3 determinants

Is $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ invertible?

Let's assume $a \neq 0$. Then we row reduce:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{bmatrix} \to \begin{bmatrix} a & b & c \\ 0 & ae-bd & af-cd \\ 0 & ah-bg & ai-cg \end{bmatrix}$$

This matrix is invertible iff the bottom two rows are linearly independent; this happens iff:

3×3 determinants

The bottom two rows of the previous matrix are linearly independent iff

$$\begin{aligned} 0 &\neq \begin{vmatrix} ae - bd & af - cd \\ ah - bg & ai - cg \end{vmatrix} \\ &= (ae - bd)(ai - cg) - (af - cd)(ah - bg) \\ &= a^2ei - aceg - abdi + bcdg - a^2fh + abfg + acdh - bcdg \\ &= a(aei + bfg + cdh - afh - ecg - bdi) \end{aligned}$$

As we'd assumed $a \neq 0$, this is true iff

$$0 \neq aei + bfg + cdh - afh - ecg - bdi$$

The quantity on the right is called the determinant of the matrix.

3×3 determinants

We saw that the matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is invertible if and only if its determinant

$$aei + bfg + cdh - afh - ecg - bdi$$

is not zero. (Strictly speaking, we saw this assuming $a \neq 0$; this assumption can easily be removed.)

Terms in the determinant

In the 2x2 case:

$$\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} \\ \boxed{c} & d \end{bmatrix}$$

$$+ad \qquad\qquad -bc$$

Terms in the determinant

In the 3x3 case:

$$\begin{bmatrix} \boxed{a} & b & c \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} & c \\ d & e & \boxed{f} \\ \boxed{g} & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & \boxed{c} \\ \boxed{d} & e & f \\ g & \boxed{h} & i \end{bmatrix}$$

$$+aei \qquad +bfg \qquad +cdh$$

$$\begin{bmatrix} \boxed{a} & b & c \\ d & e & \boxed{f} \\ g & \boxed{h} & i \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} & c \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a & b & \boxed{c} \\ d & \boxed{e} & f \\ \boxed{g} & h & i \end{bmatrix}$$

$$-afh \qquad -bdi \qquad -ceg$$

Terms in the determinant

The determinant of an $n \times n$ matrix is the sum

over all ways of choosing n entries of the matrix such that one is in each row and each column

of the product of those entries

with some signs.

Terms in the determinant

In the 2x2 case:

$$\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} \\ \boxed{c} & d \end{bmatrix}$$

$$+ad \qquad\qquad -bc$$

Terms in the determinant

In the 3x3 case:

$$\begin{bmatrix} \boxed{a} & b & c \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix} \quad \begin{bmatrix} a & \boxed{b} & c \\ d & e & \boxed{f} \\ \boxed{g} & h & i \end{bmatrix} \quad \begin{bmatrix} a & b & \boxed{c} \\ \boxed{d} & e & f \\ g & \boxed{h} & i \end{bmatrix}$$

$$+aei \qquad +bfg \qquad +cdh$$

$$\begin{bmatrix} \boxed{a} & b & c \\ d & e & \boxed{f} \\ g & \boxed{h} & i \end{bmatrix} \quad \begin{bmatrix} a & \boxed{b} & c \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix} \quad \begin{bmatrix} a & b & \boxed{c} \\ d & \boxed{e} & f \\ \boxed{g} & h & i \end{bmatrix}$$

$$-afh \qquad -bdi \qquad -ceg$$

What are the signs?

Given a choice of n entries in an $n \times n$ matrix with one in each row and each column,

An inversion is a pair of these entries such that the row of the first is before the row of the second but the column of the first is after the column of the second.

In other words, it's the number of $\nearrow$'s you can draw connecting one entry to another.

Number of inversions

In the 2x2 case:

$$\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} \\ \boxed{c} & d \end{bmatrix}$$

0 1

Number of inversions

In the 3x3 case:

$$\begin{bmatrix} \boxed{a} & b & c \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} & c \\ d & e & \boxed{f} \\ \boxed{g} & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & \boxed{c} \\ \boxed{d} & e & f \\ g & \boxed{h} & i \end{bmatrix}$$

$$0 \qquad\qquad\qquad 2 \qquad\qquad\qquad 2$$

$$\begin{bmatrix} \boxed{a} & b & c \\ d & e & \boxed{f} \\ g & \boxed{h} & i \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} & c \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a & b & \boxed{c} \\ d & \boxed{e} & f \\ \boxed{g} & h & i \end{bmatrix}$$

$$1 \qquad\qquad\qquad 1 \qquad\qquad\qquad 3$$

The formula for the determinant

The determinant of an $n \times n$ matrix is the sum

over all ways of choosing n entries of the matrix such that one is in each row and each column

of the product of those entries

times $(-1)^{\#\text{inversions}}$.

Terms in the determinant

In the 2x2 case:

$$\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} \\ \boxed{c} & d \end{bmatrix}$$

$$+ad \qquad\qquad -bc$$

Terms in the determinant

In the 3x3 case:

$$\begin{bmatrix} \boxed{a} & b & c \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} & c \\ d & e & \boxed{f} \\ \boxed{g} & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & \boxed{c} \\ \boxed{d} & e & f \\ g & \boxed{h} & i \end{bmatrix}$$

$$+aei \qquad\qquad +bfg \qquad\qquad +cdh$$

$$\begin{bmatrix} \boxed{a} & b & c \\ d & e & \boxed{f} \\ g & \boxed{h} & i \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} & c \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a & b & \boxed{c} \\ d & \boxed{e} & f \\ \boxed{g} & h & i \end{bmatrix}$$

$$-afh \qquad\qquad -bdi \qquad\qquad -ceg$$

Try it yourself!

Compute the determinant of the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Computing the determinant

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 4 & \boxed{5} & 6 \\ 7 & 8 & \boxed{9} \end{bmatrix} \qquad \begin{bmatrix} 1 & \boxed{2} & 3 \\ 4 & 5 & \boxed{6} \\ \boxed{7} & 8 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & \boxed{3} \\ \boxed{4} & 5 & 6 \\ 7 & \boxed{8} & 9 \end{bmatrix}$$

$$+45 \qquad\qquad +84 \qquad\qquad +96$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 4 & 5 & \boxed{6} \\ 7 & \boxed{8} & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & \boxed{2} & 3 \\ \boxed{4} & 5 & 6 \\ 7 & 8 & \boxed{9} \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & \boxed{3} \\ 4 & \boxed{5} & 6 \\ \boxed{7} & 8 & 9 \end{bmatrix}$$

$$-48 \qquad\qquad -72 \qquad\qquad -105$$

$$45 - 48 + 84 - 72 + 96 - 105 = -3 + 12 - 9 = 0$$

Example

Compute the determinant of the matrix:

$$\begin{bmatrix} 1 & 4 & 7 & 3 & 2 & 1 \\ 5 & 2 & 9 & 3 & 2 & 4 \\ 1 & 6 & 3 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 6 & 1 & 5 & 2 \\ 1 & 2 & 6 & 3 & 3 & 6 \end{bmatrix}$$

Example

Compute the determinant of the matrix:

$$\begin{bmatrix} 1 & 4 & 7 & 3 & 2 & 1 \\ 5 & 2 & 9 & 3 & 2 & 4 \\ 1 & 6 & 3 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 6 & 1 & 5 & 2 \\ 1 & 2 & 6 & 3 & 3 & 6 \end{bmatrix}$$

There's no way to choose an entry from each row and each column without choosing a zero. So every term in the determinant is zero. So the determinant is zero.

Try it yourself!

Compute the determinant of the matrix:

$$\begin{bmatrix} 1 & 4 & 7 & 3 & 2 & 1 \\ 0 & 2 & 9 & 3 & 2 & 4 \\ 0 & 0 & 3 & 4 & 3 & 8 \\ 0 & 0 & 0 & 4 & 1 & 7 \\ 0 & 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

There's only one way to choose an entry from each row and each column without choosing a zero – choosing the entries along the diagonal. So the determinant is

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$$

Rescaling a row

What's the determinant of this:

$$\begin{bmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Triangular matrices

More generally, any matrix with no entries above the diagonal

$$\begin{bmatrix} \boxed{*} & * & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * \\ 0 & 0 & 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{*} \end{bmatrix}$$

has determinant equal to the product of the entries on the diagonal, since any other way of selecting one entry from each row and each column must pick a zero.

Terms in the determinant

$$\begin{bmatrix} \boxed{\lambda a} & \lambda b & \lambda c \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} \lambda a & \boxed{\lambda b} & \lambda c \\ d & e & \boxed{f} \\ \boxed{g} & h & i \end{bmatrix} \qquad \begin{bmatrix} \lambda a & \lambda b & \boxed{\lambda c} \\ \boxed{d} & e & f \\ g & \boxed{h} & i \end{bmatrix}$$

$$+\lambda aei \qquad +\lambda bfg \qquad +\lambda cdh$$

$$\begin{bmatrix} \boxed{\lambda a} & \lambda b & \lambda c \\ d & e & \boxed{f} \\ g & \boxed{h} & i \end{bmatrix} \qquad \begin{bmatrix} \lambda a & \boxed{\lambda b} & \lambda c \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} \lambda a & \lambda b & \boxed{\lambda c} \\ d & \boxed{e} & f \\ \boxed{g} & h & i \end{bmatrix}$$

$$-\lambda afh \qquad -\lambda bdi \qquad -\lambda ceg$$

Rescaling a row

$$\det \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{bmatrix} = \lambda \cdot \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

More generally, the same is true for a square matrix of any size: rescaling a row rescales the determinant by the same factor:

Each term in the determinant is a product of entries, one from each row — hence containing exactly one from the rescaled row.

Rescaling a column likewise rescales the determinant.

Linearity in the rows

What's the determinant of this:

$$\begin{bmatrix} a+a' & b+b' & c+c' \\ d & e & f \\ g & h & i \end{bmatrix}$$

Linearity in the rows

$$\begin{bmatrix} \boxed{a+a'} & b+b' & c+c' \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a+a' & \boxed{b+b'} & c+c' \\ d & e & \boxed{f} \\ \boxed{g} & h & i \end{bmatrix}$$

$$+(a+a')ei \qquad +(b+b')fg$$

$$\begin{bmatrix} a+a' & b+b' & \boxed{c+c'} \\ \boxed{d} & e & f \\ g & \boxed{h} & i \end{bmatrix} \qquad \begin{bmatrix} \boxed{a+a'} & b+b' & c+c' \\ d & e & \boxed{f} \\ g & \boxed{h} & i \end{bmatrix}$$

$$+(c+c')dh \qquad -(a+a')fh$$

$$\begin{bmatrix} a+a' & \boxed{b+b'} & c+c' \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix} \qquad \begin{bmatrix} a+a' & b+b' & \boxed{c+c'} \\ d & \boxed{e} & f \\ \boxed{g} & h & i \end{bmatrix}$$

$$-(b+b')di \qquad -(c+c')eg$$

Linearity in the rows

$$\det \begin{bmatrix} a+a' & b+b' & c+c' \\ d & e & f \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \det \begin{bmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{bmatrix}$$

More generally, the same is true for a square matrix of any size: the determinant is linear in any given row (or column):

$$\det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdots \\ a\mathbf{r}_k + b\mathbf{r}'_k \\ \cdots \\ \mathbf{r}_n \end{bmatrix} = a \det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdots \\ \mathbf{r}_k \\ \cdots \\ \mathbf{r}_n \end{bmatrix} + b \det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \cdots \\ \mathbf{r}'_k \\ \cdots \\ \mathbf{r}_n \end{bmatrix}$$

Linearity in the rows

$$\det \begin{bmatrix} \mathbf{r}_1 \\ \cdots \\ a\mathbf{r}_k + b\mathbf{r}'_k \\ \cdots \\ \mathbf{r}_n \end{bmatrix} = a \det \begin{bmatrix} \mathbf{r}_1 \\ \cdots \\ \mathbf{r}_k \\ \cdots \\ \mathbf{r}_n \end{bmatrix} + b \det \begin{bmatrix} \mathbf{r}_1 \\ \cdots \\ \mathbf{r}'_k \\ \cdots \\ \mathbf{r}_n \end{bmatrix}$$

Each term in the determinant on the left is a product of entries, one from each row, hence exactly one in the k'th row.

This entry is a times the entry of $\mathbf{r}_k$ plus b times the entry of $\mathbf{r}'_k$.

The term in the determinant to the left is the sum of a times the corresponding term in the first determinant to the right, and b times the corresponding term in the second.

Try it yourself!

Observe that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Is it true that:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \stackrel{?}{=} \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

No! That would say $1 = 0 + 0$. The determinant is linear in one row (or column) at a time, not in the whole matrix at once.

Try it yourself!

Compute the determinant of the matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Computing the determinant

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 1 & \boxed{2} & 3 \\ 4 & 5 & \boxed{6} \end{bmatrix} \qquad \begin{bmatrix} 1 & \boxed{2} & 3 \\ 1 & 2 & \boxed{3} \\ \boxed{4} & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & \boxed{3} \\ \boxed{1} & 2 & 3 \\ 4 & \boxed{5} & 6 \end{bmatrix}$$

$$+12 \qquad +24 \qquad +15$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 1 & 2 & \boxed{3} \\ 4 & \boxed{5} & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & \boxed{2} & 3 \\ \boxed{1} & 2 & 3 \\ 4 & 5 & \boxed{6} \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & \boxed{3} \\ 1 & \boxed{2} & 3 \\ \boxed{4} & 5 & 6 \end{bmatrix}$$

$$-15 \qquad -12 \qquad -24$$

Note they cancel in pairs; the determinant is zero.

Repeated rows

Any matrix with a repeated row has determinant zero.

Say the repeated rows are in positions i and j.

Then for any term in the determinant, say with selected entries in positions (i, x) and (j, y), there will be another term with the same selected entries in all rows except i and j, and in these rows, the entries (i, y) and (j, x).

The i and j rows are the same, so the terms are products of the same numbers. One of them has exactly one more inversions than the other, so they have opposite signs and hence cancel.

Try it yourself!

Compute the determinants of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

These are

$$1 \times 4 - 2 \times 3 = -2 \qquad 3 \times 2 - 4 \times 1 = 2$$

Swapping rows

$$\det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \dots \\ \mathbf{r}_n \end{bmatrix} = -\det \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{r}_1 \\ \mathbf{r}_3 \\ \dots \\ \mathbf{r}_n \end{bmatrix}$$

Consider a term in the expansion of the determinant to the left. Maybe it contains the entries $(1, x)$ and $(2, y)$.

The term on the right containing entries $(1, y)$ and $(2, x)$, and all others the same, is a product of the same numbers, and has all the same inversions, except that $(1, x)$ and $(2, y)$ is an inversion if and only if $(1, y)$ and $(2, x)$ is not. So the terms have opposite signs.

Adding a multiple of one row to another

$$\det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 + c\mathbf{r}_1 \\ \mathbf{r}_3 \\ \cdots \\ \mathbf{r}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \cdots \\ \mathbf{r}_n \end{bmatrix} + c \cdot \det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_1 \\ \mathbf{r}_3 \\ \cdots \\ \mathbf{r}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \cdots \\ \mathbf{r}_n \end{bmatrix}$$

By linearity and the vanishing of determinants with repeated rows.

Computing determinants by row reduction

We have learned the following facts:

- Swapping rows negates the sign of the determinant.
- Rescaling a row rescales the determinant by the same factor.
- Adding a multiple of a row to another preserves determinants.
- The determinant of a triangular matrix — in particular, an echelon matrix — is the product of the diagonal entries.

Computing determinants by row reduction

So to compute the determinant of a matrix, row reduce it, and keep track of any row switches or rescalings of rows.

At the end, multiply together:

- the inverses of the row rescaling factors
- the diagonal entries of the final echelon matrix
- $(-1)^{\#\text{rowswaps}}$

That's the determinant of the original matrix.

This method is much much faster than summing all the terms.

Determinants of elementary matrices

If E_{am}, E_{swap}, and $E_{scale}(\lambda)$ are elementary matrices which, respectively, add a multiple of one row to another, swap two rows, and rescale one row by λ, then:

$$\det(E_{am}) = 1 \qquad \det(E_{swap}) = -1 \qquad \det(E_{scale}(\lambda)) = \lambda$$

This follows from the previous slide: each is row reduced to the identity in one move; the determinant of the identity is 1, and the effect of that one move is as above.

Determinants of elementary matrices

If $E_1, \ldots, E_m$ are elementary matrices, and M is any (square) matrix, then

$$\det(E_m \cdots E_1 M) = \det(E_m) \cdots \det(E_1) \det(M)$$

This again follows from the previous slides: we have seen that each row operation changes the determinant by multiplying by the determinant of the corresponding elementary matrix.

Determinants and invertibility

Theorem

A matrix is invertible if and only if its determinant is nonzero.

Proof.

The determinant of a matrix is a nonzero multiple of the product of the diagonal entries of its row reduced version. This is not zero if and only if the pivots run down the diagonal, i.e. there's one pivot in each row and each column, i.e., the matrix is invertible. □

Invertibility and products

If A and B are invertible, then $B^{-1}A^{-1}AB = I = ABB^{-1}A^{-1}$, so AB is invertible and $B^{-1}A^{-1}$ is its inverse.

On the other hand, if AB is invertible, then $AB(AB)^{-1} = I$ and $(AB)^{-1}AB = I$. If A and B are square, this is enough to guarantee they are both invertible.

Determinants and products

Theorem

For square matrices A, B we have: $\det(A) \cdot \det(B) = \det(AB)$

Proof.

If any of these matrices is not invertible, then both sides are zero. If the matrices are invertible, then A and B can each be row reduced to the identity. This means we can write $A = E_1 \cdots E_a$ and $B = F_1 \cdots F_b$, where the E_i and F_j are elementary matrices. Then

$$\begin{aligned} \det(A) \cdot \det(B) &= \det(E_1 \cdots E_a) \cdot \det(F_1 \cdots F_b) \\ &= \det(E_1) \cdots \det(E_a) \cdot \det(F_1) \cdots \det(F_b) \\ &= \det(E_1 \cdots E_a F_1 \cdots F_b) \\ &= \det(AB) \end{aligned}$$

□