Opers & Integrability

(w/ Frenkel, Sage, Zeitlin), multiple papers in the past few years.

This is a story about interactions between various branches of mathematical physics

- Quantum enumerative geometry (cohomology, K-theory, elliptic cohomology) [Givental et al] [Okounkov et al]
- Geometric representation theory, in particular Geometric Langlands [Aganagic, Frenkel, Okounkov]
- Integrable systems and dualities [our work + bunch of results from string theory]

Opers help us understand these connections in a new way.

① (G,q)-opers: Works for G - simple, simply-connected, complex Lie group.

Let's begin w/ $G = GL(r+1)$ (in type A we can do more explicit calculations, in particular, connections to integrability)

Consider automorphism $M_q: \mathbb{P}^1 \to \mathbb{P}^1$
$z \mapsto qz$, $q \in \mathbb{C}^\times$

Def: A meromorphic $(GL(r+1),q)$-oper on $\mathbb{P}^1$ is a triple $(A, E, \mathcal{L}_\bullet)$, where E - vector bundle of rank $r+1$ over $\mathbb{P}^1$, $\mathcal{L}^\bullet$ — complete flag of vector subbundles

$$\text{line} - \mathcal{L}_{r+1} \subset \mathcal{L}_r \subset \dots \subset \mathcal{L}_1 = E$$

s.t. the meromorphic $(GL(r+1),q)$-connection $A \in \mathrm{Hom}_{\mathcal{O}_U}(E, E^q)$ (E^q — pullback of E under M_q; U — open Zariski-dense subset)

satisfies:

i) $A\mathcal{L}_i \subset \mathcal{L}_{i-1}$

ii) $\exists$ open dense subset $U \in \mathbb{P}^1$ s.t. the restriction of $A \in \mathrm{Hom}(\mathcal{L}_\bullet, \mathcal{L}_\bullet^q)$ to $V = U \cap M_q^{-1}(U)$ is invertible and restrictions

$$\bar{A}_i : \mathcal{L}_i / \mathcal{L}_{i+1} \xrightarrow{\sim} \left(\mathcal{L}_{i-1}/\mathcal{L}_i\right)^q \quad \text{is an isomorphism on } V.$$

If $\det A = 1 \Rightarrow (SL(r+1), q)$-oper.

Changing trivialization on E via $g(z) \in SL(r+1)(z)$ changes $A(z)$ by

$$A(z) \mapsto g(qz)\, A(z)\, g(z)^{-1}$$

Let $\mathcal{L}_{r+1} = \mathrm{Span}(s(z))$. Consider determinants

$$\mathcal{W}_i(s)(z) = \left[s(z) \wedge A(z)s(qz) \wedge \dots \wedge A(z)A(qz)\dots A(q^{i-2}z)\, s(q^{i-1}z) \right]\Big|_{\Lambda^i \mathcal{L}_{r-i+2}}$$

$i = 2, \dots, r+1$

q-Oper condition: $\mathcal{W}_i \neq 0$.

Need more structure - singularities and twists

Def: An $(SL(r+1),q)$-oper has regular singularities at roots of polynomials $\{\Lambda_i(z)\}_{i=1,\dots,r}$ if $\mathcal{W}_i(z) = \beta_i \Lambda_i(z)$

Def: An $(SL(r+1),q)$-oper is called Z-twisted if $\exists\, g(z) \in SL(r+1)(z)$ s.t.

$$g(qz)A(z)g(z)^{-1} = Z \in H \subset H(z) \text{ – regular semisimple}$$

The q-oper connection reads:

$$A(z) = \begin{pmatrix} *\Lambda_1 & & & & 0 \\ & *\Lambda_2 & & & \\ & & *\Lambda_3 & & \\ & * & & * & \\ & & & & *\Lambda_r \\ & & & & & * \end{pmatrix}$$

Def: The Miura $(SL(r+1),q)$-oper is $(A, E, \mathcal{L}_\bullet, \hat{\mathcal{L}}_\bullet)$ where $(A,E,\mathcal{L}_\bullet)$ is $(SL(r+1),q)$-oper and flag $\hat{\mathcal{L}}_\bullet$ is preserved by the q-connection $A(z)$.

Z-twisted Miura oper can be diagonalized by $v(z) \in B_+(z)$.

Prop: There are exactly $|S_{r+1}|$ $(r+1)!$ Miura opers for a given Z-twisted q-oper if Z is regular semisimple (char poly free of squares)

$$A(z) = \prod_i (y_i(z))^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{y_i(z)} e_i}.$$

Can pick the order in roots s.t. Miura oper

$$A(z) = \begin{pmatrix} y_1\Lambda_1 & & & 0 \\ & \frac{y_2}{y_1}\Lambda_2 & & \\ & & \ddots & \\ 0 & & & \frac{y_r}{y_{r-1}}\Lambda_r \\ & & & & \frac{1}{y_r} \end{pmatrix}$$

Miura conditions in detail:

In the standard basis for $\hat{\mathcal{L}}_\bullet$ $e_1, e_2, \dots, e_{r+1}$ in the space of sections relative position of $\mathcal{L}_\bullet$ and $\hat{\mathcal{L}}_\bullet$ can be expressed as follows:

$$\mathcal{D}_u(s) = e_1 \wedge \dots e_{r+1-u} \wedge s(z) \wedge Z s(M_q z) \wedge \dots \wedge Z^{u-1} s(M_q^{u+1} z)$$

$\mathcal{D}_u$ have a subset of zeros coinsident w/ that of $W_u(s)(z)$. The rest of zeros correspond to points when flags are not in generic position: $\left(\mathcal{F}_{B_+,x} = aB_+, \mathcal{F}_{B_-,x} = bB_- \quad B\backslash G/B_+ = W_G \quad ab^{-1}=1 \right.$ in (G,q)-opers$\left.\right)$

$$D_u(s) =$$

Minura conditions:

$$\det_{i,j}\left[\xi_i^{\,j-1} s^{(j-1)}_{r+1-u+i}(z)\right] = \beta_u W_u \cdot V_u, \qquad s_i^{(m)}(z) = s_i(M_q^m z)$$

$$V_u(z) = \prod_{a=1}^{v_u}(z - v_{u,a})$$

② QQ-System:

Theorem: [KSZ] Polynomials $\{V_u(z)\}$, $u=1,\dots,r$ give the solution to the QQ-system.

$$\xi_{i+1} Q_i^+(M_q z) Q_i^-(z) - \xi_i Q_i^+(z) Q_i^-(M_q z) = (\xi_{i+1} - \xi_i)\Lambda_i(z) Q_{i-1}(M_q z) Q_{i+1}(z)$$

s.t. $V_i = Q_i^+(z)$. Moreover

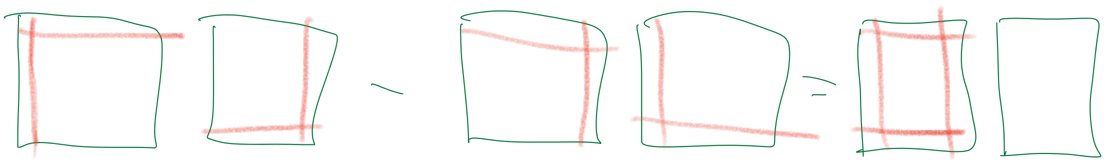
(don't write the whole formula, just ...)

$$Q^+_j(z) = \frac{1}{M_q^{i-r} W_{r-i}(z)} \frac{\det M_{1\ldots j}}{\det V_{1\ldots j}}, \qquad Q^-_j(z) = \frac{1}{M_q^{i-r} W_{r-i}(z)} \frac{\det M_{1\ldots j-1, j+1}}{\det V_{1\ldots j-1,j+1}},$$

where

$$M_{i_1\ldots i_j} = \begin{pmatrix} s_{i_1} & \xi_{i_1} s^{(1)}_{i_1} & \cdots & \xi^{j-1}_{i_1} s^{(j-1)}_{i_1} \\ \vdots & \vdots & & \vdots \\ s_{i_j} & \xi_{i_j} s^{(1)}_{i_j} & \cdots & \xi^{j-1}_{i_j} s^{(j-1)}_{i_j} \end{pmatrix}, \qquad V_{i_1\ldots i_j} = \begin{pmatrix} 1 & q\xi_{i_1} & \cdots & q^{j-1}\xi^{j-1}_{i_1} \\ \vdots & & & \\ 1 & q\xi_{i_j} & \cdots & q^{j-1}\xi^{j-1}_{i_j} \end{pmatrix},$$

Lewis-Carroll identity: a particular example.



Difference operator

$$v(z) = \begin{pmatrix} Q_+(z) & \frac{Q_-(z)}{Q_+(z)} \\ 0 & Q_+(z)^{-1} \end{pmatrix}$$

$$D_q(S) = AS$$

$$D_q(S_1) = \Lambda S_2$$

eliminate:

$$\left(D_q^2 - T(qz) D_q - \frac{\Lambda(qz)}{\Lambda(z)}\right) S_1 = 0$$

Z-twisted Miura oper can be diagonalized by $v(z) \in B_+(z)$.

3 Generalized Minors:

For (G,q)-opers the LC identity is generalized as follows

Gauss decomposition

$$G = N_- H N_+$$

$$g = n_- h\, n_+$$

V_i^+ – irrep of G w/ highest weight ω_i wrt B^+

$$h\, v^+_{\omega_i} = [h]^{\omega_i} v^+_{\omega_i}$$

Principal minors [Fomin-Zelevinski]

$$\Delta: G \to \mathbb{C}^\times$$

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \qquad i = 1, \ldots, r$$

Generalized minors for $u, v \in W_G$ ~ Weyl group

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g\, \tilde{v})$$ ~ lift to the loop group

Proposition: Let $A(z) = v(qz) Z v(z)^{-1}$, $v(z) \in B_+(z)$

$$\Delta_{w\cdot\omega_i, \omega_i}(v^{-1}(z)) = Q_+^{w,i}(z) \qquad \left(= s^i(z) \text{ for } G = SL(N)\right)$$

The QQ-system is equivalent to the following quadratic identity by Fomin and Zelevinski

$$\Delta_{u\omega_i, v\omega_i} \Delta_{u w_i \omega_i, v w_i \omega_i} - \Delta_{u w_i \omega_i, v\omega_i} \Delta_{u\cdot\omega_i, v w_i \omega_i} = \prod_{j \neq i} \Delta^{-a_{ji}}_{u\omega_j, v\omega_j}$$

Extended QQ-system:

q-connection can be expressed as follows: $A(z) = \prod g(z)^{\check{\lambda}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}$

$$g_i(z) = \xi_i \frac{Q_i^+(qz)}{Q_i^+(z)}$$

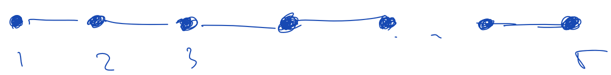
Bäcklund transformations.

Prop: $A \mapsto A^{(i)} = e^{\mu_i(qz) f_i} A(z) e^{-\mu_i(z) f_i}$, $\quad \mu_i(z) = \dfrac{Q_{i-1}^+(z)\, Q_{i+1}^+(z)}{Q_+^i(z)\, Q_-^i(z)}$

then $A^{(i)}$ is obtained from A by substitution

$$Q_+^j(z) \mapsto Q_+^j(z), \quad j \neq i$$
$$Q_+^i(z) \mapsto Q_-^i(z), \quad Z \mapsto s_i(Z)$$

One can keep iterating



4) (G,q)-opers:

(G,q)-oper on $\mathbb{P}^1$ is $(\mathcal{F}_G, A, \mathcal{F}_{B_+})$, where A - meromorphic connection

$\mathcal{F}_{B_+}$ - reduction of $\mathcal{F}_G$ to B_+ s.t. restriction of $A: \mathcal{F}_G \to \mathcal{F}_G^q$ on V

takes values in $B_+(\mathbb{C}[V]) \subset B_+(\mathbb{C}[U])$

c - Coxeter element $\quad c = \prod s_i$

Locally $\quad A(z) = n'(z) \prod_i (\phi_i(z)^{\check{\alpha}_i} s_i)\, n(z), \qquad n, n'(z) \in N_+(z)$

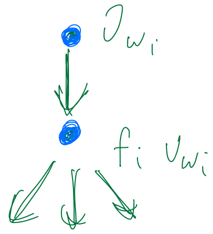
Miura (G,q)-oper $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$

preserves q-connection

Th: Every Miura (G,q)-oper w/ regular singularities can be written as

$$A(z) = \prod_i g_i(z)^{-\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} f_i}, \qquad g_i(z) \in \mathbb{C}(z)^\times$$

Miura-Plücker (G,q)-oper:



v_{ω_i} - highest weight vector of $L_i \subset V_i$ — irrep of G with highest weight ω_i wrt $H \subset B_+$

W_i - 2d subspace spanned by $\{v_{\omega_i}, f_i \cdot v_{\omega_i}\}$ - B_+ invariant

Associated vector bundle $\mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i$

contains rank-2 subbundle $\mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$

line subbundle $\mathcal{L}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$

Associate to every $i = 1 \dots r$ an $(SL(2), q)$-oper

Def: $\exists\, v(z) \in B_+(z)$ s.t.

$$A_i(z) = v(qz) Z v(z)^{-1}\big|_{W_i} \qquad A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ij}} \\ 0 & g_i(z)^{-1} \prod_{j \neq i} g_j(z)^{-a_{ij}} \end{pmatrix}$$

When. $\quad MPOp \subseteq MOp$?

Th: $\forall$ w_0-generic Miura–Plücker q-oper is Miura q-oper.

${}^LG$ QQ system $\Longleftrightarrow$ (G, q) opers

(5) Integrability & Dualities:

Consider $(SL(2), q)$-oper

$$s(z) \wedge Z\, s(qz) = \beta \Lambda(z)$$

Let $s(z) = \begin{pmatrix} Q_+(z) \\ Q_-(z) \end{pmatrix}$, $Z = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$, $\Lambda(z) = \prod_{n=1}^{L} \prod_{\ell=0}^{K_n - 1} (z - q^{\ell} z_n)$

$$M(z) = \begin{pmatrix} Q_+(z) & \xi Q_+(qz) \\ Q_-(z) & \xi^{-1} Q_-(qz) \end{pmatrix} \qquad \det M = \beta \cdot \Lambda(z)$$

1) Calculate the determinant $\Rightarrow$ QQ-system (is equivalent to $\Lambda(z) \sim Z$)

$$\xi^{-1} Q_+(z) Q_-(qz) - \xi Q_+(qz) Q_-(z) = (\xi^{-1} - \xi) \Lambda(z)$$

XXZ Bethe equations: $Q_+ = \prod_{k=1}^{m} (z - w_k)$

$$\prod_{n=1}^{L} \frac{w_k - q z_n}{w_k - z_n} = \xi^2 \prod_{j=1}^{m} \frac{q w_k - w_j}{w_k - q w_j}, \qquad k = 1 \dots m$$

2) Def: We call Z-twisted Miura $(SL(r+1), q)$-oper canonical if

1) Z – regular semisimple
2) $\deg D_k = k$
3) No other singularities except for the roots of $\Lambda(z) = D_{r+1}(z)$ which are distinct.

Proposition: $\{s_i(z)\}_{i=1,\dots,r+1}$ are of degree 1.

Consider canonical $(SL(2), q)$-oper

$Q_+(z) = z - p_1$ $\qquad \Lambda_2(z) = (z - a_1)(z - a_2)$

$Q_-(z) = z - p_2$,

$$M(z) = \begin{pmatrix} z - p_1 & \xi(zq - p_1) \\ z - p_2 & \xi^{-1}(zq - p_2) \end{pmatrix} = z \begin{pmatrix} 1 & q\xi \\ 1 & q\xi^{-1} \end{pmatrix} - \begin{pmatrix} p_1 & p_1 \xi \\ p_2 & p_2 \xi^{-1} \end{pmatrix}$$

$$= z \cdot V + M(0)$$

$$\det M(z) = \det V \cdot \Lambda(z)$$

$$\det \left(z + M(0) \cdot V^{-1} \right) = \Lambda(z) \leftarrow \text{Spectral curve for tRS!}$$

Claim: $L = -M(0) V^{-1} = \begin{pmatrix} \frac{\xi - q\xi^{-1}}{\xi - \xi^{-1}} p_1 & \frac{(q-1)\xi}{q(\xi - \xi^{-1})} p_1 \end{pmatrix}$

$$\begin{pmatrix} \cdots & \cdots \\ -\frac{(q-1)\xi p_2}{q(\xi-\xi^{-1})} & \frac{\xi q-\xi^{-1}}{\xi-\xi^{-1}}\, p_2 \end{pmatrix}$$

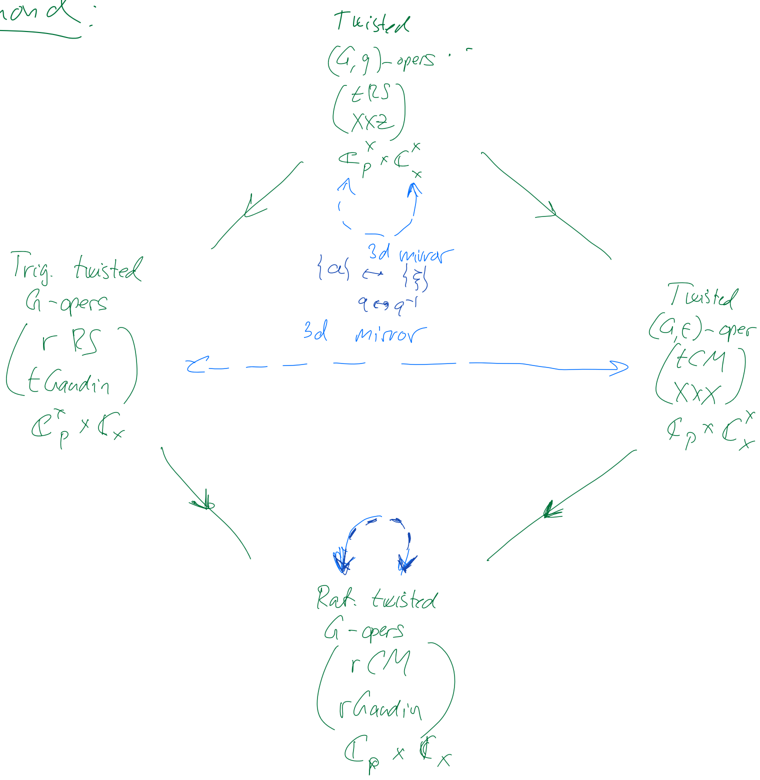
is the Lax matrix for the tRS model

Works for $SL(N)$.

$$\det(z-L)=\sum_{k=0}^{r} z^k(-1)^k H^{tRS}_{n-k}$$

$$\mathrm{Fun}(q\mathrm{Op}^1_2) \cong \frac{\mathbb{C}(q, \xi_i, p_i, a_i)}{\{H^{tRS}_k = e_k(a_1 \dots a_{r+1})\}} \cong \frac{\mathbb{C}(q, \xi_i, s_{k,i}, a_i)}{\text{Bethe}} \cong \frac{\mathbb{C}(q, \xi_i, p_i, a_i)}{q\mathrm{Wr}}$$

Diamond:



Elliptic generalizations: