Twisted Ruijsenaars model

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EIS Workshop, Berkeley

9-12 March, 2022

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1 Ruijsenaars model (trigonometric case)

$$H_k = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = k}} \left(\prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \right) T_I$$

Ruijsenaars '87

also known as Macdonald operators

$T_i : x_i \mapsto q x_i$

$$T_I = \prod_{i \in I} T_i$$

$H_1, H_2, \dots, H_n$

$[H_i, H_j] = 0$

quantum integrable system

$$H_1 = \sum_{i=1}^{n} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_i$$

② Main result

$\boxed{\begin{array}{l}\text{For } \ell \geq 2, \ \exists \text{ a completely integrable model with commuting} \\ \text{hamiltonians } \tilde{H}_1, \ldots, \tilde{H}_n \text{ such that} \\ \qquad\qquad \tilde{H}_k \approx (H_k)^\ell \end{array}}$

$\tilde{H}_k$ are quite complicated

First implicit construction given in

C.: Appendix in C.–Etingof '13

Example (C.-Fairon '17)

$$\ell = 2 : \quad \widetilde{H}_k \approx (H_k)^2$$

$$\widetilde{H}_1 = \sum_i a_i T_i^2 + \sum_{i<j} b_{ij} T_i T_j$$

$$a_i = \prod_{j \neq i} \frac{(t x_i - x_j)(q t x_i - x_j)}{(x_i - x_j)(q x_i - x_j)}$$

$$b_{ij} = \prod_{k \neq i,j} \frac{(t x_i - x_k)(t x_j - x_k)}{(x_i - x_k)(x_j - x_k)}$$

$$\times\, q^{1/2}(t-1)(t-q)\, \frac{(x_i + x_j)(x_i x_j)^{1/2}}{(x_i - q x_j)(x_j - q x_i)} \longleftrightarrow \boxed{-2qt(x_i^2 + x_j^2) + \big((q+t)(qt+1) + (q-t)^2\big) x_i x_j}$$

$\widetilde{H}_1$ twisted, $\ell = 2$ $\longleftrightarrow$ $(H_1)^2$ non-twisted

③ 1st approach: BA functions

Change notation: $x_i \rightsquigarrow q^{x_i}$

Key fact: For $t = q^{-m}$, $m \in \mathbb{Z}_+$

C. '02

$\exists$ common eigenfunction of $H_1, \dots, H_n$ of the form

True for all root systems

$$\psi(\lambda, x) = q^{\langle \lambda, x \rangle} \underbrace{P(q^{\lambda}, q^{x})}_{\text{polynomial}}$$

$$\langle \lambda, x \rangle = \sum_{i=1}^{n} \lambda_i x_i$$

$$q^{\lambda} = (q^{\lambda_1}, \dots, q^{\lambda_n})$$

$$q^{x} = (q^{x_1}, \dots, q^{x_n})$$

ψ is called BA function

non-symmetric

Cherednik-Macdonald-Mehta identity for BA function

$\psi(\lambda, x) \qquad \lambda, x \in \mathbb{C}^n = \mathbb{R}^n \oplus i\,\mathbb{R}^n$

THM (CE13): For large $\xi \in \mathbb{R}^n$,

$$\int_{\xi + i\mathbb{R}^n} \frac{\psi(\lambda,x)\psi(\mu,x)}{\Delta(x)\,\Delta(-x)}\, q^{-\frac{\langle x,x\rangle}{2}}\, dx = q^{\frac{\langle\lambda,\lambda\rangle+\langle\mu,\mu\rangle}{2}}\,\psi(\lambda,\mu).$$

$$\Delta(x) = \prod_{i<j}\prod_{k=1}^{m}\left(q^{-k}q^{x_i} - q^{x_j}\right)$$

ξ is large if $|\xi_i - \xi_{i+1}| > m$

Twisted BA function $\tilde\psi$

Given $\ell \in \mathbb{N}$, define $\tilde\psi(\lambda, x)$ by

$$\int_{\xi + i\mathbb{R}^n} \frac{\psi(\lambda,x)\psi(\mu,x)}{\Delta(x)\,\Delta(-x)}\, q^{-\ell\frac{\langle x,x\rangle}{2}}\, dx = q^{\frac{\langle\lambda,\lambda\rangle+\langle\mu,\mu\rangle}{2\ell}}\,\tilde\psi(\lambda,\mu)$$

Then (C'13):

- $\tilde\psi(\lambda,x) \in q^{\frac{\langle\lambda,x\rangle}{\ell}}\, \underbrace{\tilde P\left(q^{\frac{\lambda}{\ell}}, q^{\frac{x}{\ell}}\right)}_{\text{polynomial}}$
- $\tilde\psi$ is a joint eigenfunction of $\tilde H_i \approx (H_i)^\ell \qquad i = 1 \ldots n$

④ 2nd approach : Hamiltonian reduction

(a) Rujsenaars- Schneider model

$$\begin{tikzcd} \mathbb{C} \arrow[r, shift left, "v"] & \mathbb{C}^n \arrow[l, shift left, "w"] \arrow[loop above, "X"] \arrow[loop below, "Y"] \end{tikzcd}$$

$X, Y \in \mathrm{Mat}_{n,n}(\mathbb{C})$

$v \in \mathrm{Mat}_{1,n}$

$w \in \mathrm{Mat}_{n,1}$

$$\mathcal{M} = \{ (X, Y, v, w) \} \cong \mathbb{C}^{2n^2+2n}$$

$$\cong T^* \mathbb{C}^{n^2+n}$$

$(\mathcal{M}, \omega, G, \mu)$ Hamiltonian G-space

GL_n moment map

$$\mu = XY - YX + vw$$

$$C_n = \mu^{-1}(I_n) /\!/ G$$

$$\dim C_n = 2n$$

[KKS'78]: Rational CM system by Hamiltonian reduction on $\mathcal{M}$.

Fock–Rosly '99: To get RS system, one needs to choose different Poisson structure on $\mathcal{M}$.

Alternatively, use quasi-Poisson picture:

$$(\mathcal{M}, \Pi, G, \Phi)$$

quasi-Poisson bivector

GL_n

$\Phi : \mathcal{M} \to G$ multiplicative moment map

$$\Phi = X^{-1} Y^{-1} X Y (I_n + vw)$$

$$C_{n,t} = \Phi^{-1}(t I_n) /\!\!/ G$$

Poisson variety of dim $= 2n$

This is a particular example of a multiplicative quiver variety

Crawley-Boevey, Shaw '05

Van den Bergh '08

$$h_i = \operatorname{tr} Y^i \qquad i = 1 \dots n$$

These are G-invariant so well-defined on $C_{n,t}$

$$\{h_i, h_j\} = 0$$

To see connection to RS system, assume X is diagonalisable. Then

$$X = \operatorname{diag}(x_1, \dots, x_n)$$

$$Y = (Y_{ij}), \quad v = \dots, w = \dots.$$

$$Y_{ij} = \frac{t-1}{t - x_i/x_j} \prod_{k \neq j} \frac{t x_j - x_k}{x_j - x_k} \sigma_j$$

$$h_1 = \operatorname{tr} Y = \sum_i \prod_{k \neq i} \frac{t x_i - x_k}{x_i - x_k} \sigma_i$$

$$\{x_i, x_j\} = \{\sigma_i, \sigma_j\} = 0$$

$$\{x_i, \sigma_j\} = \delta_{ij} x_i \sigma_j$$

Ruijsenaars-Schneider '86
Y = Lax matrix for RS

(b) Cyclic quiver generalisation
(C.–Fairon '17)

$$
\begin{tikzcd}
 & & \mathbb{C}^n \arrow[dl, shift left, "X_1"] \arrow[dr, shift left, "Y_2"] & \\
\mathbb{C} \arrow[r, shift left, "v"] & \mathbb{C}^n \arrow[l, shift left, "w"] \arrow[ur, shift left, "Y_1"] \arrow[dr, shift left] & & \mathbb{C}^n \arrow[ul, shift left, "X_2"] \arrow[dl, shift left] \\
 & & \mathbb{C}^n \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}
$$

$\ell = 4$

$\mathcal{M} \cong \mathbb{C}^{2\ell n^2 + 2n}$

$G = (GL_n)^\ell$ acts on $\mathcal{M}$

$(\mathcal{M}, \Pi, \Phi)$ quasi-Ham. G-space

$\Phi : \mathcal{M} \to G$
mult. moment map

$\Phi = (\varphi_1, \dots, \varphi_\ell)$

$\varphi_1 = (Y_\ell X_\ell)^{-1} X_1 Y_1 (1_n + vw)$

$\varphi_2 = (Y_1 X_1)^{-1} X_2 Y_2$

$\dots$

$\varphi_\ell = (Y_{\ell-1} X_{\ell-1})^{-1} X_\ell Y_\ell$

$C^{(\ell)}_{n,t} = \Phi^{-1}(t_1 I_n, \dots, t_\ell I_n) /\!/ G$

$t_1, \dots, t_\ell \neq 0$ arbitrary

$t := t_1 \dots t_\ell$

$\dim C^{(\ell)}_{n,t} = 2n$ smooth symplectic

$$\tilde{h}_k = \mathrm{tr}\,(Y_\ell \cdots Y_1)^k \qquad k = 1 \ldots n$$

$$\{\tilde{h}_i, \tilde{h}_j\} = 0 \quad \Rightarrow \quad IS$$

Next, modulo G-action,

$$X_1 = \ldots = X_{\ell-1} = I_n, \quad X_\ell = X$$

$$Y_r = (t_1 \ldots t_r)\, Y \qquad r = 1 \ldots \ell-1$$

$$Y_\ell = t X^{-1} Y$$

where X, Y satisfy

$$X^{-1} Y^{-1} X Y (1 + vw) = t I_n$$

The same phase space as for RS system.

$$\boxed{\tilde{h}_k = \mathrm{tr}(Y_\ell \ldots Y_1)^k \simeq \mathrm{tr}(X^{-1} Y^\ell)^k}$$

up to constant factor

Thus, on the same space $C_{n,t}$ we have RS system with $h_k = \mathrm{tr}\, Y^k$, and a twisted RS system with $\tilde{h}_k = \mathrm{tr}\,(X^{-1} Y^\ell)^k$.

Remark : On $C_{n,t}$

$$\widetilde{H}_k = \mathrm{tr}(X^{-1}Y^{\ell})^k \longleftrightarrow \widetilde{G}_k := \mathrm{tr}(X^{-\ell}Y)^k$$

under Ruijsenaars duality

complicated

simple

Same is true at quantum level.

E.g. $$\widetilde{G}_1 = \sum_i x_i^{-\ell} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_i$$

(cf. Di Francesco–Kedem '17)

5 3rd approach : via DAHA

(Braverman - Etingof - Finkelberg 2016)

This allows to connect 1st and 2nd approaches

$\mathbb{H}_{q,t}$ DAHA (GL_n-type)

It contains:

- $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$
- $\mathbb{C}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$
- Hecke algebra of S_n

$$\begin{cases} T_i \quad i = 1 \dots n-1 \\ (T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \\ + \text{ braid relations} \end{cases}$$

(plus relations between X_i, Y_i, T_i)

$\mathbb{H}_{q,t}$ has a representation by difference - reflection operators in var's $X_1, \dots, X_n$:

$$X_i \mapsto X_i$$

$$T_i \mapsto t^{1/2} s_i + \frac{t^{1/2} - t^{-1/2}}{X_i / X_{i+1} - 1}(s_i - 1)$$

where $s_i = \sigma_{i,i+1}$

$$Y_i \mapsto T_i \dots T_n \pi \, T_1^{-1} \dots T_{i-1}^{-1}$$

where

$$\pi : (X_1, \dots, X_n) \mapsto (X_2, \dots, X_n, q X_1)$$

In this representation, symmetric comb. of Y_i after restriction onto $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$ give Ruijsenaars hamiltonians. (aka Macdonald operators)

BEF'16 define cyclotomic DAHA

$\mathbb{H}^{\ell}_{q,t} \subset \mathbb{H}_{q,t}$ (skip $\mathrm{def}^{\underline{n}}$)

Part of their results
(relevant to this story):

1. Define $D_i^{(\ell)}$ $(i=1 \dots n)$ by

$$D_i^{(\ell)} = T_{i-1}^{-1} \dots T_1^{-1} \underbrace{X_1^{-1} Y_1^{\ell}}\, T_1^{-1} \dots T_{i-1}^{-1}$$

Then these elements commute.

2. Symmetric combinations of $D_1^{(\ell)}, \dots, D_n^{(\ell)}$ after restriction onto $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}$ give quantum hamiltonians $\tilde{H}_1, \dots, \tilde{H}_n$ from 1st approach.

3. Classical limits of $\tilde{H}_i$ coincide with $\tilde{h}_i$ from cyclic quiver story.

⑥ 4th approach : via elliptic Y's

(C. : in preparation)

$$H_1 = \sum_i \prod_{j \neq i} \sigma_\mu(x_i - x_j)\, e^{\beta\, \partial/\partial x_i}$$

$$\sigma_\mu(z) = \frac{\theta(z-\mu)\,\theta'(0)}{\theta(z)\,\theta(-\mu)}$$

$\theta = \theta_1(z \mid \tau)$ odd θ-function

Ruijsenaars '87 : $[H_i, H_j] = 0$

H_k can be constructed via elliptic Cherednik operators (Komori–Hikami '98)

Trig. limit : $\tau \to +i\infty$

$$t = e^{\mu} \qquad q = e^{\beta}$$

$$R_{ij} = \sigma_\mu(x_i - x_j) - \sigma_{\xi_i - \xi_j}(x_i - x_j) s_{ij}$$

Shibukawa – Ueno '92

$$\boxed{R_{ij} R_{ik} R_{jk} = R_{jk} R_{ik} R_{ij}}$$

$i \neq j \neq k$

$\xi = (\xi_1, \dots, \xi_n)$

ξ "spectral" variables

$$Y_1 = R_{12} R_{13} \dots R_{1n} e^{\beta \partial / \partial x_1}$$

[KH'98]: Let $\rho = \frac{1}{2} \sum_{i<j} e_i - e_j$.

Then: for $\xi = -\mu\rho$ we have

$Y_1 : \mathbb{C}(x)^{S_n} \longrightarrow \mathbb{C}(x)^{S_n}$ and

$Y_1 |_{\mathbb{C}(x)^{S_n}} = H_1$.

Similar construction works for other root systems, see Komori-Hikami'98

Higher H_k constructed similarly.

$Y_1 = Y_1(\xi)$ as above, $\xi \in \mathbb{C}^n$. Given $\ell \geq 2$ and c s.t. $\ell c = a + b\tau$, $a, b \in \mathbb{Z}$, define twisted Y_1 as

$$\tilde{Y}_1 = Y_1(\xi)\, Y_1(\xi + c e_1) \ldots Y_1(\xi + (\ell-1) c e_1)\, e^{-2\pi i b x_1}$$

THM: For $\xi = -\mu\rho$, $\tilde{Y}_1$ preserves $\mathbb{C}(x)^{S_n}$ and

$\tilde{H}_1 := \tilde{Y}_1 \Big|_{\mathbb{C}(x)^{S_n}}$ is integrable, i.e.

$\exists\ \tilde{H}_k$ $k = 1 \ldots n$, $[\tilde{H}_i, \tilde{H}_j] = 0$.

Trig. limit: Take $c = \frac{\tau}{\ell}$, $\tau \to \infty$

Then $\tilde{Y}_1 \longrightarrow (Y_1)^\ell X_1^{-1}$

Higher $\tilde{H}_k$ constructed similarly:

$$\omega_k = e_1 + \dots + e_k$$

$$Y_{\omega_k} = \left(R_{k,k+1} \dots R_{1,k+1}\right)\left(R_{k,k+2} \dots R_{1,k+2}\right) \dots$$
$$\times \left(R_{k,n} \dots R_{1,n}\right) \exp\left(\beta \sum_{i=1}^{k} \frac{\partial}{\partial x_i}\right)$$

$$\tilde{Y}_{\omega_k} = Y_{\omega_k}(\xi) Y_{\omega_k}(\xi + c\omega_k) \dots Y_{\omega_k}(\xi + (\ell-1)c\omega_k)\, e^{-2\pi i b \langle \omega_k, x \rangle}$$

Set $\xi = -\mu\rho$ and $\tilde{H}_k = \tilde{Y}_{\omega_k} \Big|_{\mathbb{C}(x)^{S_n}}$.

This also works for other root system (e.g. van Diejen system)

THANKS!