The Diamond of Integrability

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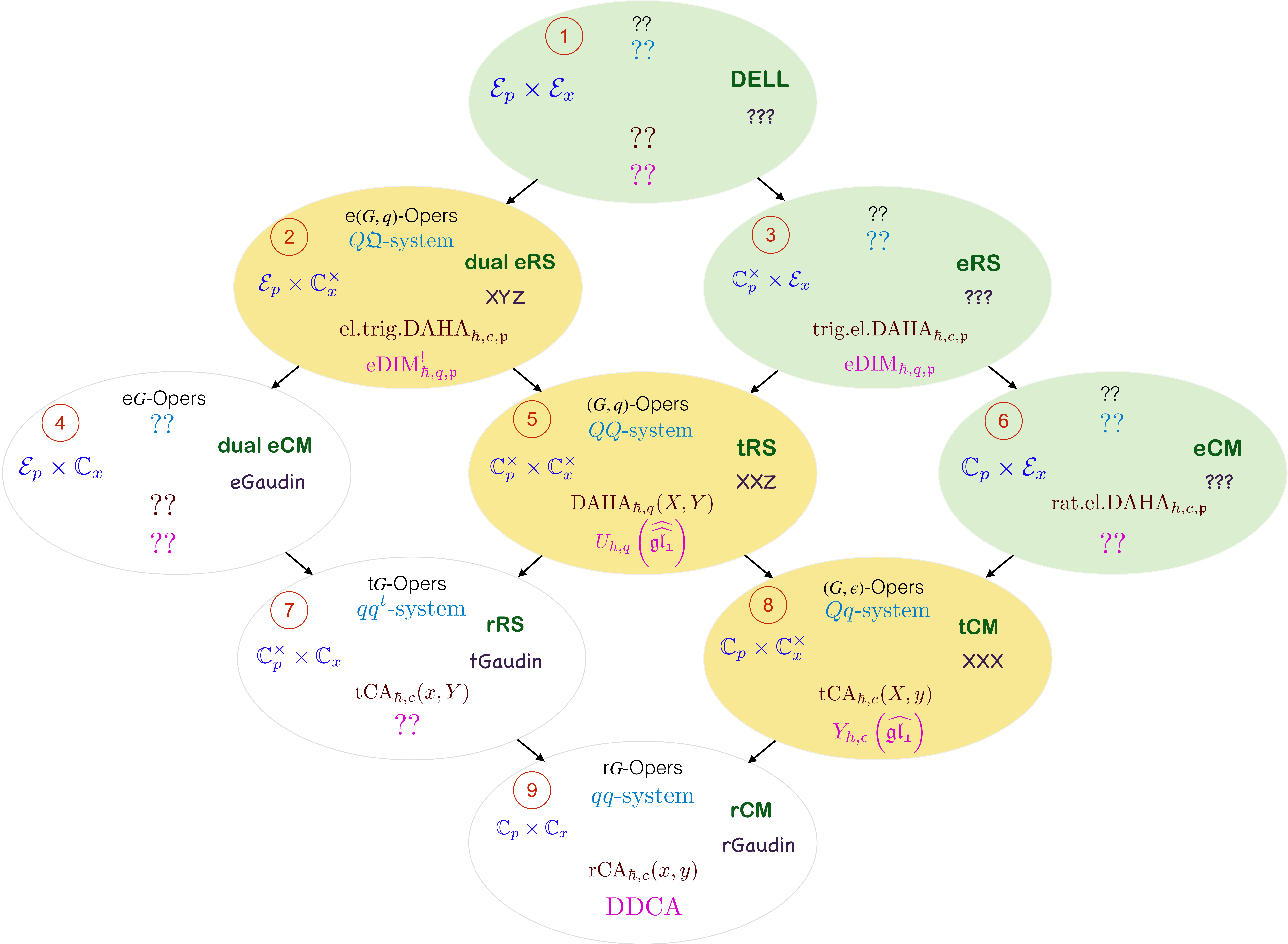

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$\mathcal{E}_p \times \mathcal{E}_x$
DELL
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2
e(G,q)-Opers
$Q\mathfrak{Q}$-system
dual eRS
$\mathcal{E}_p \times \mathbb{C}_x^\times$
XYZ
el.trig.DAHA$_{\hbar,c,\mathfrak{p}}$
eDIM$^!_{\hbar,q,\mathfrak{p}}$
3
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eRS
$\mathbb{C}_p^\times \times \mathcal{E}_x$
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trig.el.DAHA$_{\hbar,c,\mathfrak{p}}$
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4
eG-Opers
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dual eCM
$\mathcal{E}_p \times \mathbb{C}_x$
eGaudin
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5
(G,q)-Opers
QQ-system
tRS
$\mathbb{C}_p^\times \times \mathbb{C}_x^\times$
XXZ
DAHA$_{\hbar,q}(X,Y)$
$U_{\hbar,q}\left(\widehat{\widehat{\mathfrak{gl}}}_1\right)$
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7
tG-Opers
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8
(G,ϵ)-Opers
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tCA$_{\hbar,c}(X,y)$
$Y_{\hbar,\epsilon}\left(\widehat{\mathfrak{gl}}_1\right)$
9
rG-Opers
qq-system
rCM
$\mathbb{C}_p \times \mathbb{C}_x$
rGaudin
rCA$_{\hbar,c}(x,y)$
DDCA

The Diamond

- I shall review the current status of integrable models of the Calogero-Ruijsenaars family in the context of recent developments in algebraic geometry and representation theory. This study is strongly motivated by progress in string theory and *supersymmetric quantum field theories* beginning in the early 90s [Seiberg Witten, circa 1994]
- One particular theory stands out — the $\mathcal{N} = 2*$ theory in four dimensions. It has few parameters — the gauge coupling, the gauge group type, and SUSY breaking mass
- Relative simplicity of the parameter space provides a perfect playground for mathematicians — algebraic geometers, representation theorist, and, more recently, even number theorists.

Legend

Type of Oper

Equation on Baxter functions

Many-body system

Phase Space

Spin chain

Algebra for n particles

Algerba for $n \to \infty$

Classical Integrability

Equations of motion

$$\frac{df}{dt} = \{H_1, f\}$$

Integrability — family of n conserved quantities which Poisson commute with each other

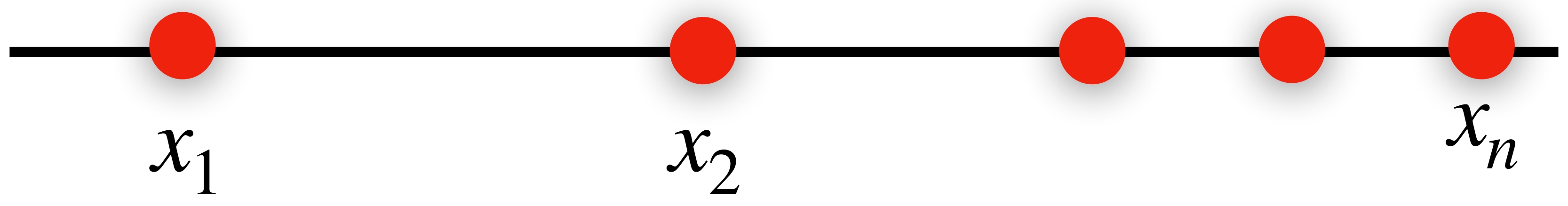
$$\{H_i, H_j\} = 0 \quad i, j = 1, \ldots, n$$

Liouville-Arnold Theorem   Compact Lagrangians $\mathscr{L}$: $\{H_i = E_i\}$ are isomorphic to tori

Evolution in the neighborhood of $\mathscr{L}$ is linearized in action/angle variables

Calogero in 1971 introduced a new integrable system. Moser in 1975 proved its integrability using Lax pair

$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$



rational CM → trigonometric CM → elliptic CM

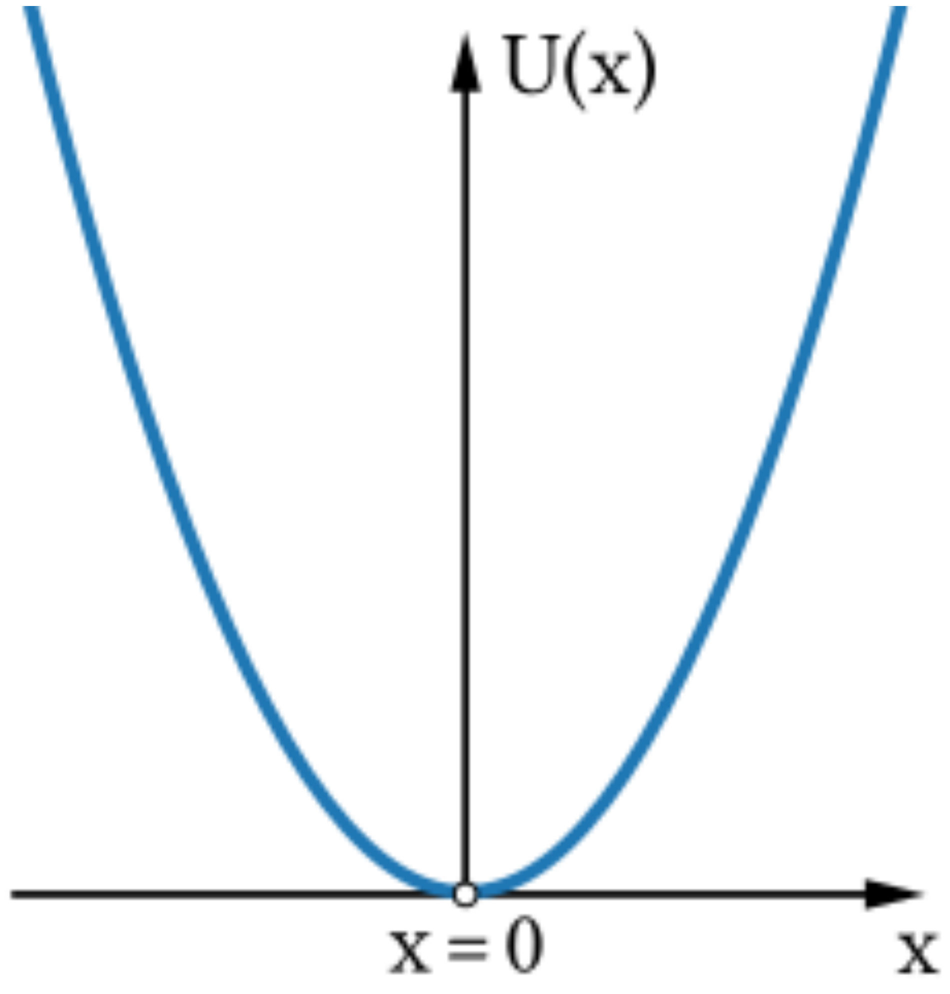
Another relativistic generalization called **Ruijsenaars-Schneider (RS)** family

rRS → tRS → eRS

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

Symplectic Manifold

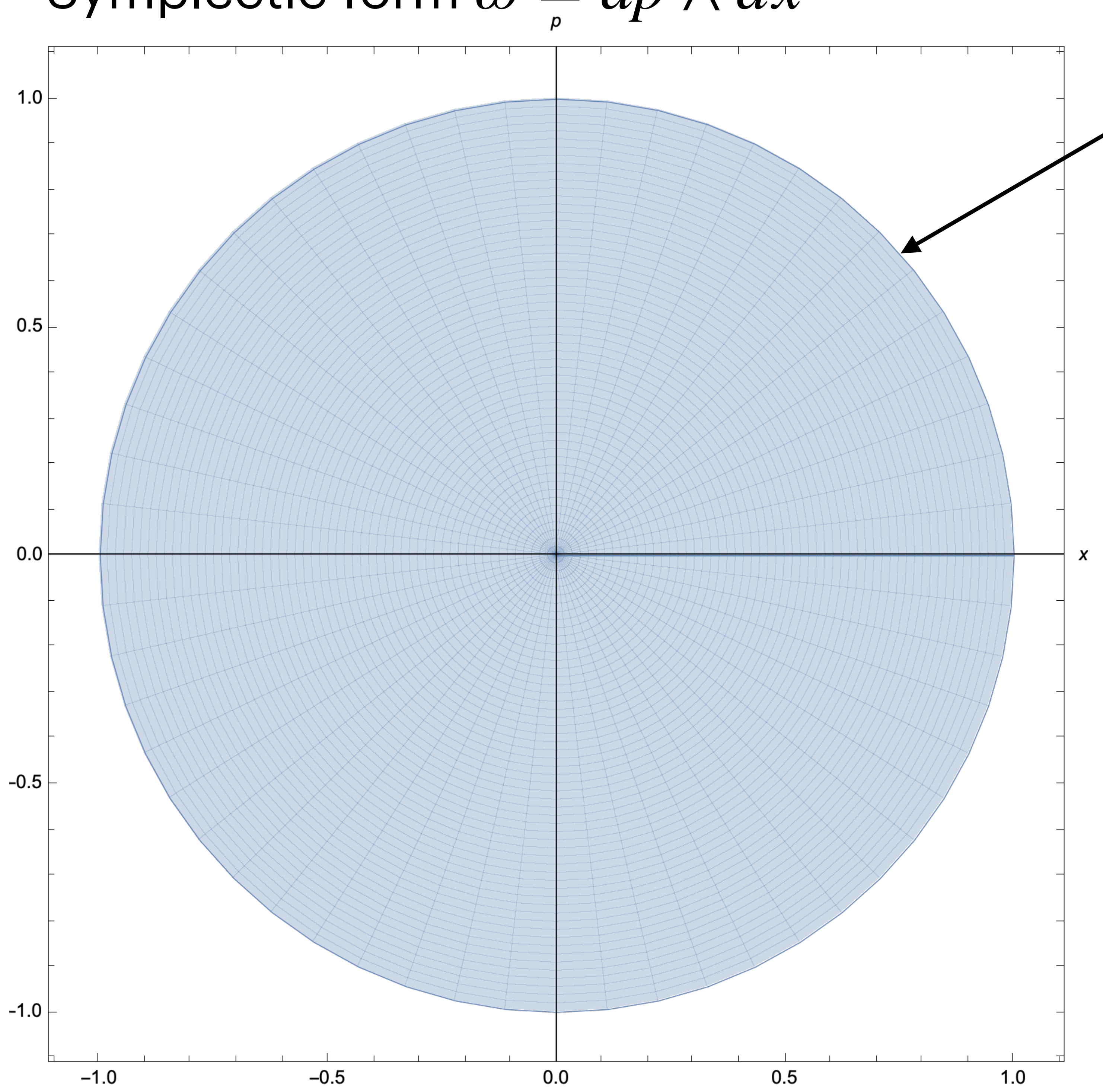
Harmonic oscillator $H = \frac{p^2}{2} + \frac{x^2}{2}$



Phase space — symplectic manifold $\mathscr{M}$

Symplectic form $\omega = dp \wedge dx$

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$



Lagrangian $\mathscr{L} \subset \mathscr{M}$ is a middle-dimensional submanifold and such that the restriction of the symplectic form on $\mathscr{L}$ vanishes

$$\omega|_{\mathcal{L}} = 0$$

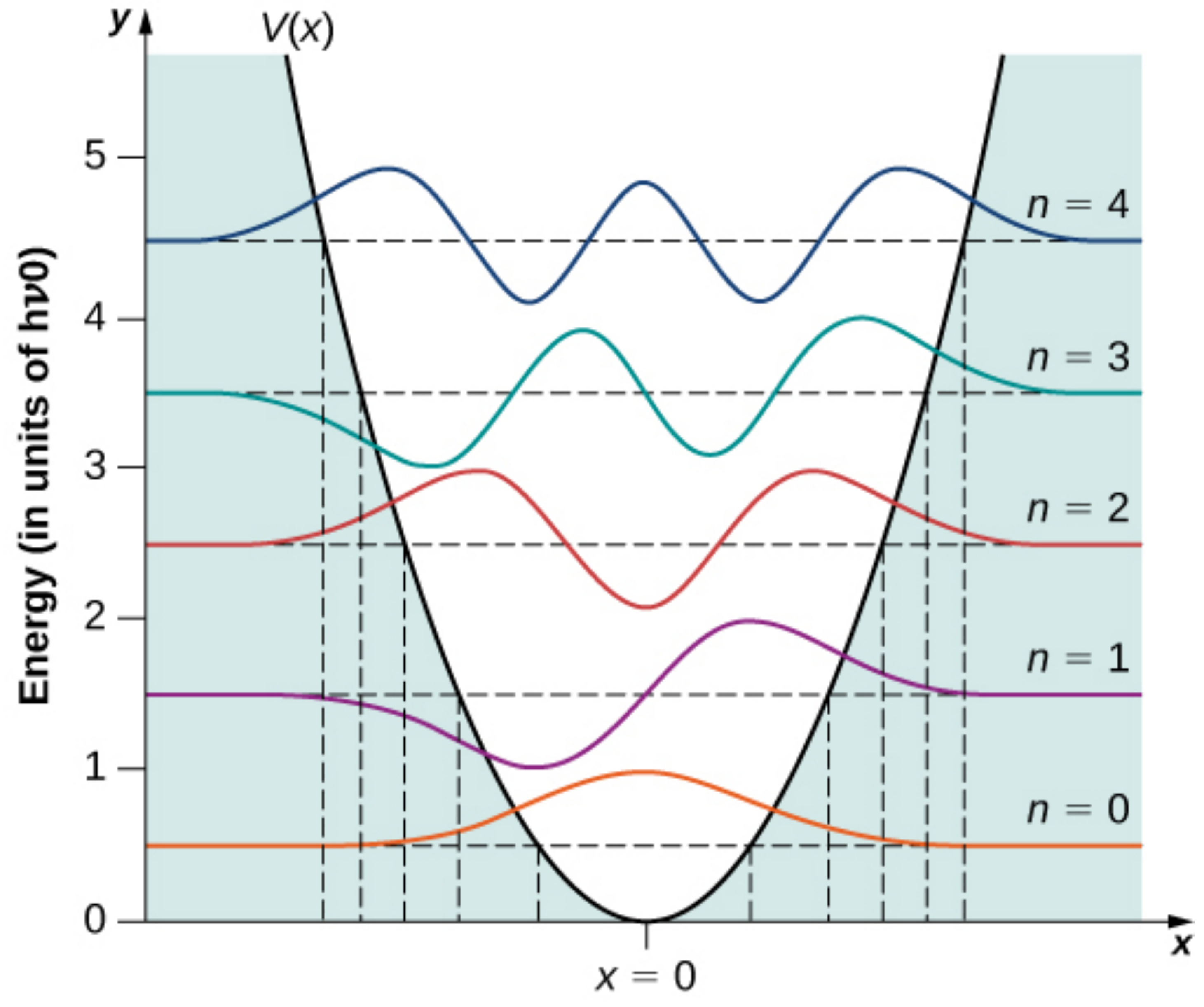
Symplectic form ω is locally exact on $\mathscr{L}$

$$\theta = d^{-1}\omega = pdx$$

Quantization as Symplectic Geometry

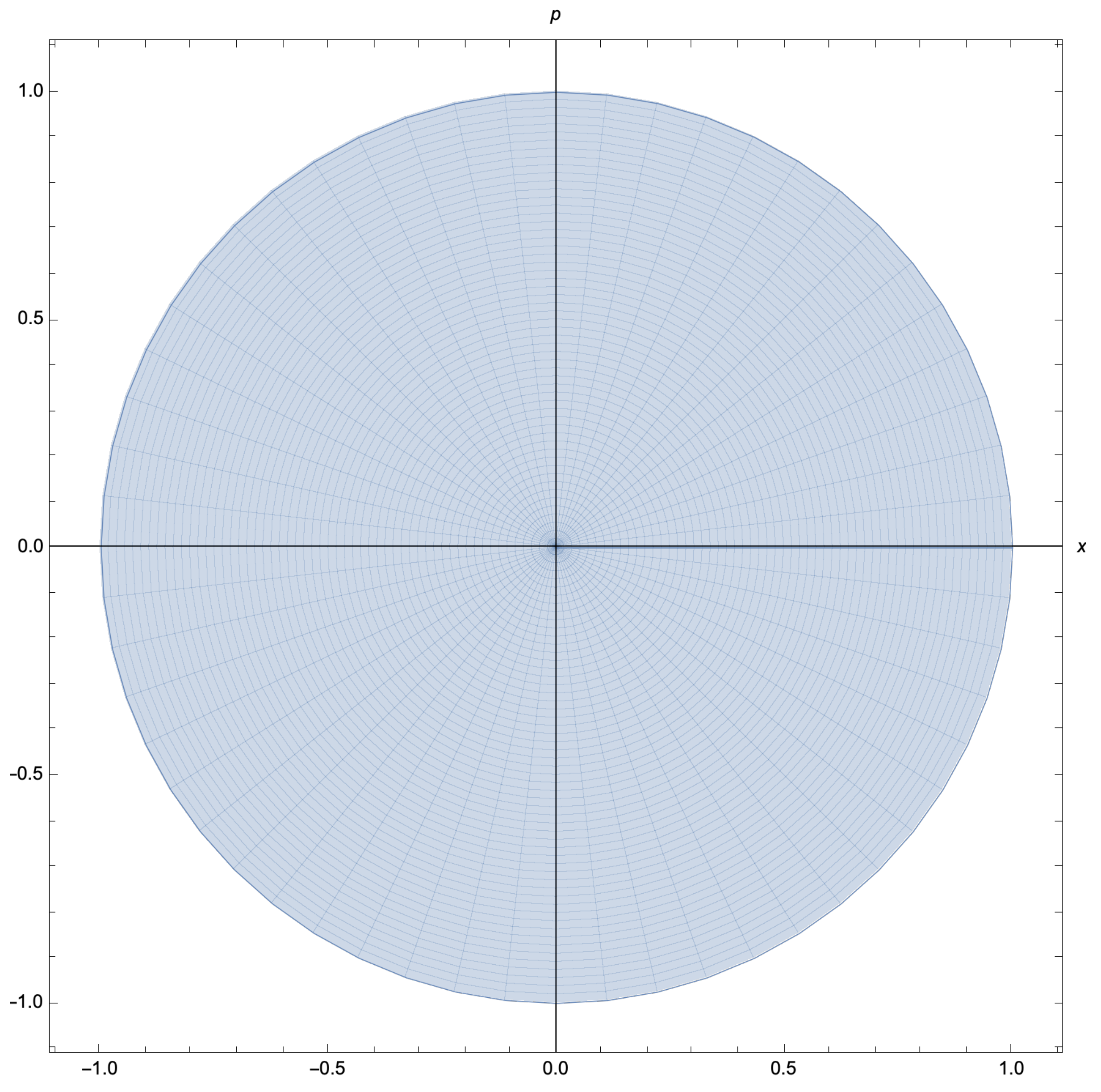
Quantum oscillator energy states

$$E_n = \hbar\left(n + \frac{1}{2}\right)$$



Symplectic area

$$E_n = \frac{1}{2\pi}\int dp \wedge dx \sim \oint_{\mathcal{L}} \theta$$



Quantization

Coordinates and momenta become operators

$$p, x \mapsto \hat{p}, \hat{x}$$

Poisson brackets associated to ω become commutators

$$\{A, B\}_{P.B.} \mapsto [A, B]$$

Heisenberg algebra

$$[\hat{p}, \hat{x}] = -i\hbar$$

$$\hat{x} f(x) = x f(x)$$
$$\hat{p} f(x) = -i\hbar f'(x)$$

Lagrangian constraint

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$

Replaced by operator

$$\left(\frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} - E\right) Z(x) = 0$$

This ODE has square integrable solutions only for special values of E

$$E_n = \hbar\left(n + \frac{1}{2}\right)$$

e.g. for $n = 0$ $\quad Z(x) \sim e^{-\frac{1}{2\hbar}x^2}$

The Art of Quantization

Symplectic manifold $(\mathscr{M}, \omega)$ ⟶ Hilbert space $\mathscr{H}$ — Representations

Algebra of functions on $\mathscr{M}$ ⟶ Algebra of operators on $\mathscr{H}$ — Algebra (i.e. DAHA)

Lagrangian submanifolds $\mathscr{L} \subset \mathscr{M}$ ⟶ States in Hilbert space $\mathscr{H}$ — Highest weight vectors

$\{f_i\}$ $\qquad$ $\hat{f}_i \, \mathcal{Z} = 0$

Trigonometric-Trigonometric

All ingredients are known in the middle of the diamond

5

(G, q)-Opers

QQ-system

tRS

$\mathbb{C}_p^\times \times \mathbb{C}_x^\times$

XXZ

$\mathrm{DAHA}_{\hbar,q}(X, Y)$

$U_{\hbar,q}\left(\widehat{\widehat{\mathfrak{gl}_1}}\right)$

Double Affine Hecke Algebra of Rank One

Let $\mathfrak{g}$ be Lie algebra. The (Iwahori)-**Hecke** algebra is defined as the deformation of the group algebra of the Weyl group of $\mathfrak{g}$

For $\mathfrak{sl}(2)$ it is generated by T with relation $(T-t)(T+t^{-1})=0$ where $t \in \mathbb{C}^{\times}$

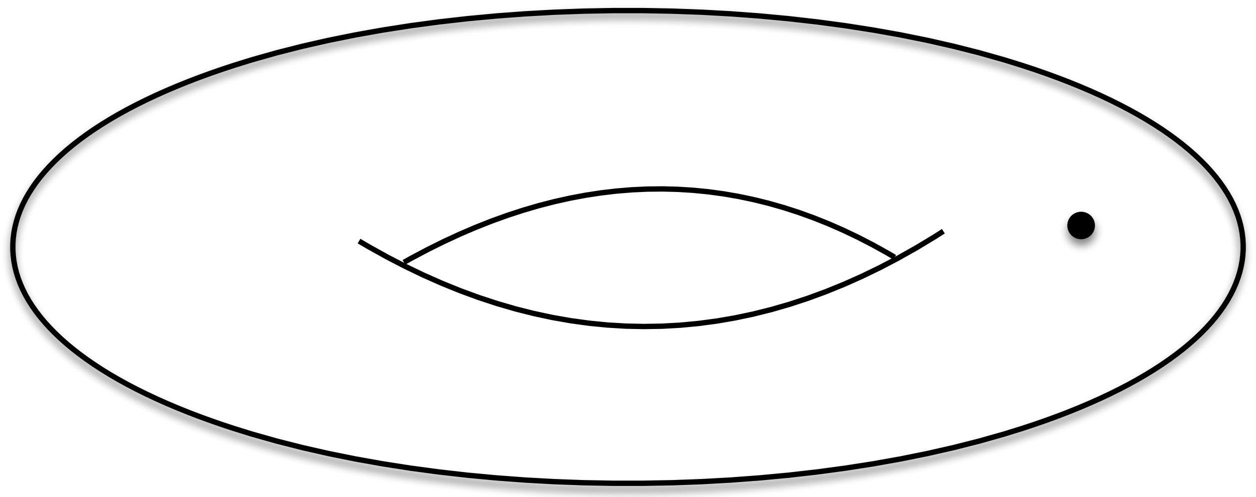
Affine Hecke algebra (AHA) for $\mathfrak{sl}(2)$:

$$\frac{\mathbb{C}(t^{\pm 1}) \otimes \mathbb{C}[X^{\pm 1}, T]}{\left(TXT - X^{-1}, (T-t)(T-t^{-1})\right)}$$

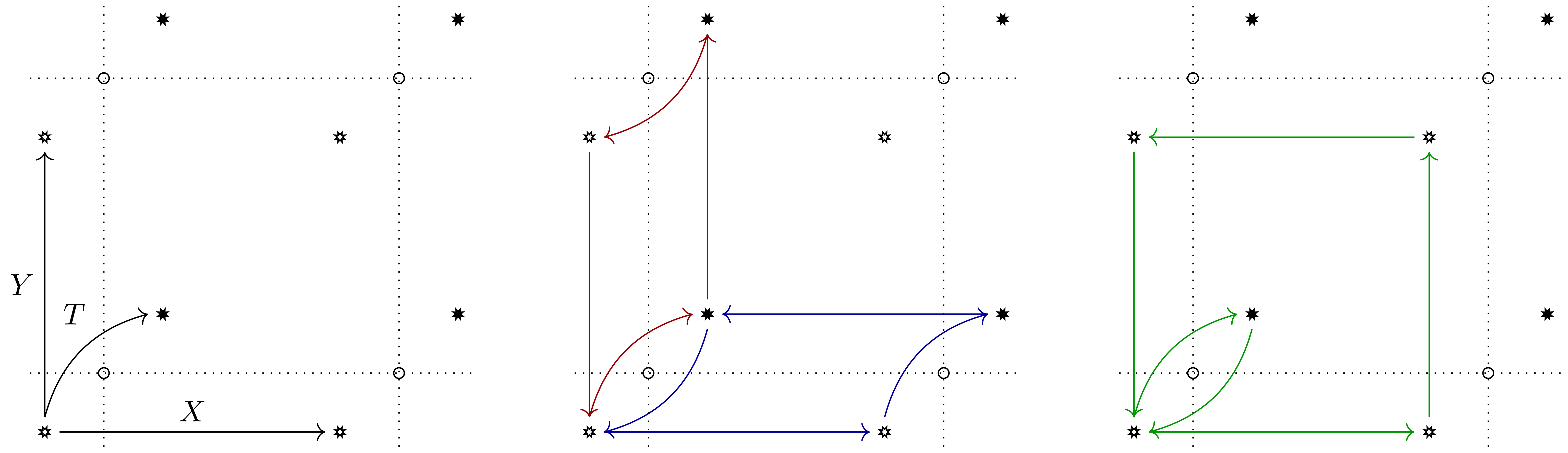
Double affine Hecke algebra for $\mathfrak{sl}(2)$ — two copies of AHA (X,T) and (Y,T) in the presence of additional relation and parameter $q \in \mathbb{C}^{\times}$

$$\ddot{H}(\mathbb{Z}_2) = \frac{\mathbb{C}(q^{\pm 1}, t^{\pm 1}) \otimes \mathbb{C}[X^{\pm 1}, Y^{\pm 1}, T]}{\left(TXT - X^{-1}, TYT - Y^{-1}, Y^{-1}X^{-1}YX - q^{-1}, (T-t)(T+t^{-1})\right)}$$

DAHA from Affine Braid Group



Orbifold fundamental group
of the torus with puncture $(T^2\backslash p)/\mathbb{Z}_2$



(a) (b) (c)

Generated by X, T, Y modulo relations $TXT = X^{-1}$, $TY^{-1}T = Y$, and $Y^{-1}X^{-1}YXT^2 = 1$

Its central extension is known as elliptic braid group is obtained by deforming the last relation to

$$Y^{-1}X^{-1}YXT^2 = q^{-1}$$

The full $\mathfrak{sl}(2)$ DAHA is obtained by imposing Hecke relation

$$\ddot{H}(\mathbb{Z}_2) = \mathbb{C}_{q,t}\left[T^{\pm 1}, X^{\pm 1}, Y^{\pm 1}\right] \Big/ \left\{ \begin{array}{ll} TXT = X^{-1}\ , & Y^{-1}X^{-1}YXT^2 = q^{-1}\ , \\ TY^{-1}T = Y\ , & (T-t)(T+t^{-1}) = 0 \end{array} \right\}$$

Spherical DAHA

Idempotent element $\mathbf{e} = (T + t^{-1})/(t + t^{-1})$

Spherical subalgebra $S\ddot{H} := \mathbf{e}\ddot{H}\mathbf{e}$

Generators of spherical DAHA

$$x = X + X^{-1}$$
$$y = Y + Y^{-1}$$
$$z = q^{-\frac{1}{2}} Y^{1} X + q^{\frac{1}{2}} X^{-1} Y$$

q-commutator

$$[a, b]_q := q^{-\frac{1}{2}} ab - q^{\frac{1}{2}} ba$$

Relations

$$[x, y]_q = (q^{-1} - q) z$$
$$[y, z]_q = (q^{-1} - q) x$$
$$[z, x]_q = (q^{-1} - q) y$$
$$q^{-1} x^2 + q y^2 + q^{-1} z^2 - q^{-\frac{1}{2}} xyz = (q^{-\frac{1}{2}} t - q^{\frac{1}{2}} t^{-1})^2 + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2$$

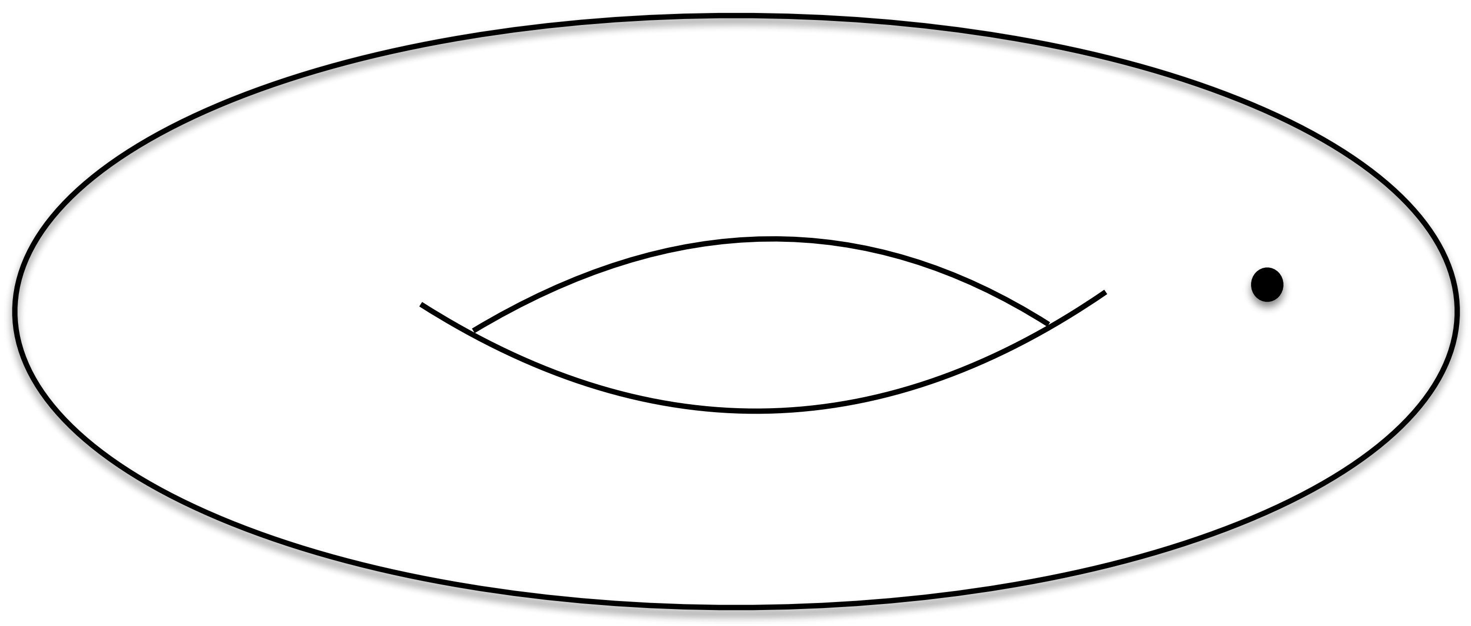
\`Classical' limit

$$S\ddot{H} \xrightarrow[q \to 1]{} \mathscr{O}(\mathcal{M}_{\text{flat}}(C_p, \mathrm{SL}(2, \mathbb{C})))$$

Coordinate ring of the moduli space of $SL(2,\mathbb{C})$ flat connections on punctured torus

$$\mathcal{M}_{\text{flat}}(C_p, \mathrm{SL}(2, \mathbb{C})) = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^2 + z^2 - xyz - 2 = \mathrm{Tr}(\rho(\mathfrak{c})) = \tilde{t}^2 + \tilde{t}^{-2}\}$$

SL(2,$\mathbb{C}$) Flat Connection on Punctured Torus



Fundamental group $\pi_1(C_p) = \langle \mathfrak{m}, \mathfrak{l}, \mathfrak{c} | \mathfrak{m}\mathfrak{l}\mathfrak{m}^{-1}\mathfrak{l}^{-1} = \mathfrak{c} \rangle$

Let $\rho : \pi_1(C_p) \to \mathrm{SL}(2, \mathbb{C})$

$x = \mathrm{Tr}(\rho(\mathfrak{m})),\ y = \mathrm{Tr}(\rho(\mathfrak{l})),\ \text{and}\ z = \mathrm{Tr}\big(\rho(\mathfrak{m}\mathfrak{l}^{-1})\big)$

Markov cubic $\mathcal{M}_{\text{flat}}(C_p, \mathrm{SL}(2, \mathbb{C})) = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^2 + z^2 - xyz - 2 = \mathrm{Tr}(\rho(\mathfrak{c})) = \tilde{t}^2 + \tilde{t}^{-2}\}$

Elliptic fibration of Kodaira type I_0^*

[Oblomkov]

Theorem. Spherical DAHA is a ***deformation quantization*** of the coordinate ring of the moduli space of flat $SL(2,\mathbb{C})$ connections $\mathfrak{X} = \mathscr{M}_{\text{flat}}(C_p, SL(2,\mathbb{C}))$ with respect to Poisson structure Ω_J

$$\Omega_J = \frac{1}{2\pi i}\frac{dx \wedge dy}{\partial f/\partial z} = \frac{1}{2\pi i}\frac{dx \wedge dy}{2z - xy}$$

1) Representations of (spherical) DAHA — $Rep(\ddot{H})$ $\quad \dim V_i \sim \mathrm{Vol}(\mathfrak{D}_i)$

2) Lagrangian submanifolds of $\mathfrak{X}$ whose quantization yields these representations — $\mathscr{F}uk(\mathfrak{X}, \omega_{\mathfrak{X}})$

Brane quantization

Brane quantization for DAHA

[2206.03565 SpringerBriefs Monograph with S. Gukov, S. Nawata, D. Pei, and I. Saberi]

Derived Equivalence

$$\mathbf{R}\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, -) : D^b A\text{-Brane}(\mathfrak{X}, \omega_{\mathfrak{X}}) \to D^b\,\mathrm{Rep}(\mathscr{O}^q(\mathfrak{X}))$$

Let C_p be a punctured genus-one Riemann surface, $\boldsymbol{\mathfrak{X}} = \mathscr{M}_{\mathrm{flat}}(C_p, SL(2,\mathbb{C}))$ the moduli space of flat $SL(2,\mathbb{C})$ connections with prescribed monodromy at the puncture, and $S\ddot{H}(\mathbb{Z}_2)$ be the spherical subalgebra of DAHA of type A_1. Then the above functor restricts to a derived equivalence of the **subcategory of compact Lagrangian A-branes of $\mathfrak{X}$** and the **category of finite-dimensional $S\ddot{H}(\mathbb{Z}_2)$-modules**.

DAHA Representations

We will talk about polynomial representations of DAHA $\mathscr{P} := \mathbb{C}_{q,t}[X^{\pm}]^{\check{\mathbb{Z}}_2}$

$$\mathrm{pol} : S\ddot{H} \to \mathrm{End}(\mathscr{P}), \quad \begin{aligned} x &\mapsto X + X^{-1}, \\ y &\mapsto \frac{tX - t^{-1}X^{-1}}{X - X^{-1}}\varpi + \frac{t^{-1}X - tX^{-1}}{X - X^{-1}}\varpi^{-1}, \\ z &\mapsto q^{\frac{1}{2}}X\frac{tX - t^{-1}X^{-1}}{X - X^{-1}}\varpi + q^{\frac{1}{2}}X^{-1}\frac{t^{-1}X - tX^{-1}}{X - X^{-1}}\varpi^{-1} \end{aligned}$$

Shift operator

$$\varpi^{\pm}(X) = q^{\pm}X$$

Highest weight representation for y $\quad y\,\mathcal{Z} = (Y + Y^{-1})\mathcal{Z} = (a + a^{-1})\mathcal{Z}$

For arbitrary value of a the eigenvector is a series of hypergeometric type which arises in enumerative geometry (see later)

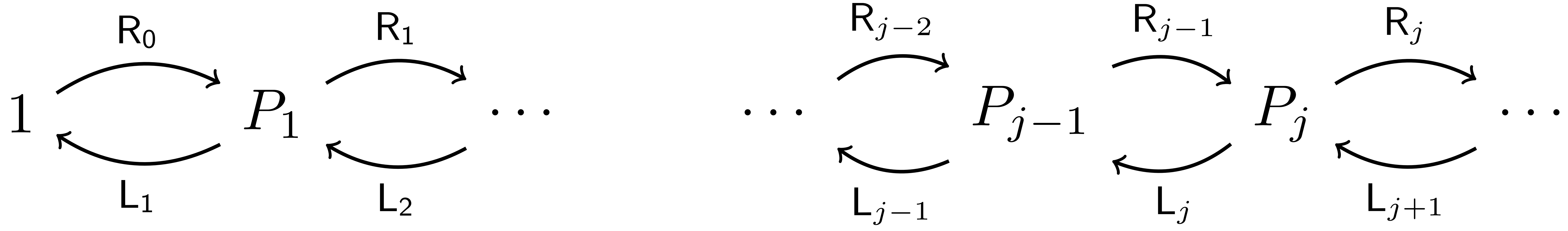
When $a = q^j t$ we get Macdonald polynomials of type A_1 labelled spin-j/2 representation

$$P_j(X;q,t) := X^j\,{}_2\phi_1(q^{-2j}, t^2; q^{-2j+2}t^{-2}; q^2; q^2t^{-2}X^{-2})$$

Polynomial Representation

Macdonald Polynomials generate the ring $\mathscr{P}$ over $\mathbb{C}[q^{\pm 1}, t^{\pm 1}]$

Raising and lowering operators



Action

$$\mathrm{pol}(\mathsf{R}_j) \cdot P_j(X;q,t) = (1 - q^{2j}t^2)P_{j+1}(X;q,t) \ ,$$

$$\mathrm{pol}(\mathsf{L}_j) \cdot P_j(X;q,t) = \frac{(1-q^{2j})(1-q^{2(j-1)}t^4)}{q^{2j}t^2(q^{2(j-1)}t^2-1)} P_{j-1}(X;q,t)$$

Finite-Dimensional Representations

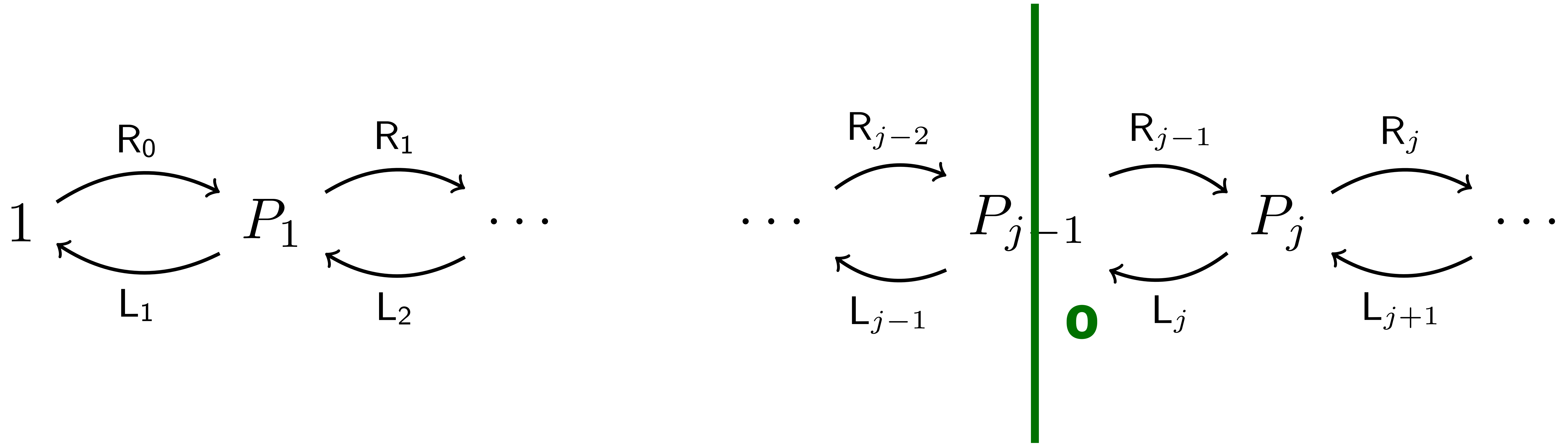
Shortening condition $\mathrm{pol}(\mathsf{L}_j) \cdot P_j = 0$

$$\frac{(1-q^{2j})(1-q^{(j-1)}t^2)(1+q^{(j-1)}t^2)}{q^{2j}t^2(q^{2(j-1)}t^2-1)}$$

must vanish

Raising operator will never be null due to $(1-q^{2j}t^2)$

$$q^{2n} = 1\ ,$$
$$t^2 = -q^{-k}\ ,$$
$$t^2 = q^{-(2\ell-1)}$$



Short exact sequence of modules $0 \to S \to V \to V/S \to 0$

Higgs Bundles

Nonabelian Hodge correspondence relates representations of the fundamental group of smooth projective algebraic varieties with Higgs bundles (E, φ)

$$\mathfrak{X} \simeq \mathcal{M}_H(C_p, SU(2))$$

Hitchin moduli space is hyperKahler

Holomorphic $SU(2)$ vector bundle over C_p with holomorphic section φ (Higgs field) of $K_{C_p} \otimes \mathrm{ad}(E) \otimes \mathcal{O}(p)$

Tame ramification at p

$$A = \alpha_p \, d\vartheta + \cdots$$

$$\varphi = \frac{1}{2}(\beta_p + i\gamma_p)\frac{dz}{z} + \cdots$$

Hitchin moduli space is the space of solutions of Hitchin equations modulo gauge transformations

$$F - [\varphi, \overline{\varphi}] = 0$$

$$\overline{D}_A \, \varphi = 0$$

NAHC:

$$\mathcal{A} = A + i(\varphi + \bar{\varphi})$$

Hitchin equations equivalent to flatness condition

$$F_{\mathcal{A}} = 0$$

Complex and Kähler Structures

The space $\mathscr{M}_H(C_p, SU(2))$ is hyperKähler

$$\omega_I = -\frac{i}{2\pi}\int_C |d^2z|\ \mathrm{Tr}\Big(\delta A_{\bar z}\wedge\delta A_z - \delta\bar\varphi\wedge\delta\varphi\Big)\ ,$$
$$\omega_J = \frac{1}{2\pi}\int_C |d^2z|\ \mathrm{Tr}\Big(\delta\bar\varphi\wedge\delta A_z + \delta\varphi\wedge\delta A_{\bar z}\Big)\ ,$$
$$\omega_K = \frac{i}{2\pi}\int_C |d^2z|\ \mathrm{Tr}\Big(\delta\bar\varphi\wedge\delta A_z - \delta\varphi\wedge\delta A_{\bar z}\Big)\ .$$

Triplet of holomorphic symplectic forms

$$\Omega_I = \omega_J + i\omega_K,\ \Omega_J = \omega_K + i\omega_I,\ \ \Omega_k = \omega_I + i\omega_J$$

Complex structure	Complex modulus	Kähler modulus
I	$\beta_p + i\gamma_p$	α_p
J	$\gamma_p + i\alpha_p$	β_p
K	$\alpha_p + i\beta_p$	γ_p

Geometry of $\mathfrak{X}$

Hitchin fibration $\pi : \mathcal{M}_H(C_p, SU(2)) \to \mathcal{B}_H$ whose fibers are Abelian varieties (Liouville tori)

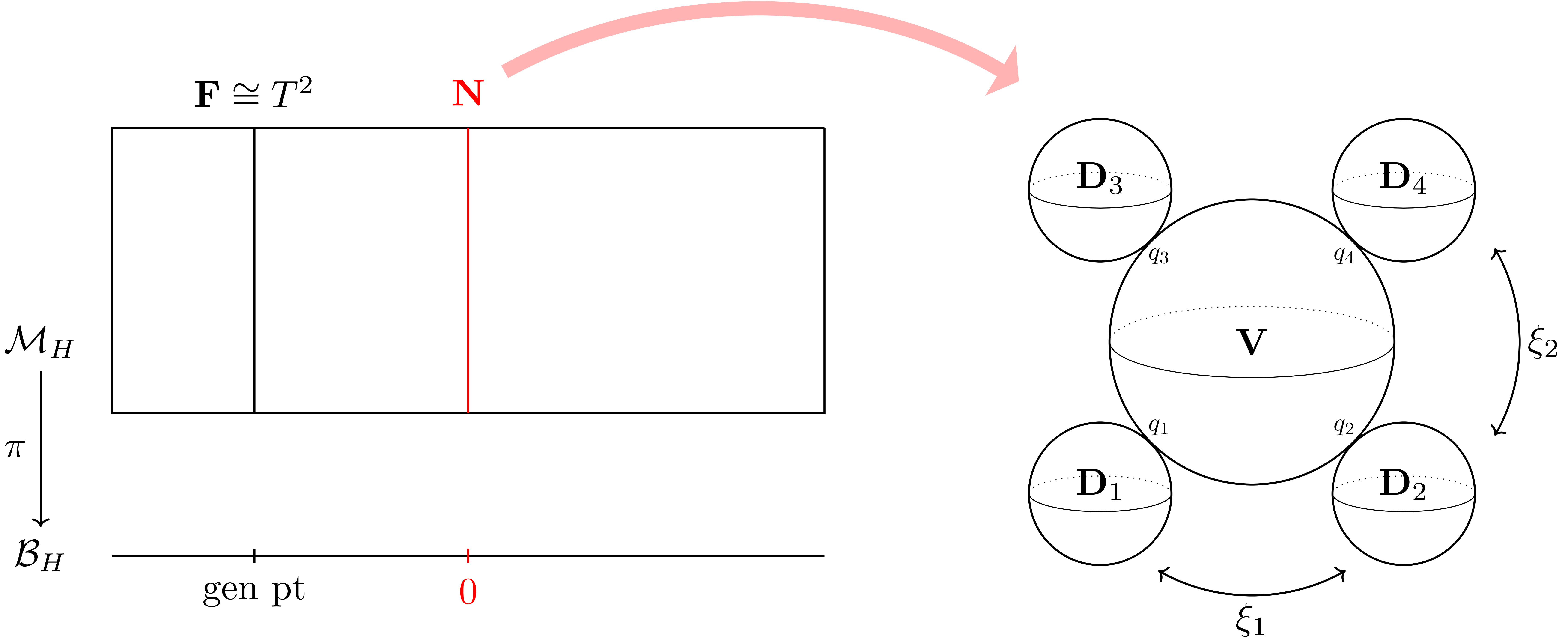
$(E, \varphi) \mapsto \mathrm{Tr}\varphi^2$ Holomorphic in complex structure I

Singular fiber $\mathbf{N} = \pi^{-1}(0)$

$$\mathbf{N} = \mathbf{V} \cup \bigcup_{i=1}^{4} \mathbf{D}_i$$

`Pillowcase' for $\alpha_p = \beta_p = \gamma_p = 0$

$$\mathbf{V} \cong (S^1 \times S^1)/\mathbb{Z}_2$$



Away from $\beta_p = 0$ locus — resolution of A_1 singularities (exceptional divisors).

β_p — Kahler structure parameter in J

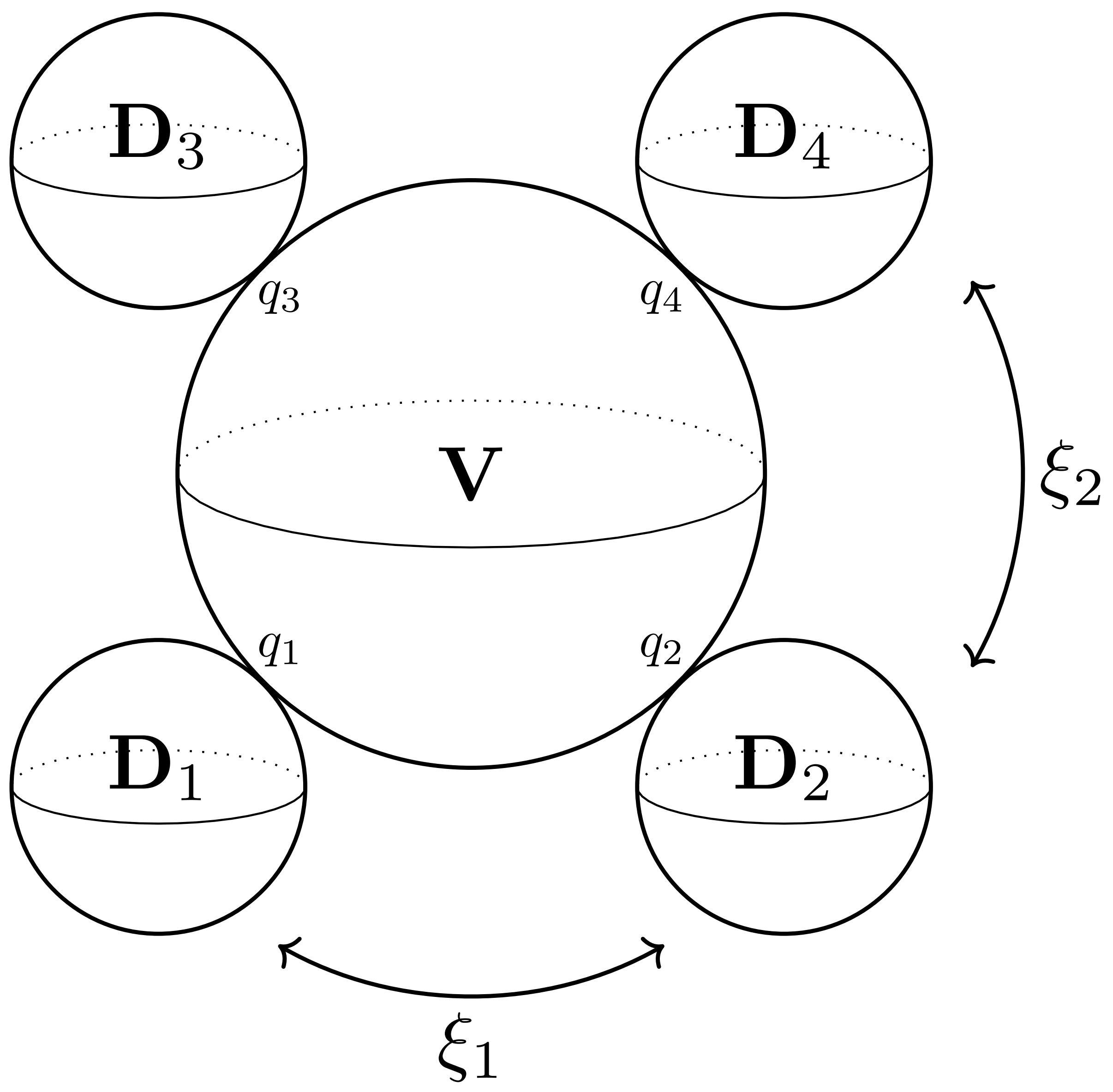
$H_2(\mathcal{M}_H(C_p, G), \mathbb{Z})$

Null vector of intersection form

$$[\mathbf{F}] = 2[\mathbf{V}] + \sum_{i=1}^{4} [\mathbf{D}_i]$$

$\widehat{D}_4$ Dynkin diagram

Cycles



Pillowcase

$$\int_{\mathbf{V}} \frac{\omega_I}{2\pi} = \frac{1}{2} - |\alpha_p|$$
$$\int_{\mathbf{V}} \frac{\omega_J}{2\pi} = -\beta_p \ ,$$
$$\int_{\mathbf{V}} \frac{\omega_K}{2\pi} = -\gamma_p \ ,$$

Hitchin fiber

$$\int_{\mathbf{F}} \frac{\omega_I}{2\pi} = 1 \ , \qquad \int_{\mathbf{F}} \frac{\omega_J}{2\pi} = 0 = \int_{\mathbf{F}} \frac{\omega_K}{2\pi}$$

Exceptional divisors

$$\frac{\alpha_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_I}{2\pi} \ , \qquad \frac{\beta_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_J}{2\pi} \ , \qquad \frac{\gamma_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_K}{2\pi}$$

$$i = 1, 2, 3, 4.$$

Symmetries

$$\xi_1 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_2 \quad \text{and} \quad \mathbf{D}_3 \leftrightarrow \mathbf{D}_4$$
$$\xi_2 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_3 \quad \text{and} \quad \mathbf{D}_2 \leftrightarrow \mathbf{D}_4$$
$$\xi_3 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_4 \quad \text{and} \quad \mathbf{D}_2 \leftrightarrow \mathbf{D}_3$$

Canonical Coisotropic Brane

[Gukov Witten]
[Kapustin Orlov]

Canonical coisotropic brane $\mathfrak{B}_{\mathrm{cc}}: \begin{array}{c}\mathcal{L}\\ \downarrow\\ \mathfrak{X}\end{array}$ $c_1(\mathcal{L}) = [F/2\pi] \in H^2(\mathfrak{X}, \mathbb{Z})$

2d sigma model into $\mathfrak{X}$

Family of $\mathfrak{B}_{cc}$ branes parameterized by $\hbar$ on symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$

$$\hbar = |\hbar| e^{i\theta}$$

Quantization $q = e^{2\pi i \hbar}$

Values of the B-field are determined by equation $\Omega := F + B + i\omega_{\mathfrak{X}} = \dfrac{\Omega_J}{i\hbar}$

$B \in H^2(\mathfrak{X}, \mathrm{U}(1))$

Needed for generic $\hbar$

$$F + B = \mathrm{Re}\ \Omega = \frac{1}{|\hbar|}(\omega_I \cos\theta - \omega_K \sin\theta)\ ,$$

$$\omega_{\mathfrak{X}} = \mathrm{Im}\ \Omega = -\frac{1}{|\hbar|}(\omega_I \sin\theta + \omega_K \cos\theta)\ .$$

HyperKähler condition $F + B = \omega_{\mathfrak{X}}\, J$

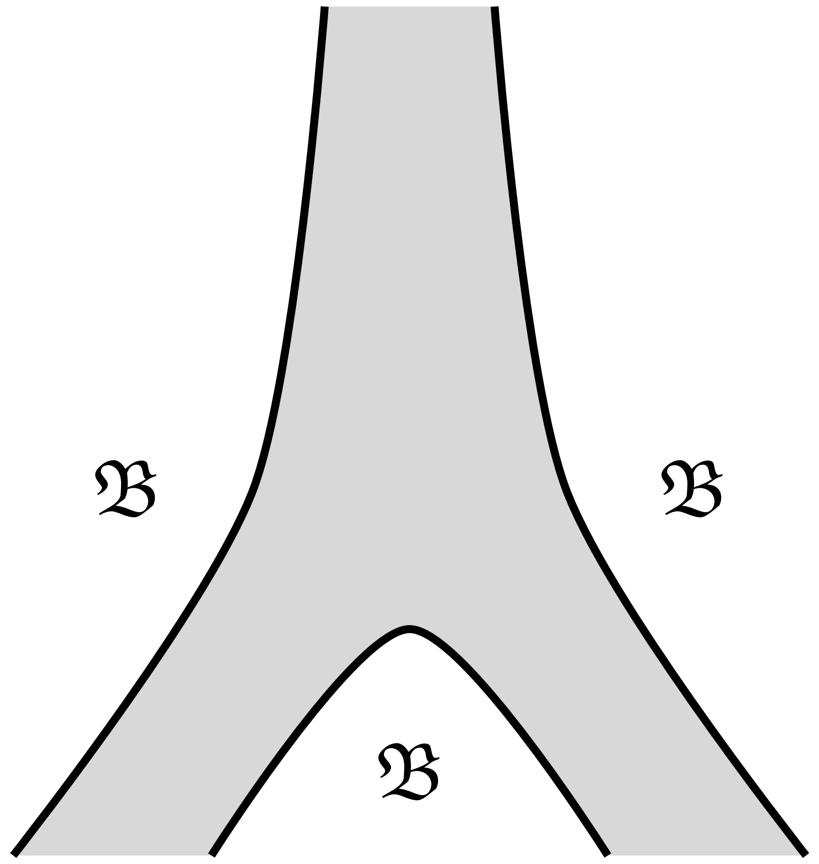
$$\left(\omega_{\mathfrak{X}}^{-1}(B+F)\right)^2 = J^2 = -1$$

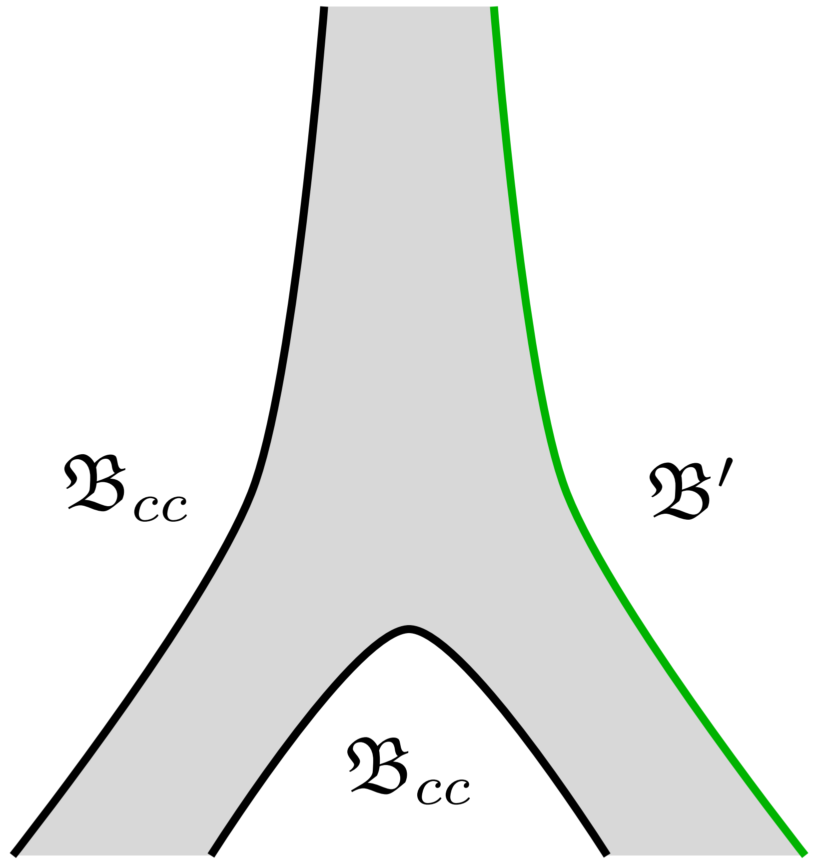
E.g. for real $\hbar$ we have $\omega_{\mathfrak{X}} = \omega_K$ and $\mathfrak{B}_{cc}$ brane is of type (B, A, A), for purely imaginary of type (A, A, B)

Branes and Quantization

$\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ parameterized by $\hbar$ provides deformation of the space of holomorphic functions on $\mathfrak{X}$ which is spherical DAHA $= S\ddot{H}$

$$\left(\omega_{\mathfrak{X}}^{-1}(B+F)\right)^2 = J^2 = -1 \qquad \int_{\mathbf{F}} \frac{\Omega}{2\pi} = \frac{1}{\hbar} \qquad \frac{1}{2\pi}\int_{\mathbf{D}_i} F + B + i\omega_{\mathfrak{X}} = \int_{\mathbf{D}_i} \frac{\Omega_J}{2\pi i\hbar} = \frac{\gamma_p + i\alpha_p}{2i\hbar} = -c + \frac{1}{2}$$





$$\begin{array}{ccc} \mathscr{O}^q(\mathfrak{X}) & = & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}}) \\ \circlearrowleft & & \circlearrowleft \\ \mathscr{B}' & = & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}') \end{array}$$

$$\mathrm{End}(\mathfrak{B}_{\mathrm{cc}}) \cong S\ddot{H}$$

Lagrangian Branes

Lagrangian A brane — unitary bundle, flat Spin^c-structure on L and grade lift

$$\mathfrak{B}_{\mathbf{L}} : \quad \begin{array}{c} \mathcal{L}' \otimes K_{\mathbf{L}}^{-1/2} \\ \downarrow \\ \mathbf{L} \end{array}$$

Flatness condition $F'_{\mathbf{L}} + B|_{\mathbf{L}} = 0$

Representation space $\mathscr{L} := \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{L}})$

Hirzebruch-Riemann-Roch formula
(B-model analysis to compute dimension of open strings)

$$\begin{aligned} \dim \mathscr{L} &= \dim H^0(\mathbf{L}, \mathfrak{B}_{\mathrm{cc}} \otimes \mathfrak{B}_{\mathbf{L}}^{-1}) \\ &= \int_{\mathbf{L}} \mathrm{ch}(\mathfrak{B}_{\mathrm{cc}}) \wedge \mathrm{ch}(\mathfrak{B}_{\mathbf{L}}^{-1}) \wedge \mathrm{Td}(T\mathbf{L}) \end{aligned}$$

For a Lagrangian in two dimensions

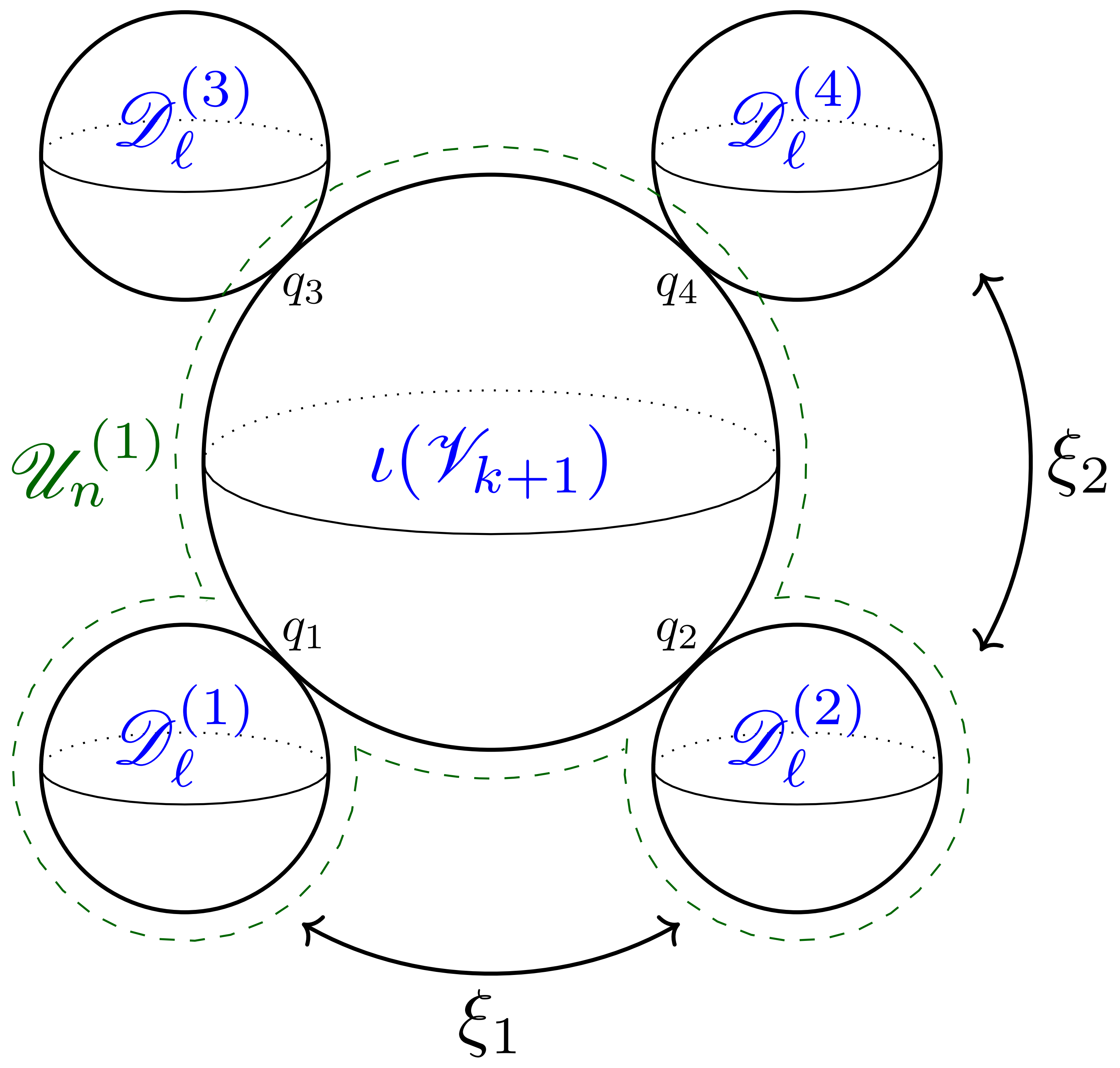
$$\mathrm{Td}(T\mathbf{L}) = \mathrm{ch}(K_{\mathbf{L}}^{-1/2})\widehat{A}(T\mathbf{L})$$

So the dimension reads

$$\dim \mathscr{L} = \int_{\mathbf{L}} \mathrm{ch}(\mathfrak{B}_{\mathrm{cc}}) = \int_{\mathbf{L}} \frac{F+B}{2\pi}$$

Lagrangian branes are objects in Fukaya category $\mathsf{Fuk}(\mathfrak{X}, \omega_{\mathfrak{X}})$

Compact Lagrangians

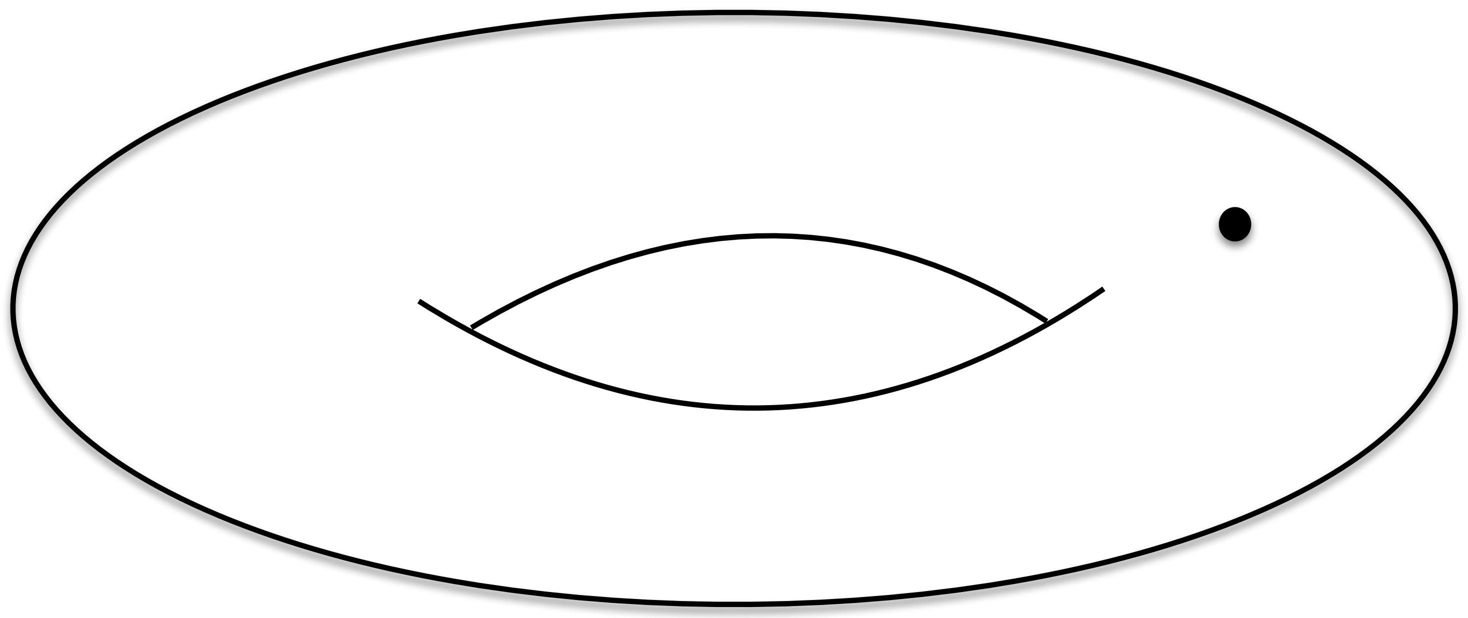


finite-dim rep	shortening condition	A-brane condition
$\mathscr{F}_m^{(x_m,y_m)}$	$q^m = 1$	$m = \frac{1}{\hbar}$
$\mathscr{U}_n$	$q^{2n} = 1$	$n = \frac{1}{2\hbar}$
$\mathscr{V}_{k+1}$	$t^2 = -q^{-k}$	$k = \frac{1}{2\hbar} + \frac{\gamma_p + i\alpha_p}{i\hbar}$
$\mathscr{D}_\ell$	$t^2 = q^{-\ell+1/2}$	$\ell = \frac{\gamma_p + i\alpha_p}{2i\hbar}$

Many-Body Systems

5
(G,q)-Opers
QQ-system
tRS
$\mathbb{C}_p^\times \times \mathbb{C}_x^\times$
XXZ
$\mathrm{DAHA}_{\hbar,q}(X,Y)$
$U_{\hbar,q}\left(\widehat{\widehat{\mathfrak{gl}_1}}\right)$

Coming back to flat connections on pictured T^2



$$\mathcal{M}_n = \{A, B, C\}/GL(n;\mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

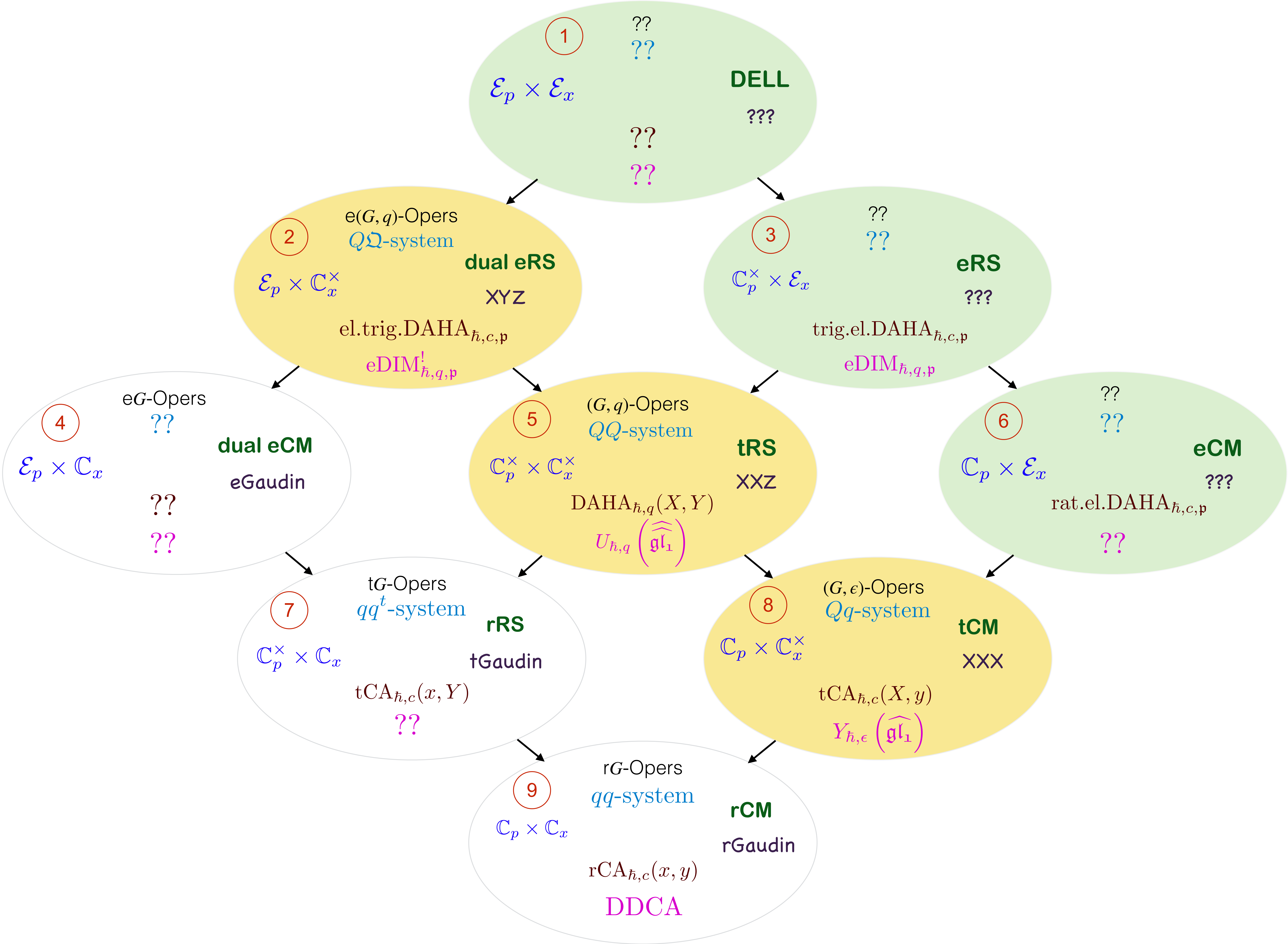
$$C = \mathrm{diag}(t, \ldots, t, t^{n-1})$$

In the basis where $A = \mathrm{diag}(a_1, \ldots, a_n)$ is diagonal characteristic polynomial of B yields trig. Ruijsenaars Hamiltonians (Macdonald operators — center of spherical DAHA)

$$\det(u - B) = \sum_k (-1)^k H_k u^{n-k}$$

For n=2

$$H_1 = \frac{a_1 - t a_2}{a_1 - a_2} p_1 + \frac{a_2 - t a_1}{a_2 - a_1} p_2$$



1
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$\mathcal{E}_p \times \mathcal{E}_x$
DELL
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2
e(G,q)-Opers
$Q\mathfrak{Q}$-system
dual eRS
$\mathcal{E}_p \times \mathbb{C}_x^\times$
XYZ
el.trig.DAHA$_{\hbar,c,\mathfrak{p}}$
eDIM$^!_{\hbar,q,\mathfrak{p}}$
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trig.el.DAHA$_{\hbar,c,\mathfrak{p}}$
eDIM$_{\hbar,q,\mathfrak{p}}$
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$\mathbb{C}_p \times \mathcal{E}_x$
eCM
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rat.el.DAHA$_{\hbar,c,\mathfrak{p}}$
??
7
tG-Opers
qq^t-system
rRS
$\mathbb{C}_p^\times \times \mathbb{C}_x$
tGaudin
tCA$_{\hbar,c}(x,Y)$
??
8
(G,ϵ)-Opers
Qq-system
tCM
$\mathbb{C}_p \times \mathbb{C}_x^\times$
XXX
tCA$_{\hbar,c}(X,y)$
$Y_{\hbar,\epsilon}\left(\widehat{\mathfrak{gl}}_1\right)$
9
rG-Opers
qq-system
rCM
$\mathbb{C}_p \times \mathbb{C}_x$
rGaudin
rCA$_{\hbar,c}(x,y)$
DDCA

Enumerative AG/Representation Theory

- Yellow diagonal. The eigenfunctions of the *tCM*, *tRS*, and *dual eRS* models coincide with quasimap vertex functions of quantum equivariant *cohomology*, *K-theory*, and *elliptic cohomology* of the cotangent bundle to the complete flags in $\mathbb{C}^n$ respectively (also in χ(fin. Laumon))
- Green Diagonal. The eigenfunctions of the e*CM*, e*RS*, and *DELL* models are holomorphic equivariant Euler characteristics of affine Laumon spaces in *cohomology*, *K-theory*, and *elliptic cohomology* respectively

The Abelian nature of Lagrangian fibers in Hitchin system suggests that coordinates and momenta take values in $\mathbb{C},\ \mathbb{C}^\times,\ \mathcal{E}=\mathbb{C}^\times/q^{\mathbb{Z}}$

The DELL

[PK Shakirov]

N-particle DELL Hamiltonians $$\widehat{\mathcal{H}}_a = \widehat{\mathcal{O}}_0^{-1}\widehat{\mathcal{O}}_a$$

$$\widehat{\mathcal{O}}(z) = \sum_{n\in\mathbb{Z}} \widehat{\mathcal{O}}_n\, z^n = \sum_{n_1,\dots,n_N=-\infty}^{\infty} (-z)^{\sum n_i}\, w^{\sum \frac{n_i(n_i-1)}{2}} \prod_{i<j} \theta\left(t^{n_i-n_j}\widehat{x}_i/\widehat{x}_j|p\right)\, \widehat{p}_1^{n_1}\dots\widehat{p}_N^{n_N}$$

Eigenvalue equation

$$\mathcal{O}(z)\mathcal{Z}_{inst}^{DELL}(p,x_1,\dots,x_N) = \lambda(z,\boldsymbol{a},w,p)\ \mathcal{O}_0\mathcal{Z}_{inst}^{DELL}(p,x_1,\dots,x_N)$$

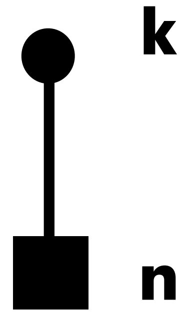
Euler characteristic

$$\mathcal{Z}_{inst}^{DELL}(p,x_1,\dots,x_N) = \sum_{\boldsymbol{d}} \mathfrak{q}^{\boldsymbol{d}} \int_{\mathcal{L}_{\boldsymbol{d}}} 1$$

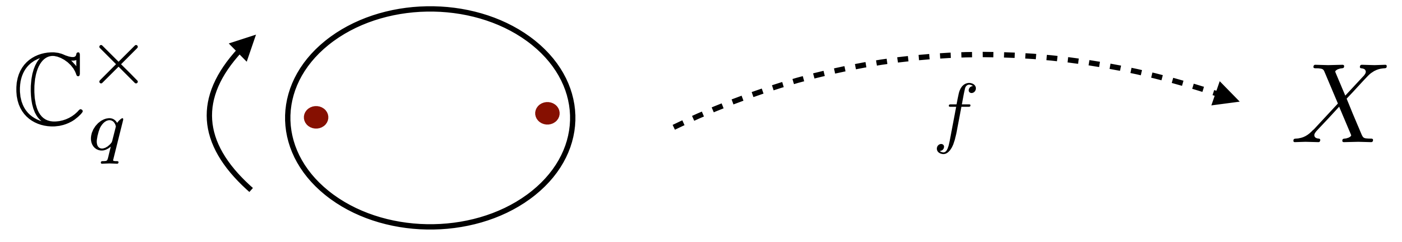
Quantum K-theory

Classical K-theory of a quiver variety X is generated by tensorial polynomials of tautological bundles on X and their duals

For quantum deformation parameterized by z we study quasimaps from $\mathbb{P}^1$ to X



$p_1 = 0, \;\; p_2 = \infty$



Vertex functions are eigenfunctions of quantum tRS difference operators!

$T_i(a)V(z,a) = e_i(z)V(z,a)$ 3d Mirror symmetry $T_i(z)V(z,a) = e_i(a)V(z,a)$

[PK Zeitlin [arXiv:1802.04463]
Math.Res.Lett. **28** (2021) 435]

Saddle point approximation yields Bethe equations

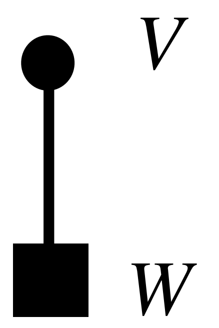
$q \to 1$

$$\prod_{j=1}^{n} \frac{s_i - a_j}{t a_j - s_i} = z \prod_{j=1}^{k} \frac{s_i t - s_j}{s_i - t s_j}$$

The QQ-System

Baxter Q-operator $Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$ has eigenvalue $Q(u) = \prod_{i=1}^{k} (u - s_i)$

Short exact sequence of bundles for $T^*Gr_{k,n}$ $\quad 0 \to V \to W \to V^\vee \to 0$



Eigenvalues of operators Q and $\widetilde{Q}$ (generated by $V^\vee$) satisfy the QQ-relation

$$z\widetilde{Q}(tu)Q(u) - Q(tu)\widetilde{Q}(u) = \prod_{i=1}^{n} (u - a_i)$$

which is equivalent to Bethe equations

Also:

Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

Relations between generalized minors (Jacobi-like identities)

[Fomin, Zelevinski] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_t(\hat{g})$

[Frenkel, Hernandez] ....

Spectral determinants in the QDE/IM correspondence

[Frenkel, PK, Zeitlin, to appear][Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

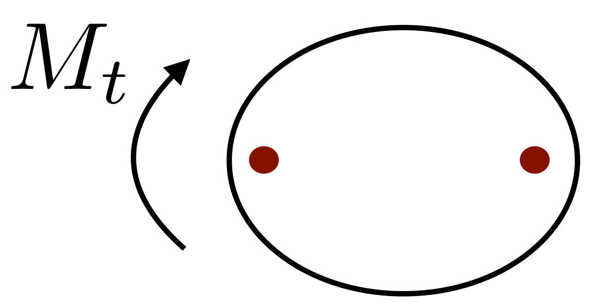
Describes (t-)oper bundles

[Frenkel, PK, Zeitlin, Sage]

[PK, Sage, Zeitlin, Commun.Math.Phys. **381** (2021) 641]

(G,t)-Opers

$$M_t : \mathbb{P}^1 \to \mathbb{P}^1$$
$$u \mapsto tu$$



Principal bundle $\mathscr{F}_G$ over $\mathbb{P}^1$

(G,t)-connection A is a meromorphic section of $Hom_{\mathscr{O}_{\mathbb{P}^1}}(\mathscr{F}_G, \mathscr{F}^t_G)$

t-gauge transformation $\quad A(u) \mapsto g(tu)A(u)g(u)^{-1} \qquad g(u) \in G(\mathbb{C}(u))$

(SL(2),q)-oper

Triple $(E, A, \mathscr{L})$

(E, A) is the $(SL(2), t)$ connection

$\mathscr{L} \subset E$ is a line subbundle

The induced map $\bar{A} : \mathscr{L} \to (E/\mathscr{L})^t$ is an isomorphism

in a trivialization $\mathscr{L} =$ Span(s) $\quad s(tu) \wedge A(u)s(u) \neq 0$

Chose trivialization of $\mathscr{L}$ $\quad s(u) = \begin{pmatrix} Q(u) \\ \widetilde{Q}(u) \end{pmatrix}$

Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

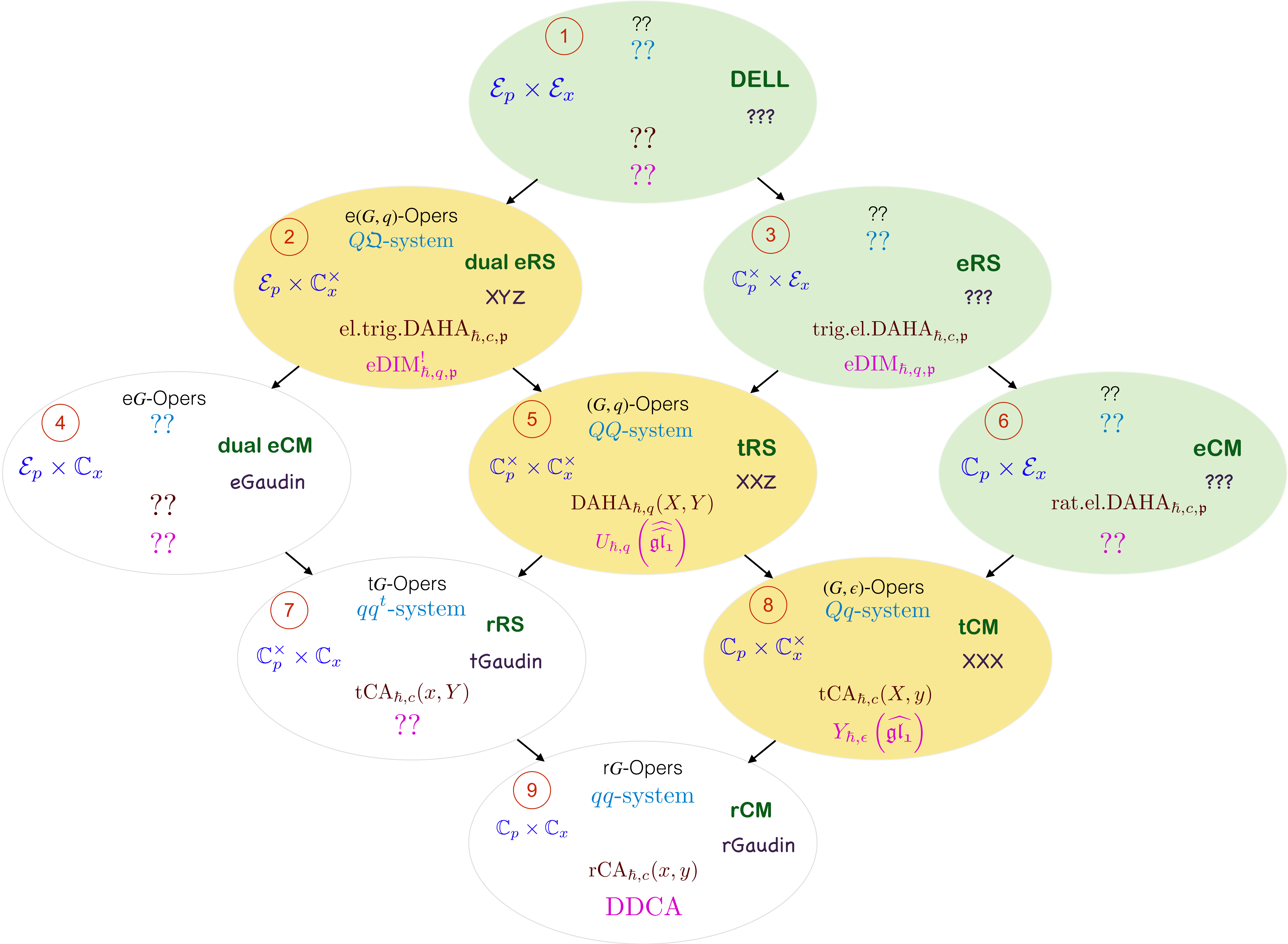
t-Oper condition with $A(u) = Z$ — SL(2) QQ-system

$$z\widetilde{Q}(tu)Q(u) - Q(tu)\widetilde{Q}(u) = \prod_{i=1}^{n}(u - a_i)$$

q-Opers, QQ-System & Bethe Ansatz

[Frenkel, PK, Sage, Zeitlin **JEMS 2023**]

Theorem: There is a 1-to-1 correspondence between the set of nondegenerate Z-twisted (G,t)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$



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DAHA as $n \to \infty$

[PK]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from $\mathbb{P}^1$ to X

$a_k = q^k t^{n-k}$ restricts us to the Fock space representation of (q,t)-Heisenberg algebra which is a DAHA module