Hello and welcome to class!

Today

We talk about coordinates and change of basis.

Review: Spanning sets

Given a vector space V,
find a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$
such that every element of V
is a linear combination of the $\mathbf{v}_i$.

In other words, find a collection of vectors which span V.

Such a collection is called a spanning set.

Review: Linear indepedendence

Definition

Vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ in a vector space V are linearly indepedendent if none is a linear combination of the others.

Equivalently, if, whenever $\sum_i c_i \mathbf{v}_i = 0$ for some constants $c_i \in \mathbb{R}$, all the c_i must be zero.

Review: Bases

Definition

A subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ of a vector space V is a basis for V if it is linearly independent and spans V

Another way to write the definition: a subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ of a vector space V is a basis for V if there's one (spanning) and only one (linear independence) way to write any element $\mathbf{v} \in V$ as a linear combination of the $\mathbf{v}_i$.

Review: Bases

Example

$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ is a basis for $\mathbb{R}^3$

Example

$\{1, x, x^2\}$ is a basis for polynomials of degree at most 2 (aka $\mathbb{P}_2$)

Review: Linear transformations

Definition

If V and W are vector spaces, a function $T : V \to W$ is said to be a linear transformation if

$$T(c\mathbf{v} + c'\mathbf{v}') = cT(\mathbf{v}) + c'T(\mathbf{v}')$$

for all c, c' in $\mathbb{R}$ and all $\mathbf{v}, \mathbf{v}'$ in V.

Review: one-to-one and onto

A function $f : X \to Y$ is said to be:

- onto if every element of Y is f of ≥ 1 element of X.
- one-to-one if every element of Y is f of ≤ 1 element of X.

For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, we know that it is one-to-one if and only if the columns of the associated matrix are linearly independent, and onto if and only if the columns of the associated matrix span the codomain.

Review: the identity function

For a set X, there's a function from X to itself which does nothing.

$$\begin{aligned} id_X : X &\mapsto X \\ x &\mapsto x \end{aligned}$$

When X is a vector space, id_X is a linear transformation.

When $X = \mathbb{R}^n$, the matrix of id_X is the identity matrix.

Review: Invertibility

A function $f : X \to Y$ is invertible if there's some $g : Y \to X$ with

$$g \circ f = id_X \qquad f \circ g = id_Y$$

When X, Y are vector spaces and f, g are linear, the matrices of f and g are inverses; i.e., they multiply to the identity.

As with matrices, if f has an inverse, it's unique. We write it f^{-1}.

Review: Isomorphism

Invertible linear transformations are also called isomorphisms.

If there's an isomorphism $f : V \to W$, we say V and W are isomorphic vector spaces.

Isomorphic vector spaces look the same to linear algebra.

More precisely, any question which can be asked just in terms of operations which make sense in any vector space must have the same answer in both.

You use the isomorphism f to translate back and forth.

Review: Invertibility and bases

For a linear transformation $T : V \to W$, these are equivalent:

- T is one-to-one and onto
- T is invertible (its inverse is linear)
- T takes some basis of V to a basis of W.
- T takes any basis of V to a basis of W.

Bases, coordinates, parameterizations

Given a vector space V and a basis $\{\mathbf{v}_i\}$ of V, and a vector space W and any elements $\{\mathbf{w}_i\}$ of W there's a unique linear transformation

$$\begin{aligned} T : V &\to W \\ \mathbf{v}_i &\mapsto \mathbf{w}_i \\ \sum a_i \mathbf{v}_i &\mapsto \sum a_i \mathbf{w}_i \end{aligned}$$

Bases, coordinates, parameterizations

For any elements $\{\mathbf{w}_i\}$ of W, there's a linear transformation

$$\begin{aligned} T : \mathbb{R}^n &\to W \\ e_i &\mapsto \mathbf{w}_i \\ (a_1, a_2, \ldots, a_n) &\mapsto \sum a_i \mathbf{w}_i \end{aligned}$$

If the $\mathbf{w}_i$ form a basis, then the above map is an isomorphism.

This is called a parameterization of W. Its inverse map $W \to \mathbb{R}^n$ is called a choice of coordinates on W.

Example

The elements $\{1, x, \ldots, x^n\}$ give a basis of $\mathbb{P}_n$.

So $\mathbb{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$.

Bases, coordinates, parameterizations

A basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of a vector space V determines a choice of coordinates $V \xrightarrow{\sim} \mathbb{R}^n$.

For $\mathbf{v} \in V$, we write $[\mathbf{v}]_{\mathcal{B}}$ for the image of $\mathbf{v}$ under this map.

In other words, uniquely expanding $\mathbf{v} = \sum \beta_i \mathbf{v}_i$, we have

$$[\mathbf{v}]_{\mathcal{B}} = [\beta_1, \beta_2, \ldots, \beta_n]$$

Example

Consider the basis $\mathcal{B} = \{1, x+1, x^2+2x+1\}$ of $\mathbb{P}_2$.

Determine $[4x^2+3x+2]_{\mathcal{B}}$.

We are supposed to find c_1, c_2, c_3 such that

$$c_1 \cdot 1 + c_2(x+1) + c_3(x^2+2x+1) = 4x^2+3x+2$$

This gives a system of linear equations

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 2 \\ & & c_2 & + & 2c_3 & = & 3 \\ & & & & c_3 & = & 4 \end{array}$$

We solve to find $[4x^2+3x+2]_{\mathcal{B}} = [3, -5, 4]$.

Change of basis

A basis allows you to treat an arbitrary (finite dimensional) vector space V as if it were just $\mathbb{R}^n$.

Depending on the problem, one choice of basis may be more convenient than another.

We want to be able to "change bases" at will.

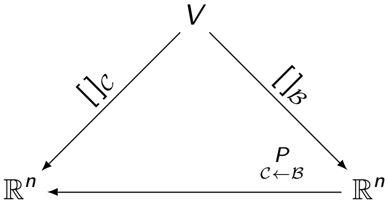
That is, given bases $\mathcal{B}$ and $\mathcal{C}$ of V, and given an expression $[\mathbf{v}]_{\mathcal{B}}$ in one of them, we would like to find the expression $[\mathbf{v}]_{\mathcal{C}}$ in the other.

Change of basis

We are looking for the transformation $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

In other words, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ completes the triangle of isomorphisms:



Change of basis

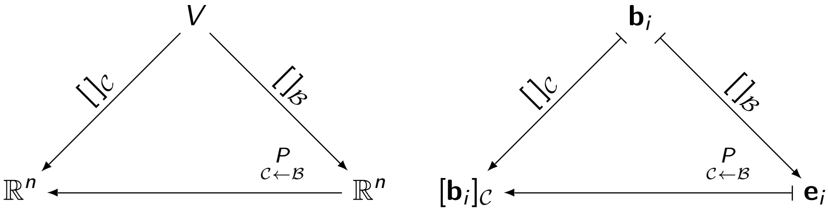
$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$$

Since it's a map from $\mathbb{R}^n \to \mathbb{R}^n$, we can describe $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ by a matrix.

To figure out which matrix, we should see where $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ sends the $\mathbf{e}_i$.

Change of basis

Say $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$.



Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}(\mathbf{e}_i) = [\mathbf{b}_i]_\mathcal{C}$.

In other words, the i'th column of the matrix of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is $[\mathbf{b}_i]_\mathcal{C}$.

Example

$\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+1, x^2+2x+1\}$ are bases of $\mathbb{P}_2$.

Write the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

The columns of $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ are $[1]_{\mathcal{B}}$, $[x+1]_{\mathcal{B}}$, $[x^2+2x+1]_{\mathcal{B}}$, so

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Try it yourself!

$\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+1, x^2+2x+1\}$ are bases of $\mathbb{P}_2$.

Write the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are $[1]_{\mathcal{C}}$, $[x]_{\mathcal{C}}$, $[x^2]_{\mathcal{C}}$, these work out to give

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x+1, x^2+2x+1\}$ are bases of $\mathbb{P}_2$.

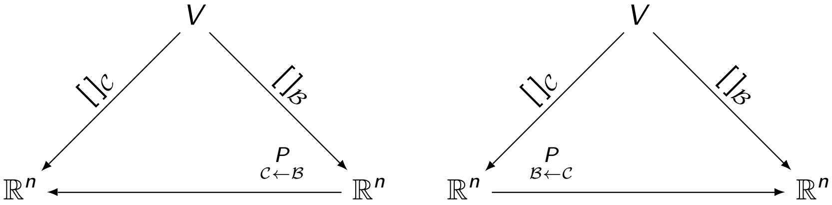
We saw

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Try multiplying these.

Change of basis

Changing back happens by the inverse:

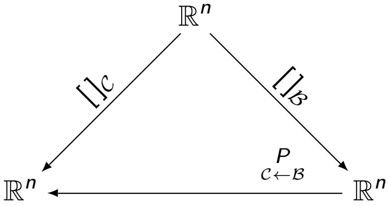


Said differently, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \mathrm{Identity}$, since this changes basis and then changes it back. So,

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

Change of basis in $\mathbb{R}^n$

Consider now the special case $V = \mathbb{R}^n$.



Now the maps $[\,]_{\mathcal{B}}$ and $[\,]_{\mathcal{C}}$ are themselves being change of bases, from the standard basis to $\mathcal{B}$ or $\mathcal{C}$, respectively.

Writing *std* for the standard basis, we have

$$[\,]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow std}{P} \qquad\qquad [\,]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow std}{P}$$

Change of basis in $\mathbb{R}^n$

$$[\,]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow std}{P} \qquad\qquad [\,]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow std}{P}$$

It is easy to write a formula for $\underset{std\leftarrow\mathcal{B}}{P}$: since $[\mathbf{v}]_{std} = \mathbf{v}$, the matrix for $\underset{std\leftarrow\mathcal{B}}{P}$ is $[\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n]$. Thus:

$$[\,]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow std}{P} = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n]^{-1}$$

$$[\,]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow std}{P} = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n]^{-1}$$

Finally we can compose these to find

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow std}{P} \cdot \underset{std\leftarrow\mathcal{B}}{P} = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n]^{-1}[\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n]$$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{B}\leftarrow std}{P} \cdot \underset{std\leftarrow\mathcal{C}}{P} = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n]^{-1}[\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n]$$

Example

Determine the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from the basis $\mathcal{B} = \{(1,2),(3,4)\}$ to the basis $\{(3,4),(5,7)\}$.

We just saw the formula

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow std}{P} \cdot \underset{std\leftarrow\mathcal{B}}{P} = [\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n]^{-1}[\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n]$$

In this case, that's

$$\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

The matrix of a linear transformation

If $\mathcal{B}$ is a basis in V and $\mathcal{C}$ is a basis in W, a linear transformation $T : V \to W$ is written in coordinates by the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ which completes the square

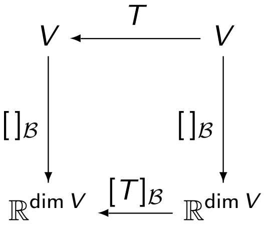
$$\begin{array}{ccc} W & \xleftarrow{\quad T \quad} & V \\ \Big\downarrow {\scriptstyle [\,]_{\mathcal{C}}} & & \Big\downarrow {\scriptstyle [\,]_{\mathcal{B}}} \\ \mathbb{R}^{\dim W} & \xleftarrow{{}_{\mathcal{C}}[T]_{\mathcal{B}}} & \mathbb{R}^{\dim V} \end{array}$$

For a new choice of basis $\mathcal{B}'$ of V and $\mathcal{C}'$ of W, we have

$$ {}_{\mathcal{C}'}[T]_{\mathcal{B}'} = \underset{\mathcal{C}' \leftarrow \mathcal{C}}{P} \; {}_{\mathcal{C}}[T]_{\mathcal{B}} \; \underset{\mathcal{B} \leftarrow \mathcal{B}'}{P} $$

The matrix of a linear transformation

In the special case when $V = W$ and $\mathcal{B} = \mathcal{C}$, we write just $[T]_{\mathcal{B}}$.

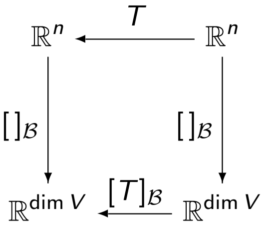


For a new choice of basis $\mathcal{B}'$ of V, we have

$$[T]_{\mathcal{B}'} = \underset{\mathcal{B}' \leftarrow \mathcal{B}}{P} [T]_{\mathcal{B}} \underset{\mathcal{B} \leftarrow \mathcal{B}'}{P} = \left(\underset{\mathcal{B} \leftarrow \mathcal{B}'}{P} \right)^{-1} [T]_{\mathcal{B}} \underset{\mathcal{B} \leftarrow \mathcal{B}'}{P}$$

Change of basis and conjugation

Finally, if $V = \mathbb{R}^n$,



we recall that $[\mathbf{v}]_{\mathcal{B}} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]^{-1} \cdot \mathbf{v}$ so

$$[T]_{\mathcal{B}} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]^{-1} \cdot T \cdot [\mathbf{b}_1, \ldots, \mathbf{b}_n]$$