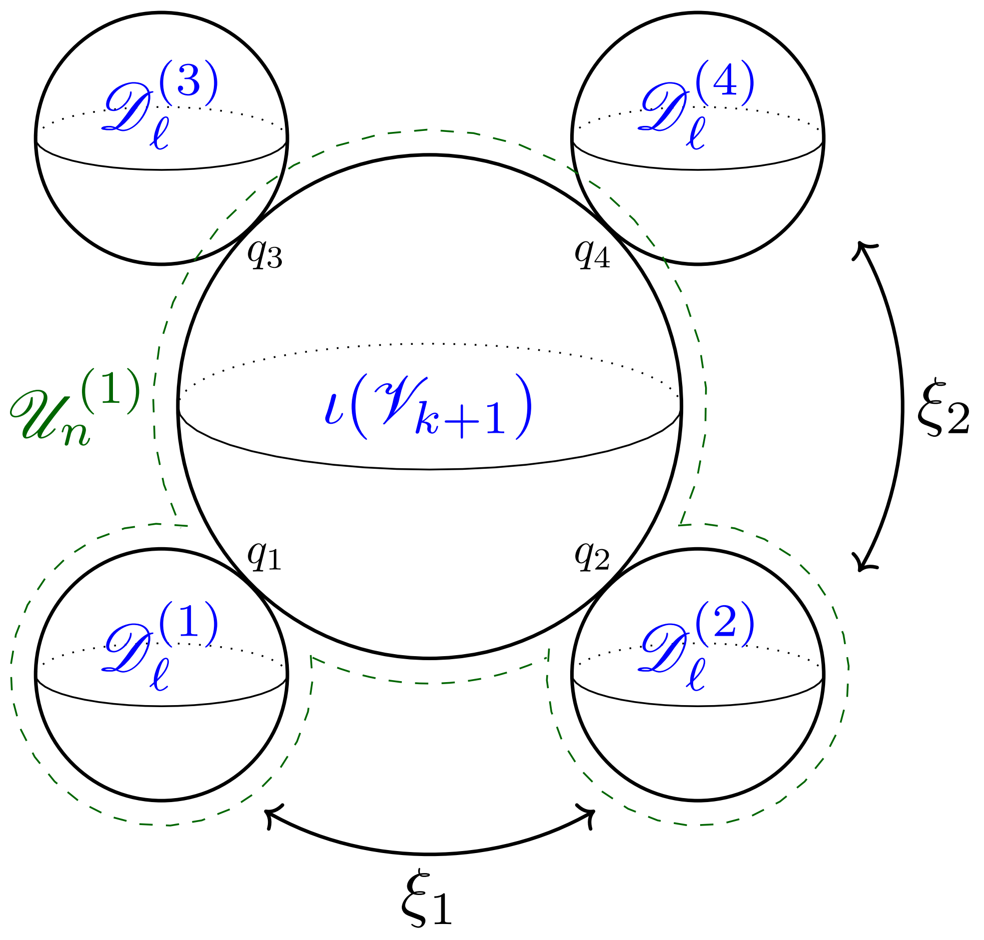
Branes & DAHA Representations

Peter Koroteev

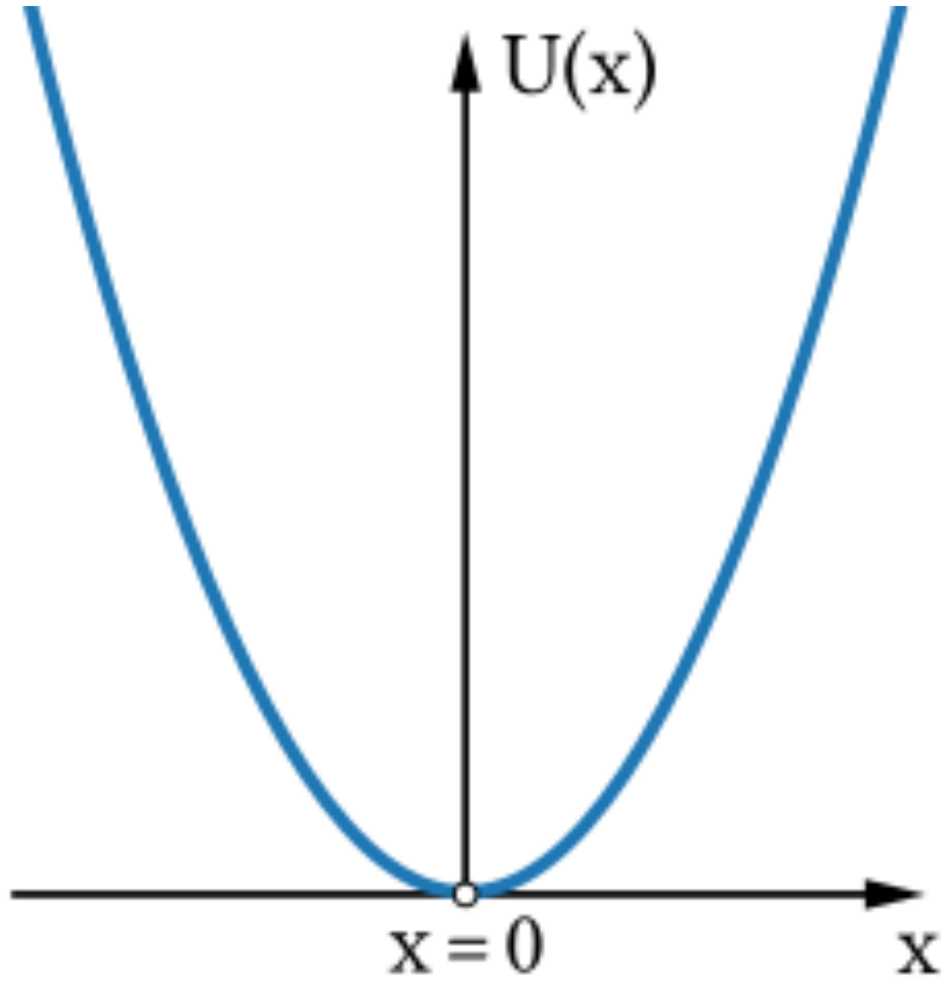
2206.03565 with S. Gukov, S. Nawata, D. Pei, and I. Saberi



$$\begin{tikzcd} 1 \arrow[r, bend left, "\mathsf{R}_0"] & P_1 \arrow[l, bend left, "\mathsf{L}_1"] \arrow[r, bend left, "\mathsf{R}_1"] & \cdots \arrow[l, bend left, "\mathsf{L}_2"] \end{tikzcd} \qquad \begin{tikzcd} \cdots \arrow[r, bend left, "\mathsf{R}_{j-2}"] & P_{j-1} \arrow[l, bend left, "\mathsf{L}_{j-1}"] \arrow[r, bend left, "\mathsf{R}_{j-1}"] & P_j \arrow[l, bend left, "\mathsf{L}_j"] \arrow[r, bend left, "\mathsf{R}_j"] & \cdots \arrow[l, bend left, "\mathsf{L}_{j+1}"] \end{tikzcd}$$

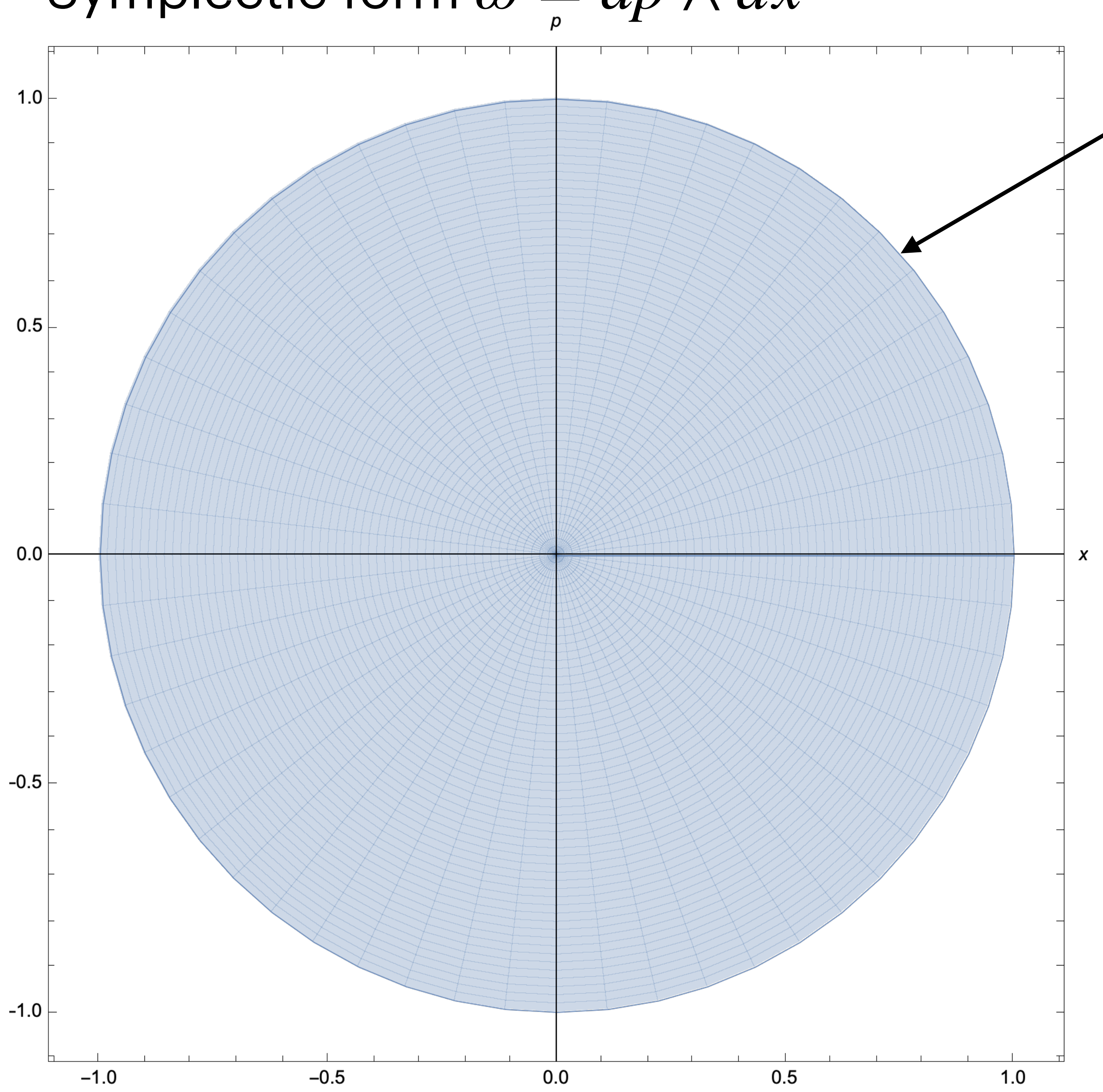
Symplectic Manifold

Harmonic oscillator $$H = \frac{p^2}{2} + \frac{x^2}{2}$$



Phase space — symplectic manifold $\mathscr{M}$

Symplectic form $\omega = dp \wedge dx$



$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$

Lagrangian $\mathscr{L} \subset \mathscr{M}$ is a middle-dimensional submanifold and such that the restriction of the symplectic form on $\mathscr{L}$ vanishes

$$\omega|_{\mathcal{L}} = 0$$

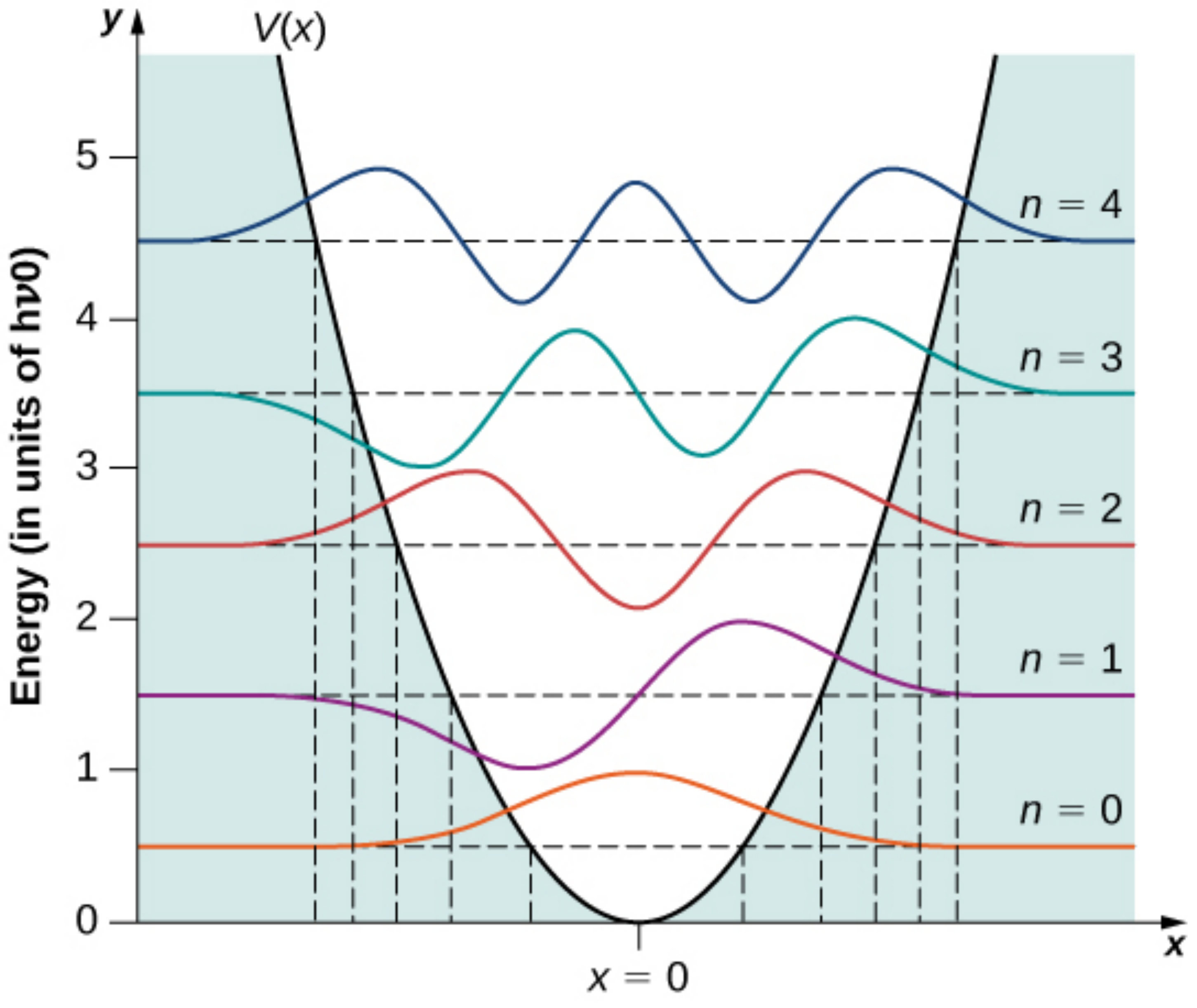
Symplectic form ω is locally exact on $\mathscr{L}$

$$\theta = d^{-1}\omega = p dx$$

Quantization as Symplectic Geometry

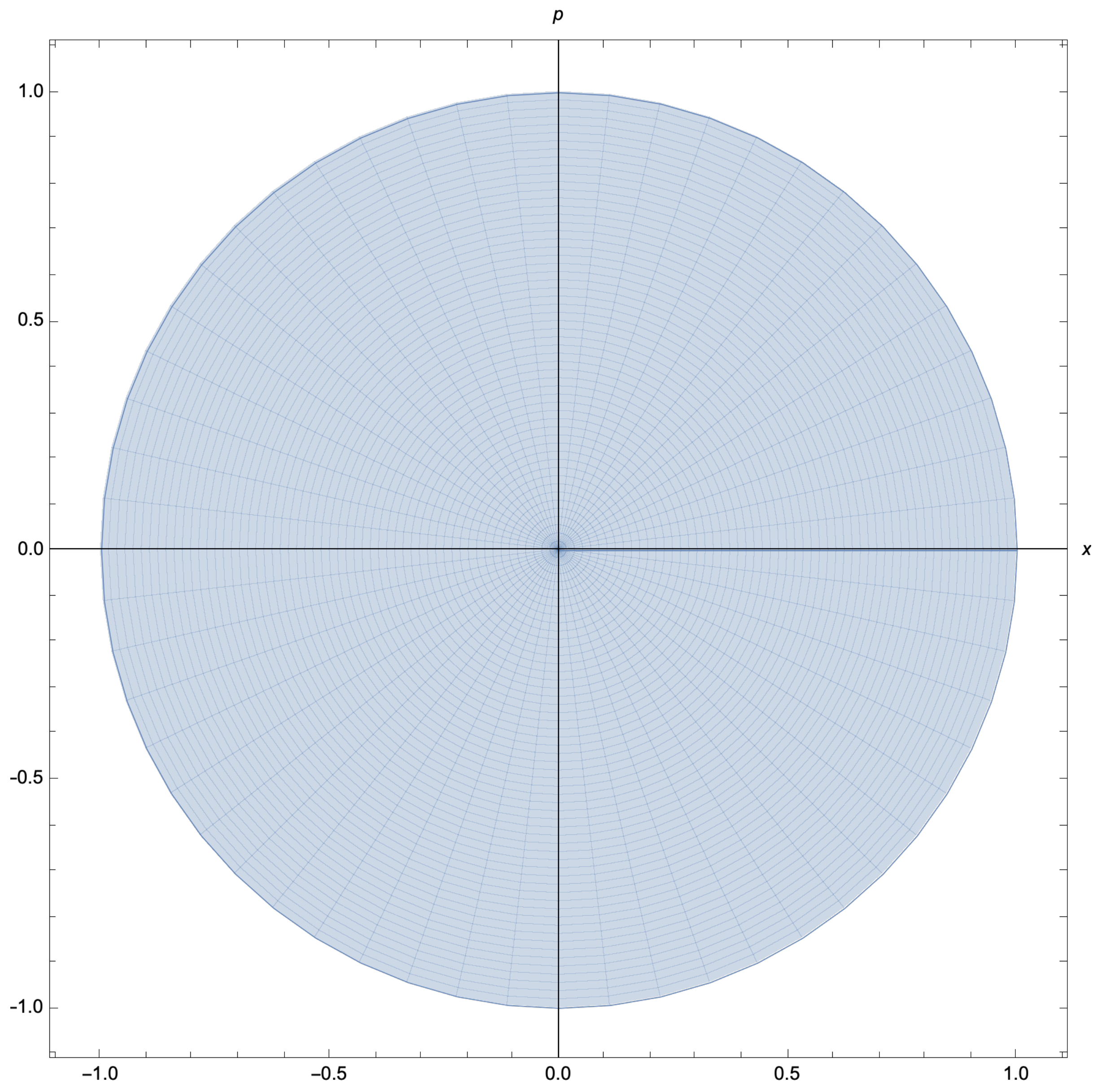
Quantum oscillator energy states

$$E_n = \hbar\left(n + \frac{1}{2}\right)$$



Symplectic area

$$E_n = \frac{1}{2\pi}\int dp \wedge dx \sim \oint_{\mathcal{L}} \theta$$



Quantization

Coordinates and momenta become operators

$$p, x \mapsto \hat{p}, \hat{x}$$

Poisson brackets associated to ω become commutators

$$\{A, B\}_{P.B.} \mapsto [A, B]$$

Heisenberg algebra

$$[\hat{p}, \hat{x}] = -i\hbar$$

$$\hat{x} f(x) = x f(x)$$
$$\hat{p} f(x) = -i\hbar f'(x)$$

Lagrangian constraint

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$

Replaced by operator

$$\left(\frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} - E\right) Z(x) = 0$$

This ODE has square integrable solutions only for special values of E

$$E_n = \hbar \left(n + \frac{1}{2}\right)$$

e.g. for $n = 0$ $\quad Z(x) \sim e^{-\frac{1}{2\hbar} x^2}$

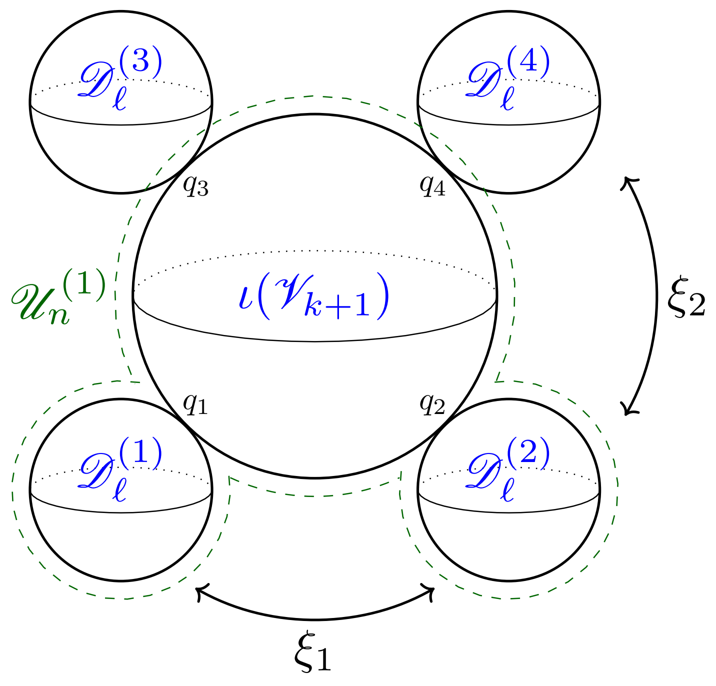
The Art of Quantization

Symplectic manifold $(\mathscr{M}, \omega)$ $\longrightarrow$ Hilbert space $\mathscr{H}$ — DAHA representations

Algebra of functions on $\mathscr{M}$ $\longrightarrow$ Algebra of operators on $\mathscr{H}$ — DAHA

Lagrangian submanifolds $\mathscr{L} \subset \mathscr{M}$ $\longrightarrow$ States in Hilbert space $\mathscr{H}$ — Highest weight vectors

$$\{f_i\} \qquad \hat{f}_i \, \mathcal{Z} = 0$$



$$\dim V_i \sim \mathrm{Vol}(\mathfrak{D}_i)$$

$$y\,\mathcal{Z} = (Y + Y^{-1})\mathcal{Z} = (a + a^{-1})\mathcal{Z}$$

Double Affine Hecke Algebra

- DAHA (and related algebras) were introduced by I. Cherednik in the study of Macdonald polynomials from the viewpoint of representation theory
- A. Oblomkov demonstrated that in Type A DAHA is flat one-parameter deformation (deformation quantization) of the Poisson structure on the Calogero-Moser (CM) space
- The CM space can be described as an SL(2,C) character variety of a torus with puncture. Using this we shall provide geometric construction of DAHA representations

Main Theorem

Let C_p be a punctured genus-one Riemann surface, $\mathfrak{X} = \mathscr{M}_{\text{flat}}(C_p, SL(2,\mathbb{C}))$ the moduli space of flat $SL(2,\mathbb{C})$ connections with prescribed monodromy at the puncture, and $S\ddot{H}(\mathbb{Z}_2)$ be the spherical subalgebra of DAHA of type A_1. Then there is a derived equivalence between the Fukaya category of $\mathfrak{X}$ and the category of finite-dimensional $S\ddot{H}(\mathbb{Z}_2)$-modules

$$D^b\mathscr{F}uk(\mathfrak{X}, \omega_{\mathfrak{X}}) \simeq D^b Rep(\ddot{H})$$

The left had side can be upgraded to a larger category of A-branes, while the right had side to all representations

Related Developments

- Holomorphic Floer Theory (generalized Riemann-Hilbert correspondence). A rigorous definition of branes and quantization. Category of holonomic D-modules. [Kontsevich, Soibelman]

- Painleve equations [Iohara et al]

- Wrapped Fukaya categories [Etgu, Lekili]

Double Affine Hecke Algebra rank 1

Let $\mathfrak{g}$ be Lie algebra. The (Iwahori)-**Hecke** algebra is defined as deformation of the group algebra of the Weyl group of $\mathfrak{g}$

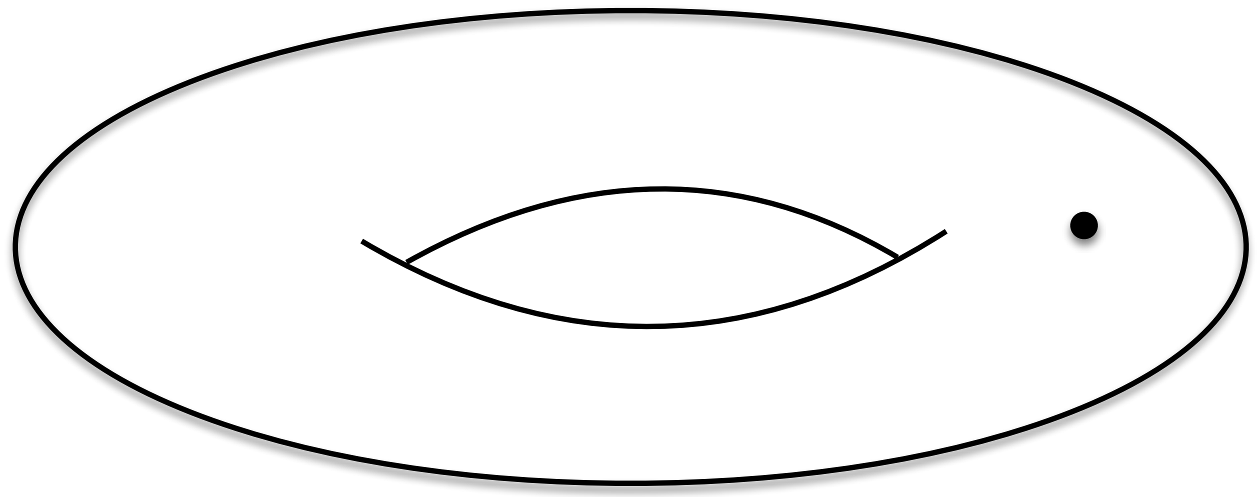
For $\mathfrak{sl}(2)$ it is generated by T with relation $(T-t)(T+t^{-1})=0$ where $t \in \mathbb{C}^{\times}$

Affine Hecke algebra (AHA) for $\mathfrak{sl}(2)$: $$\frac{\mathbb{C}(t^{\pm 1}) \otimes \mathbb{C}[X^{\pm 1}, T]}{\left(TXT - X^{-1}, (T-t)(T-t^{-1})\right)}$$

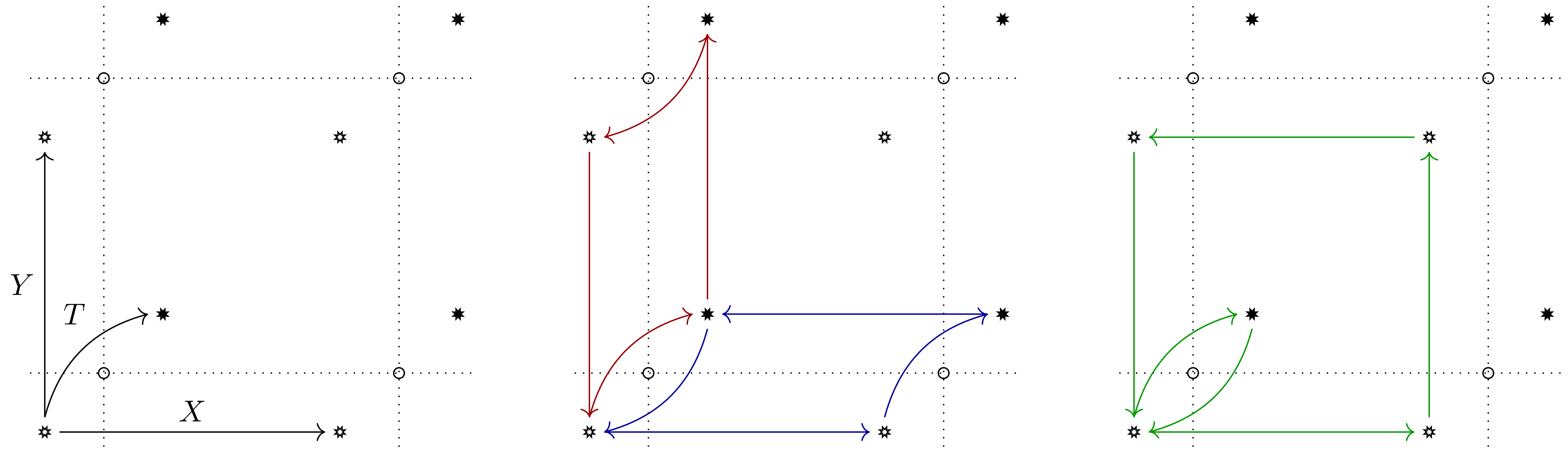
Double affine Hecke algebra for $\mathfrak{sl}(2)$ — two copies of AHA (X,T) and (Y,T) in the presence of additional relation and parameter $q \in \mathbb{C}^{\times}$

$$\ddot{H}(\mathbb{Z}_2) = \frac{\mathbb{C}(q^{\pm 1}, t^{\pm 1}) \otimes \mathbb{C}[X^{\pm 1}, Y^{\pm 1}, T]}{\left(TXT - X^{-1}, TYT - Y^{-1}, Y^{-1}X^{-1}YX - q^{-1}, (T-t)(T+t^{-1})\right)}$$

DAHA from Affine Braid Group



Orbifold fundamental group
of the torus with puncture $(T^2\backslash \mathrm{p})/\mathbb{Z}_2$



(a) (b) (c)

Generated by X, T, Y modulo relations $TXT = X^{-1}$, $TY^{-1}T = Y$, and $Y^{-1}X^{-1}YXT^2 = 1$

Its central extension is known as elliptic braid group is obtained
by deforming the last relation to

$$Y^{-1}X^{-1}YXT^2 = q^{-1}$$

The full $\mathfrak{sl}(2)$ DAHA is obtained by
imposing Hecke relation

$$\ddot{H}(\mathbb{Z}_2) = \mathbb{C}_{q,t}\left[T^{\pm1}, X^{\pm1}, Y^{\pm1}\right] \Big/ \left\{ \begin{array}{cc} TXT = X^{-1}\ , & Y^{-1}X^{-1}YXT^2 = q^{-1}\ , \\ TY^{-1}T = Y\ , & (T-t)(T+t^{-1}) = 0 \end{array} \right\}$$

Symmetries

Discrete symmetry $\Xi = \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\xi_1: \quad T \mapsto T, \quad X \mapsto -X, \quad Y \mapsto Y, \quad q \mapsto q, \quad t \mapsto t$$
$$\xi_2: \quad T \mapsto T, \quad X \mapsto X, \quad Y \mapsto -Y, \quad q \mapsto q, \quad t \mapsto t$$

Mapping class group of torus

$SL(2,\mathbb{C})$

$$\tau_+: \quad (X,Y,T) \mapsto (X, q^{-\frac{1}{2}}XY, T)$$
$$\tau_-: \quad (X,Y,T) \mapsto (q^{\frac{1}{2}}YX, Y, T)$$
$$\sigma: \quad (X,Y,T) \mapsto (Y^{-1}, XT^2, T)$$

Nonlinear involution

$$\tilde{\iota}: T \mapsto -T, \quad X \mapsto X, \quad Y \mapsto Y, \quad q \mapsto q, \quad t \mapsto t^{-1}$$

Spherical DAHA

Idempotent element $\mathbf{e} = (T + t^{-1})/(t + t^{-1})$

Spherical subalgebra $S\ddot{H} := \mathbf{e}\ddot{H}\mathbf{e}$

q-commutator

$$[a, b]_q := q^{-\frac{1}{2}} ab - q^{\frac{1}{2}} ba$$

Generators of spherical DAHA

$$x = X + X^{-1}$$
$$y = Y + Y^{-1}$$
$$z = q^{-\frac{1}{2}} Y^1 X + q^{\frac{1}{2}} X^{-1} Y$$

Relations

$$[x, y]_q = (q^{-1} - q)z$$
$$[y, z]_q = (q^{-1} - q)x$$
$$[z, x]_q = (q^{-1} - q)y$$
$$q^{-1}x^2 + qy^2 + q^{-1}z^2 - q^{-\frac{1}{2}}xyz = (q^{-\frac{1}{2}}t - q^{\frac{1}{2}}t^{-1})^2 + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2$$

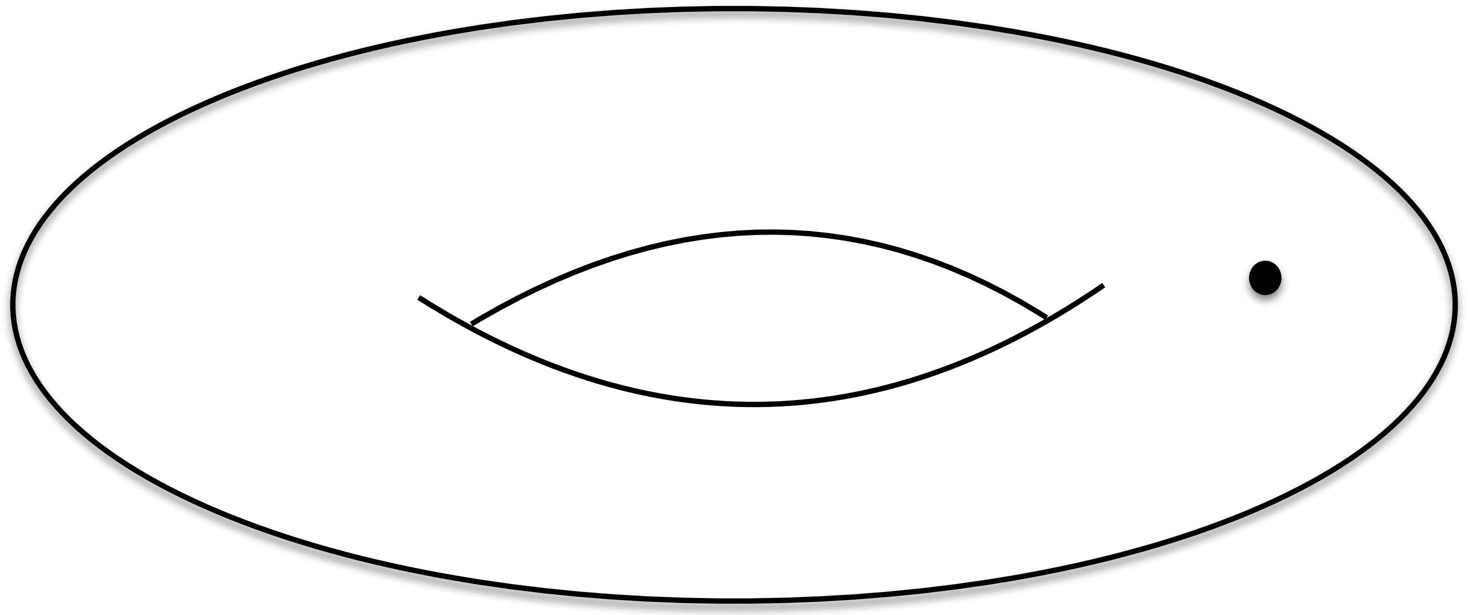
\`Classical' limit

$$S\ddot{H} \xrightarrow[q \to 1]{} \mathscr{O}(\mathcal{M}_{\text{flat}}(C_p, \mathrm{SL}(2, \mathbb{C})))$$

Coordinate ring of the moduli space of $SL(2,\mathbb{C})$ flat connections on punctured torus

$$\mathcal{M}_{\text{flat}}(C_p, \mathrm{SL}(2, \mathbb{C})) = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^2 + z^2 - xyz - 2 = \mathrm{Tr}(\rho(\mathfrak{c})) = \tilde{t}^2 + \tilde{t}^{-2}\}$$

SL(2,$\mathbb{C}$) Flat Connection on Punctured Torus



Fundamental group $\pi_1(C_p) = \langle \mathfrak{m}, \mathfrak{l}, \mathfrak{c} | \mathfrak{m}\mathfrak{l}\mathfrak{m}^{-1}\mathfrak{l}^{-1} = \mathfrak{c} \rangle$

Let $\rho : \pi_1(C_p) \to \mathrm{SL}(2, \mathbb{C})$

$$x = \mathrm{Tr}(\rho(\mathfrak{m})), \;\; y = \mathrm{Tr}(\rho(\mathfrak{l})), \;\; \text{and} \;\; z = \mathrm{Tr}\big(\rho(\mathfrak{m}\mathfrak{l}^{-1})\big)$$

Markov cubic $\mathcal{M}_{\text{flat}}(C_p, \mathrm{SL}(2,\mathbb{C})) = \{(x,y,z) \in \mathbb{C}^3 | x^2 + y^2 + z^2 - xyz - 2 = \mathrm{Tr}(\rho(\mathfrak{c})) = \tilde{t}^2 + \tilde{t}^{-2}\}$

Elliptic fibration of Kodaira type I_0^*

[Oblomkov]

Theorem. Spherical DAHA is a ***deformation quantization*** of the coordinate ring of the moduli space of flat $SL(2,\mathbb{C})$ connections $\mathfrak{X} = \mathscr{M}_{\text{flat}}(C_p, SL(2,\mathbb{C}))$ with respect to Poisson structure Ω_J

$$\Omega_J = \frac{1}{2\pi i} \frac{dx \wedge dy}{\partial f / \partial z} = \frac{1}{2\pi i} \frac{dx \wedge dy}{2z - xy}$$

$$\dim V_i \sim \mathrm{Vol}(\mathfrak{D}_i)$$

Next: 1) Representations of (spherical) DAHA — $Rep(\ddot{H})$

2) Lagrangian submanifolds of $\mathfrak{X}$ whose quantization yields these representations — $\mathscr{F}uk(\mathfrak{X}, \omega_{\mathfrak{X}})$

Brane quantization

DAHA Representations

We will talk about polynomial representations of DAHA $\qquad \mathscr{P} := \mathbb{C}_{q,t}[X^{\pm}]^{\breve{\mathbb{Z}}_2}$

$$\mathrm{pol}: S\ddot{H} \to \mathrm{End}(\mathscr{P}), \quad \begin{aligned} x &\mapsto X + X^{-1}, \\ y &\mapsto \frac{tX - t^{-1}X^{-1}}{X - X^{-1}}\varpi + \frac{t^{-1}X - tX^{-1}}{X - X^{-1}}\varpi^{-1}, \\ z &\mapsto q^{\frac{1}{2}}X\frac{tX - t^{-1}X^{-1}}{X - X^{-1}}\varpi + q^{\frac{1}{2}}X^{-1}\frac{t^{-1}X - tX^{-1}}{X - X^{-1}}\varpi^{-1} \end{aligned}$$

Shift operator

$$\varpi^{\pm}(X) = q^{\pm}X$$

Highest weight representation for y $\qquad y\,\mathcal{Z} = (Y + Y^{-1})\mathcal{Z} = (a + a^{-1})\mathcal{Z}$

For arbitrary value of a the eigenvector is a series of hypergeometric type which arises in enumerative geometry [PK, Zeitlin]

When $a = q^j t$ we get Macdonald polynomials of type A_1 labelled spin-j/2 representation

$$P_j(X; q, t) := X^j\, {}_2\phi_1(q^{-2j}, t^2; q^{-2j+2}t^{-2}; q^2; q^2t^{-2}X^{-2})$$

Macdonald Polynomials

$$P_1 = X + X^{-1}$$

$$P_2 = X^2 + X^{-2} + \frac{(q+1)(t-1)}{qt-1}$$

$$P_3 = X^3 + X^{-3} + \frac{(q^2+q+1)(t-1)}{q^2t-1}(X^{-1} + X)$$

Polynomial Representation

Macdonald Polynomials generate the ring $\mathscr{P}$ over $\mathbb{C}[q^{\pm 1}, t^{\pm 1}]$

Raising and lowering operators

$$\begin{aligned} \mathsf{R}_j &:= x - q^{j-\frac{1}{2}} t z = X(q^j t^{-1} Y - q^{2j} t^2) + X^{-1}(q^j t Y^{-1} - q^{2j} t^2) \ , \\ \mathsf{L}_j &:= x - q^{-j-\frac{1}{2}} t^{-1} z = X(q^{-j} t^{-3} Y - q^{-2j} t^{-2}) + X^{-1}(q^{-j} t^{-1} Y^{-1} - q^{-2j} t^{-2}) \end{aligned}$$

$$\begin{tikzcd} 1 \arrow[r, bend left, "\mathsf{R}_0"] & P_1 \arrow[l, bend left, "\mathsf{L}_1"] \arrow[r, bend left, "\mathsf{R}_1"] & \cdots \arrow[l, bend left, "\mathsf{L}_2"] & \cdots \arrow[r, bend left, "\mathsf{R}_{j-2}"] & P_{j-1} \arrow[l, bend left, "\mathsf{L}_{j-1}"] \arrow[r, bend left, "\mathsf{R}_{j-1}"] & P_j \arrow[l, bend left, "\mathsf{L}_j"] \arrow[r, bend left, "\mathsf{R}_j"] & \cdots \arrow[l, bend left, "\mathsf{L}_{j+1}"] \end{tikzcd}$$

Action

$$\begin{aligned} \mathrm{pol}(\mathsf{R}_j) \cdot P_j(X;q,t) &= (1 - q^{2j} t^2) P_{j+1}(X;q,t) \ , \\ \mathrm{pol}(\mathsf{L}_j) \cdot P_j(X;q,t) &= \frac{(1-q^{2j})(1-q^{2(j-1)}t^4)}{q^{2j}t^2(q^{2(j-1)}t^2-1)} P_{j-1}(X;q,t) \end{aligned}$$

Finite-Dimensional Representations

Shortening condition $\quad \mathrm{pol}(\mathsf{L}_j)\cdot P_j = 0$

Raising operator will never be null due to $\quad (1-q^{2j}t^2)$

$$\frac{(1-q^{2j})(1-q^{(j-1)}t^2)(1+q^{(j-1)}t^2)}{q^{2j}t^2(q^{2(j-1)}t^2-1)}$$

must vanish

$$q^{2n} = 1 \ ,$$
$$t^2 = -q^{-k} \ ,$$
$$t^2 = q^{-(2\ell-1)}$$

$$\begin{tikzcd}
1 \arrow[r, bend left, "\mathsf{R}_0"] & P_1 \arrow[l, bend left, "\mathsf{L}_1"] \arrow[r, bend left, "\mathsf{R}_1"] & \cdots \arrow[l, bend left, "\mathsf{L}_2"] & \cdots \arrow[r, bend left, "\mathsf{R}_{j-2}"] & P_{j-1} \arrow[l, bend left, "\mathsf{L}_{j-1}"] \arrow[r, bend left, "\mathsf{R}_{j-1}"] & P_j \arrow[l, bend left, "\mathsf{L}_j"] \arrow[r, bend left, "\mathsf{R}_j"] & \cdots \arrow[l, bend left, "\mathsf{L}_{j+1}"]
\end{tikzcd}$$

0

Short exact sequence of modules $\quad 0 \to S \to V \to V/S \to 0$

Higgs Bundles

Nonabelian Hodge correspondence relates representations of the fundamental group of smooth projective algebraic varieties with Higgs bundles (E, φ)

$$\mathfrak{X} \simeq \mathcal{M}_H(C_p, SU(2))$$

Hitchin moduli space

Holomorphic $SU(2)$ vector bundle over C_p with holomorphic section φ (Higgs field) of $K_{C_p} \otimes \mathrm{ad}(E) \otimes \mathcal{O}(p)$

Tame ramification at p

$$A = \alpha_p \, d\vartheta + \cdots$$
$$\varphi = \frac{1}{2}(\beta_p + i\gamma_p)\frac{dz}{z} + \cdots$$

Hitchin moduli space is the space of solutions of Hitchin equations modulo gauge transformations

$$F - [\varphi, \overline{\varphi}] = 0$$
$$\overline{D}_A \, \varphi = 0$$

NAHC:

$$\mathcal{A} = A + i(\varphi + \bar{\varphi})$$

Hitchin equations equivalent to flatness condition

$$F_{\mathcal{A}} = 0$$

Complex and Kähler Structures

The space $\mathscr{M}_H(C_p, SU(2))$ is hyperKahler

$$\omega_I = -\frac{i}{2\pi}\int_C |d^2z|\ \mathrm{Tr}\Big(\delta A_{\bar z}\wedge\delta A_z - \delta\bar\varphi\wedge\delta\varphi\Big)\ ,$$
$$\omega_J = \frac{1}{2\pi}\int_C |d^2z|\ \mathrm{Tr}\Big(\delta\bar\varphi\wedge\delta A_z + \delta\varphi\wedge\delta A_{\bar z}\Big)\ ,$$
$$\omega_K = \frac{i}{2\pi}\int_C |d^2z|\ \mathrm{Tr}\Big(\delta\bar\varphi\wedge\delta A_z - \delta\varphi\wedge\delta A_{\bar z}\Big)\ .$$

Triplet of holomorphic symplectic forms

$$\Omega_I = \omega_J + i\omega_K,\ \Omega_J = \omega_K + i\omega_I,\ \ \Omega_k = \omega_I + i\omega_J$$

Complex structure	Complex modulus	Kähler modulus
I	$\beta_p + i\gamma_p$	α_p
J	$\gamma_p + i\alpha_p$	β_p
K	$\alpha_p + i\beta_p$	γ_p

Geometry of $\mathfrak{X}$

$$x^2 + y^2 + z^2 - xyz - 2 - t^2 - t^{-2} = 0$$

Symplectic form $$\Omega_J = \frac{1}{2\pi i}\frac{dx \wedge dy}{\partial f/\partial z} = \frac{1}{2\pi i}\frac{dx \wedge dy}{2z - xy}$$

Kahler form $$\omega_J = \frac{i}{4\pi}(dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z})$$

$$A = \alpha_p \, d\vartheta + \cdots$$
$$\varphi = \frac{1}{2}(\beta_p + i\gamma_p)\frac{dz}{z} + \cdots$$

Holonomy around puncture $$\begin{pmatrix} t^2 & 0 \\ 0 & t^{-2} \end{pmatrix} = e^{2\pi(\gamma_p + i\alpha_p)}$$

When $t = 1$ $$\mathcal{M}_{\text{flat}}(T^2, SL(2,\mathbb{C})) \simeq \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2}$$

Real slice $$\mathcal{M}_{\text{flat}}(T^2, SU(2)) \simeq \frac{S^1 \times S^1}{\mathbb{Z}_2}$$

‘Pillow case’

Geometry of $\mathfrak{X}$

Hitchin fibration $\pi : \mathcal{M}_H(C_p, SU(2)) \to \mathcal{B}_H$ whose fibers are Abelian varieties (Liouville tori)

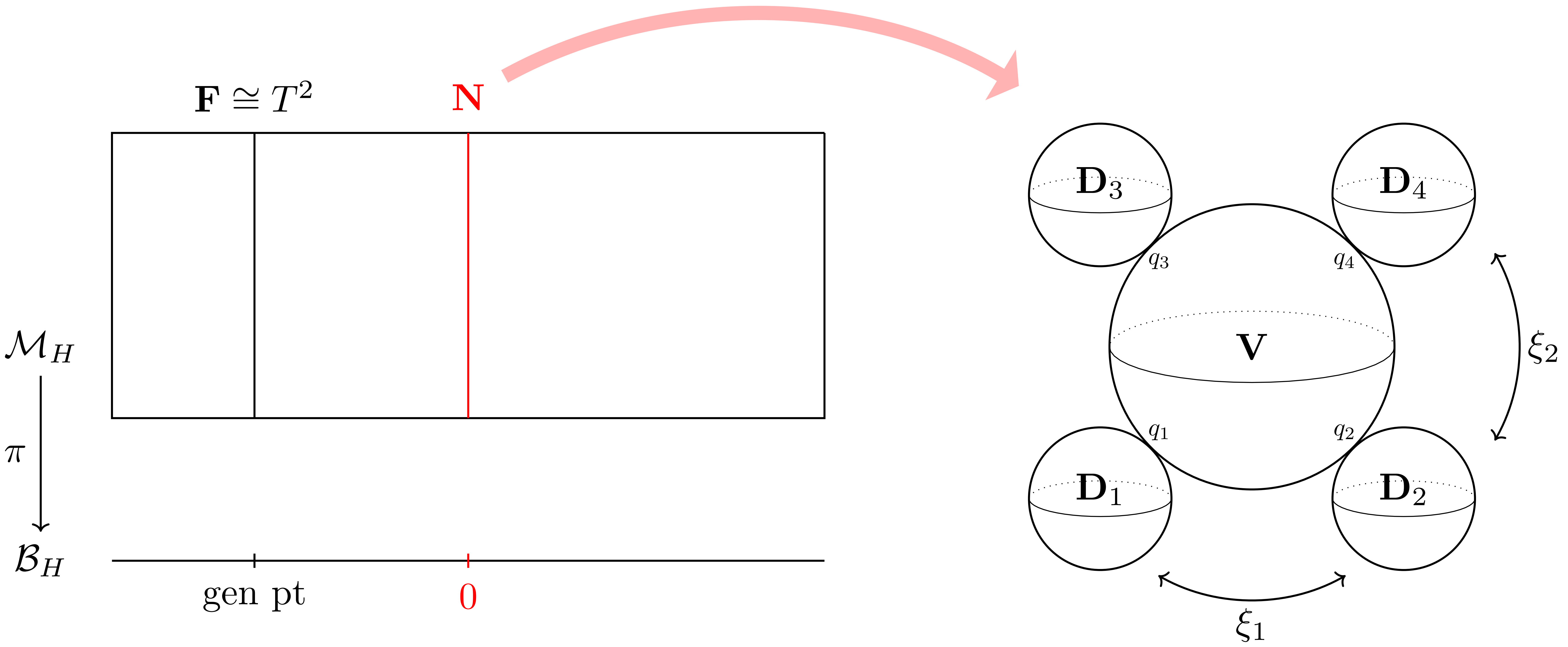
$(E, \varphi) \mapsto \mathrm{Tr}\varphi^2$ Holomorphic in complex structure I

The only singular fiber is pre-image of zero $\mathbf{N} = \pi^{-1}(0)$

$$\mathbf{N} = \mathbf{V} \cup \bigcup_{i=1}^{4} \mathbf{D}_i$$

\`Pillowcase' for $\alpha_p = \beta_p = \gamma_p = 0$

$$\mathbf{V} \cong (S^1 \times S^1)/\mathbb{Z}_2$$



Away from $\beta_p = 0$ locus —

resolution of A_1 singularities (exceptional divisors).

β_p — Kahler structure parameter in J

Holomorphic Lagrangians with respect to Ω_I

Branes of type (B,A,A)

$H_2(\mathcal{M}_H(C_p, G), \mathbb{Z})$

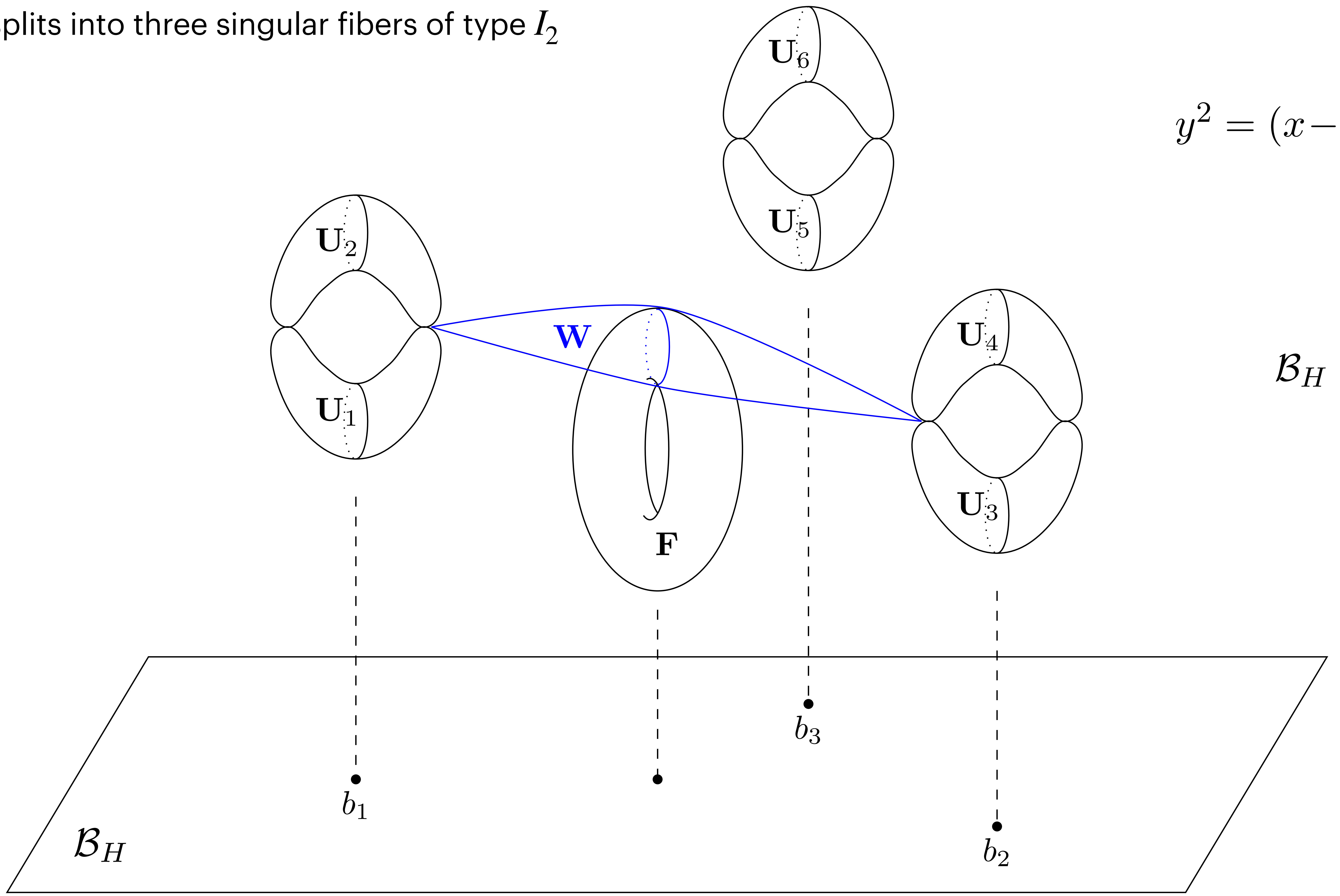
Null vector of intersection form

$$[\mathbf{F}] = 2[\mathbf{V}] + \sum_{i=1}^{4} [\mathbf{D}_i]$$

$\widehat{D}_4$ Dynkin diagram

Complex/Kähler Structure Deformations

For generic values of (β_p, γ_p) the embeddings of two-cycles $\mathbf{D}_i$, $\mathbf{V}$ intro $\mathscr{M}_H$ are no longer holomorphic w.r.t. I and singular fiber of type I_0^* splits into three singular fibers of type I_2

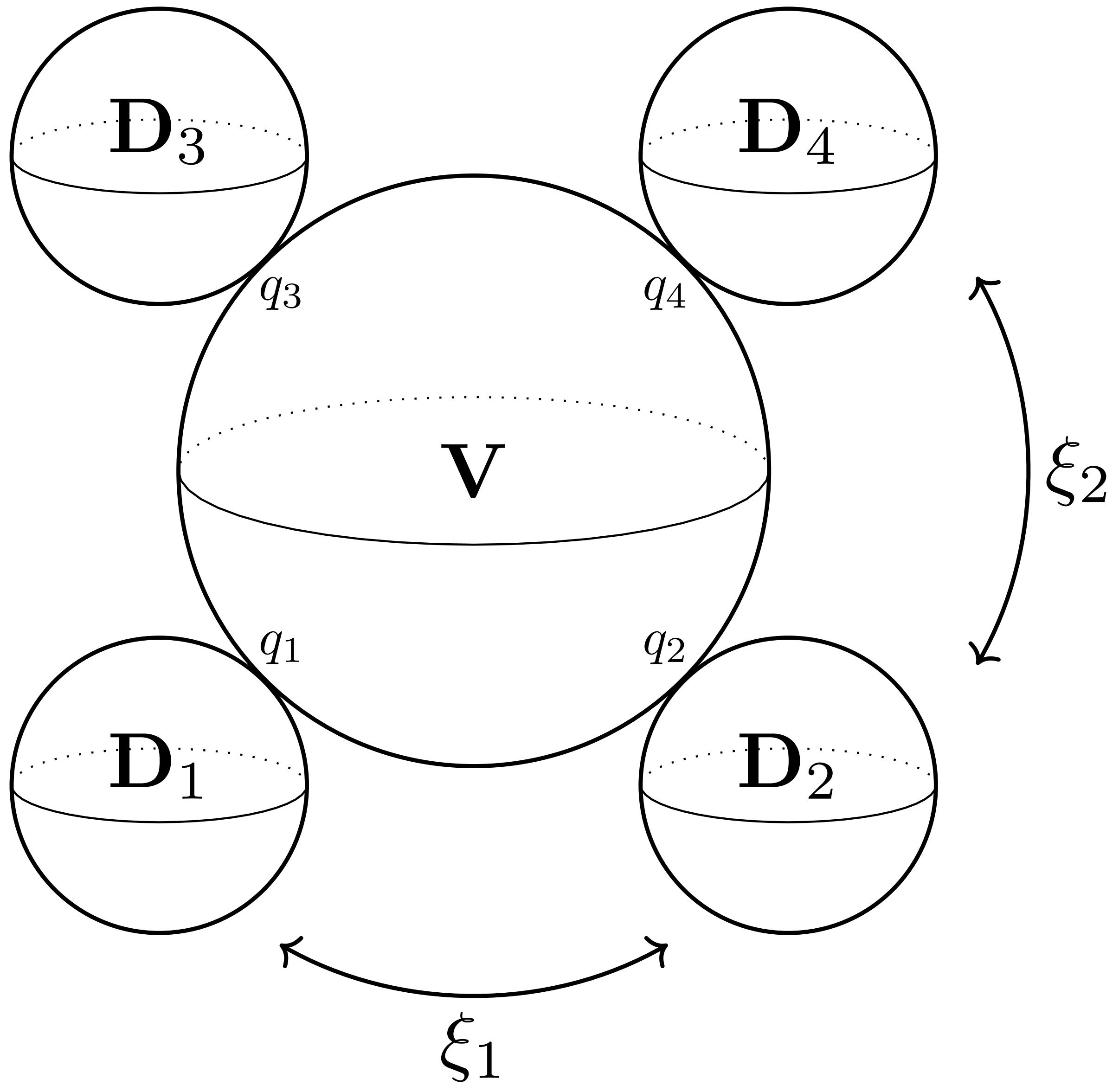


$$y^2 = (x-e_1)(x-e_2)(x-e_3) \text{ with } e_1+e_2+e_3=0$$

$$\mathcal{B}_H \ni b_i := e_i \operatorname{Tr}(\beta_p + i\gamma_p)^2 \qquad (i=1,2,3)$$

$$[\mathbf{U}_1] = [\mathbf{V}] + [\mathbf{D}_1] + [\mathbf{D}_2]$$
$$[\mathbf{U}_2] = [\mathbf{V}] + [\mathbf{D}_3] + [\mathbf{D}_4]$$

Cycles



Pillowcase

$$\int_{\mathbf{V}} \frac{\omega_I}{2\pi} = \frac{1}{2} - |\alpha_p|$$
$$\int_{\mathbf{V}} \frac{\omega_J}{2\pi} = -\beta_p \ ,$$
$$\int_{\mathbf{V}} \frac{\omega_K}{2\pi} = -\gamma_p \ ,$$

Hitchin fiber

$$\int_{\mathbf{F}} \frac{\omega_I}{2\pi} = 1 \ , \qquad \int_{\mathbf{F}} \frac{\omega_J}{2\pi} = 0 = \int_{\mathbf{F}} \frac{\omega_K}{2\pi}$$

Exceptional divisors $\quad i = 1, 2, 3, 4.$

$$\frac{\alpha_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_I}{2\pi} \ , \qquad \frac{\beta_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_J}{2\pi} \ , \qquad \frac{\gamma_p}{2} = \int_{\mathbf{D}_i} \frac{\omega_K}{2\pi}$$

Symmetries

$$\xi_1 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_2 \quad \text{and} \quad \mathbf{D}_3 \leftrightarrow \mathbf{D}_4$$
$$\xi_2 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_3 \quad \text{and} \quad \mathbf{D}_2 \leftrightarrow \mathbf{D}_4$$
$$\xi_3 : \mathbf{D}_1 \leftrightarrow \mathbf{D}_4 \quad \text{and} \quad \mathbf{D}_2 \leftrightarrow \mathbf{D}_3$$

$$\xi_i : \mathbf{U}_{2i+1} \leftrightarrow \mathbf{U}_{2i+2} \quad \text{and} \quad \mathbf{U}_{2i+3} \leftrightarrow \mathbf{U}_{2i+4}$$

Canonical Coisotropic Brane

[Gukov Witten]
[Kapustin Orlov]

Canonical coisotropic brane $\mathfrak{B}_{\mathrm{cc}}$: $\begin{array}{c}\mathcal{L}\\ \downarrow\\ \mathfrak{X}\end{array}$ $\quad c_1(\mathcal{L}) = [F/2\pi] \in H^2(\mathfrak{X}, \mathbb{Z})$

2d sigma model into $\mathfrak{X}$

A-branes are flat unitary bundles over Lagrangian submanifold with respect to $\omega_{\mathfrak{X}} = Im\left(\frac{i}{\hbar}\Omega_J\right)$

$\hbar = |\hbar| e^{i\theta}$

Quantization parameter $q = e^{2\pi i\hbar}$

Family of $\mathfrak{B}_{cc}$ branes parameterized by $\hbar$ on symplectic manifold $(\mathfrak{X}, \omega_{\mathfrak{X}})$

Values of the B-field are determined by equation $\quad \Omega := F + B + i\omega_{\mathfrak{X}} = \frac{\Omega_J}{i\hbar}$ $\quad B \in H^2(\mathfrak{X}, \mathrm{U}(1))$

$$F + B = \mathrm{Re}\ \Omega = \frac{1}{|\hbar|}(\omega_I \cos\theta - \omega_K \sin\theta)\ ,$$
$$\omega_{\mathfrak{X}} = \mathrm{Im}\ \Omega = -\frac{1}{|\hbar|}(\omega_I \sin\theta + \omega_K \cos\theta)\ .$$

HyperKahler condition $\quad F + B = \omega_{\mathfrak{X}}\, J$

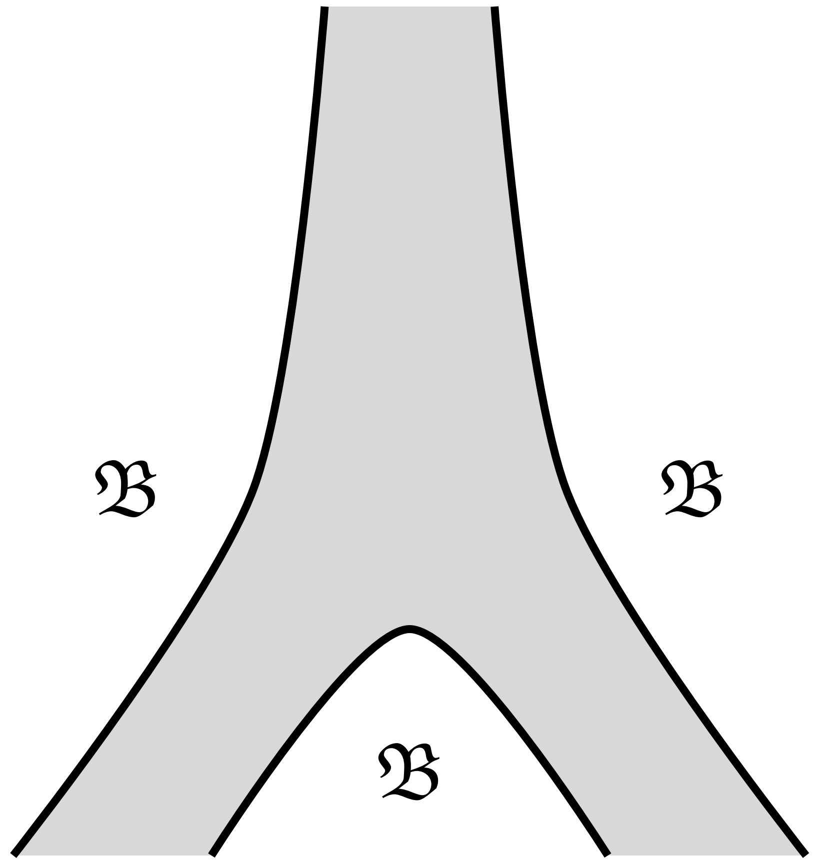
E.g. for real $\hbar$ we have $\omega_{\mathfrak{X}} = \omega_K$ and $\mathfrak{B}_{cc}$ brane is of type (B, A, A), for purely imaginary of type (A, A, B)

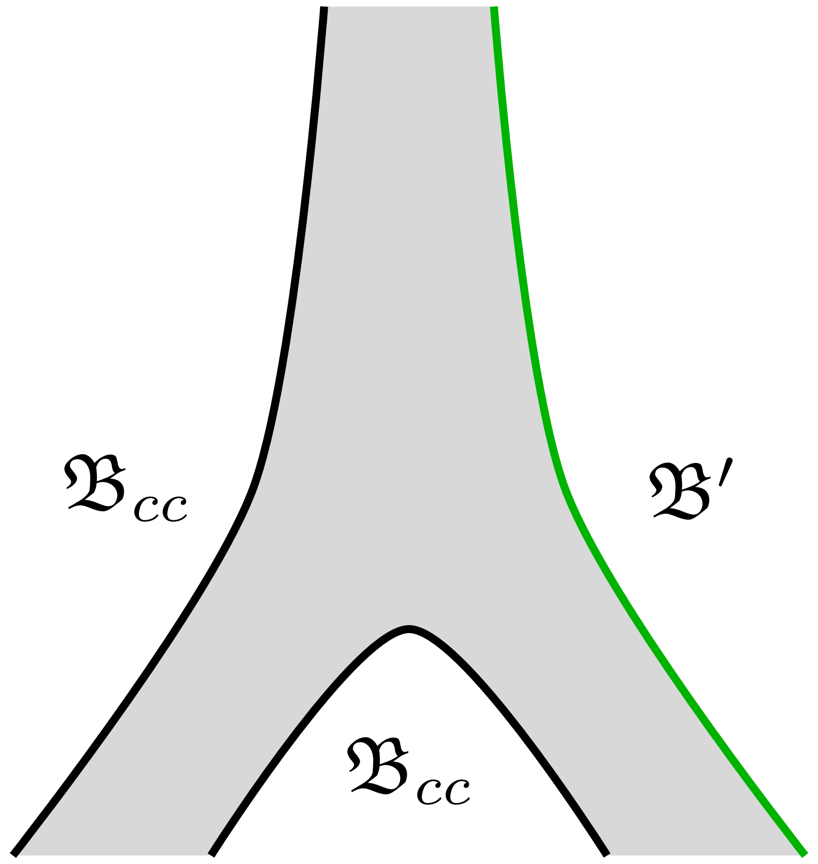
Branes and Quantization

$\mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}})$ parameterized by $\hbar$ provides deformation of the space of holomorphic functions on $\mathfrak{X}$ which is spherical DAHA $= S\ddot{H}$

$$\left(\omega_{\mathfrak{X}}^{-1}(B+F)\right)^2 = J^2 = -1 \qquad \int_{\mathbf{F}} \frac{\Omega}{2\pi} = \frac{1}{\hbar} \qquad \frac{1}{2\pi}\int_{\mathbf{D}_i} F + B + i\omega_{\mathfrak{X}} = \int_{\mathbf{D}_i} \frac{\Omega_J}{2\pi i\hbar} = \frac{\gamma_p + i\alpha_p}{2i\hbar} = -c + \frac{1}{2}$$

$$q = t^c$$





$$\begin{array}{ccc} \mathscr{O}^q(\mathfrak{X}) & = & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathrm{cc}}) \\ \circlearrowright & & \circlearrowright \\ \mathscr{B}' & = & \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}') \end{array}$$

$$\mathrm{End}(\mathfrak{B}_{\mathrm{cc}}) \cong S\ddot{H}$$

Lagrangian Branes

$$\mathfrak{B}_{\mathbf{L}}: \quad \begin{array}{c} \mathcal{L}' \otimes K_{\mathbf{L}}^{-1/2} \\ \downarrow \\ \mathbf{L} \end{array}$$

Flatness condition $F'_{\mathbf{L}} + B|_{\mathbf{L}} = 0$

Representation space $\mathscr{L} := \mathrm{Hom}(\mathfrak{B}_{cc}, \mathfrak{B}_{\mathbf{L}})$

Grothendieck-Riemann-Roch formula

$$\begin{aligned} \dim \mathscr{L} &= \dim H^0(\mathbf{L}, \mathfrak{B}_{cc} \otimes \mathfrak{B}_{\mathbf{L}}^{-1}) \\ &= \int_{\mathbf{L}} \mathrm{ch}(\mathfrak{B}_{cc}) \wedge \mathrm{ch}(\mathfrak{B}_{\mathbf{L}}^{-1}) \wedge \mathrm{Td}(T\mathbf{L}) \end{aligned}$$

For a Lagrangian in two dimensions

$$\mathrm{Td}(T\mathbf{L}) = \mathrm{ch}(K_{\mathbf{L}}^{-1/2}) \widehat{A}(T\mathbf{L})$$

So the dimension reads

$$\dim \mathscr{L} = \int_{\mathbf{L}} \mathrm{ch}(\mathfrak{B}_{cc}) = \int_{\mathbf{L}} \frac{F + B}{2\pi}$$

Lagrangian branes are objects in Fukaya category $\mathsf{Fuk}(\mathfrak{X}, \omega_{\mathfrak{X}})$

Representations vs Branes: Generic fiber F

Generic fiber $\theta = 0$

$$\omega_{\mathfrak{X}} = -\frac{\omega_K}{\hbar}\ , \quad \text{and} \quad F + B = \frac{\omega_I}{\hbar}$$

$$\dim \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}^{\lambda}_{\mathbf{F}}) = \int_{\mathbf{F}} \frac{F+B}{2\pi} = \int_{\mathbf{F}} \frac{\omega_I}{2\pi\hbar} = \frac{1}{\hbar}$$

Quantization condition $\hbar = 1/m$

Shortening condition $q = e^{2\pi i/m}$

Finite-dimensional representation $\mathscr{F}_m^{(x_m,+)} = \mathscr{P}/(X^m + X^{-m} - x_m - x_m^{-1})$

Singular fibers of type I_2

$$F'_{\mathbf{U}_1} + B|_{\mathbf{U}_1} = 0$$

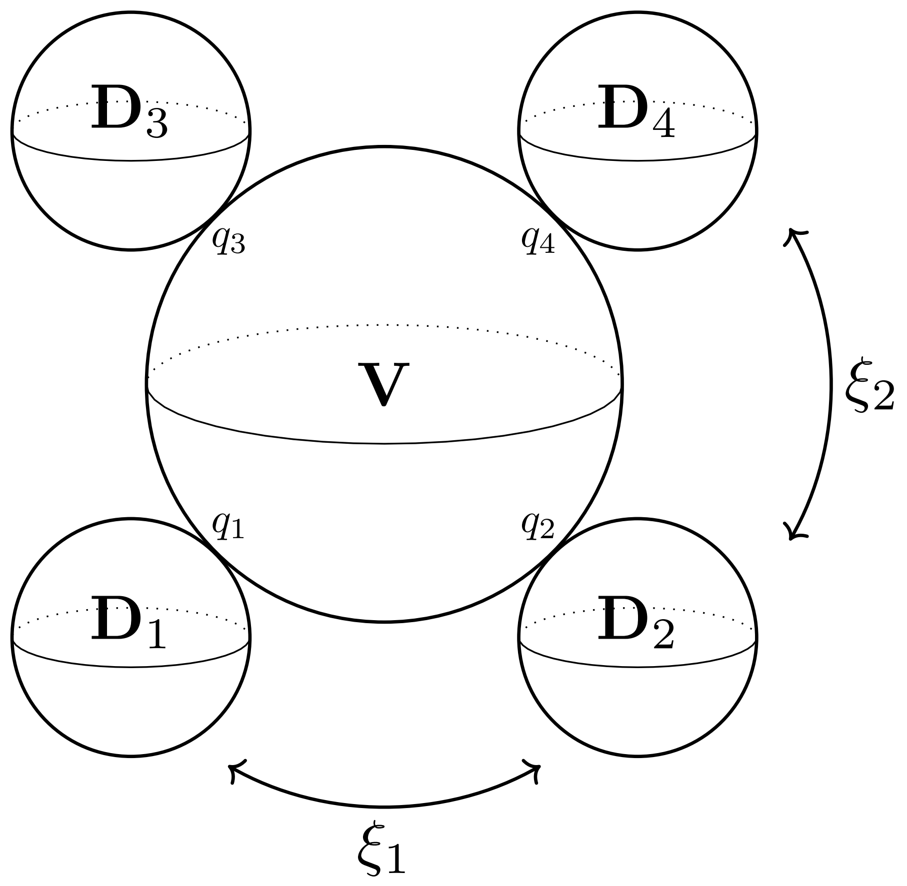
brane $\mathfrak{B}_{\mathbf{U}_1}$ can exist only at $1/(2\hbar) = n \in \mathbb{Z}_{>0}$

$$\dim \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{U}_1}) = \int_{\mathbf{U}_1} \frac{F+B}{2\pi} = \int_{\mathbf{U}_1} \frac{\omega_I}{2\pi\hbar} = \frac{1}{2\hbar}$$

Representation $\quad \mathrm{pol}(\mathsf{L}_n) \cdot P_n(X;q,t) = 0 \quad$ where $\quad P_n(X;q,t) = X^n + X^{-n}$

$$\mathscr{U}_n^{(1)} := \mathscr{P}/(P_n) \qquad\qquad \mathscr{U}_n^{(1)} = \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{U}_1})$$

Bun_G Component



Assume $\beta_p = 0$ for simplicity. For V to be Lagrangian with respect to $\omega_{\mathfrak{X}}$ the following should hold

$$\mathrm{Im}\ \frac{\left(\frac{1}{2} - \alpha_p\right) + i\gamma_p}{\hbar} \ = \ 0$$

There is no deformation parameter

$$\dim \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{V}}) = \int_{\mathbf{V}} \frac{F+B}{2\pi} = \frac{1}{2\hbar} - \frac{\gamma_p + i\alpha_p}{i\hbar} = \frac{1}{2\hbar} + 2c - 1$$

$$q = t^c$$

Shortening condition

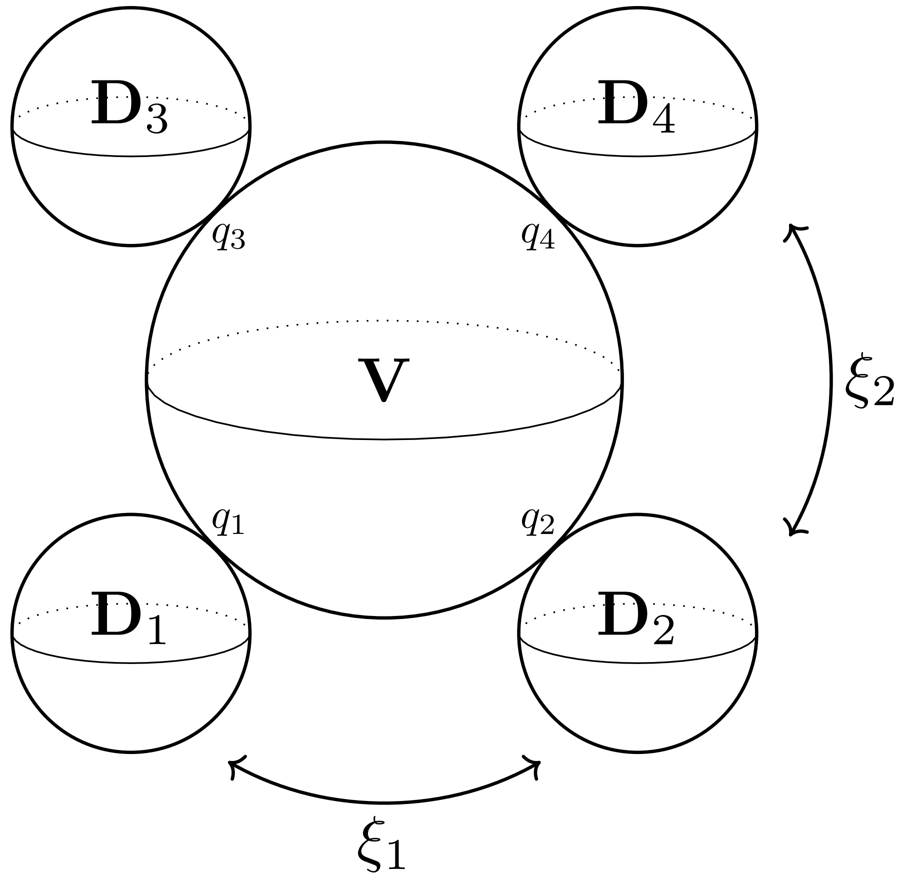
$$\frac{1}{2\hbar} + 2c - 1 = k + 1 \in \mathbb{Z}_+$$

$$t^2 = -q^{k+2}$$

Additional series [Cherednik]

$$\mathscr{V}_{k+1} := \mathscr{P}/(P_{k+1})$$

Exceptional Divisors



Exceptional divisors $\mathbf{D}_i$ are Lagrangian w.r.t. $\omega_{\mathfrak{X}}$ if deformation parameter $\gamma_i + i\alpha_p$ in complex structure J is proportional to $i\hbar$

$$\mathrm{Im}\,\frac{\gamma_p + i\alpha_p}{2i\hbar} = 0$$

Value of β_p can be arbitrary

Flatness condition $$F'_{\mathbf{D}_i} + B\big|_{\mathbf{D}_i} = 0$$

$$\int_{\mathbf{D}_i} \frac{\mathrm{Im}\ \Omega}{2\pi} = \int_{\mathbf{D}_i} \frac{\omega_{\mathfrak{X}}}{2\pi} = 0$$

Shortening condition $$t^2 = q^{-(2\ell-1)} \qquad \mathrm{pol}(\mathsf{L}_{2\ell}) \cdot P_{2\ell}(X;q,t) = 0$$

$$\dim \mathrm{Hom}(\mathfrak{B}_{\mathrm{cc}}, \mathfrak{B}_{\mathbf{D}_i}) = \int_{\mathbf{D}_i} \frac{F+B}{2\pi} = -c + \frac{1}{2} = \ell \in \mathbb{Z}_+$$

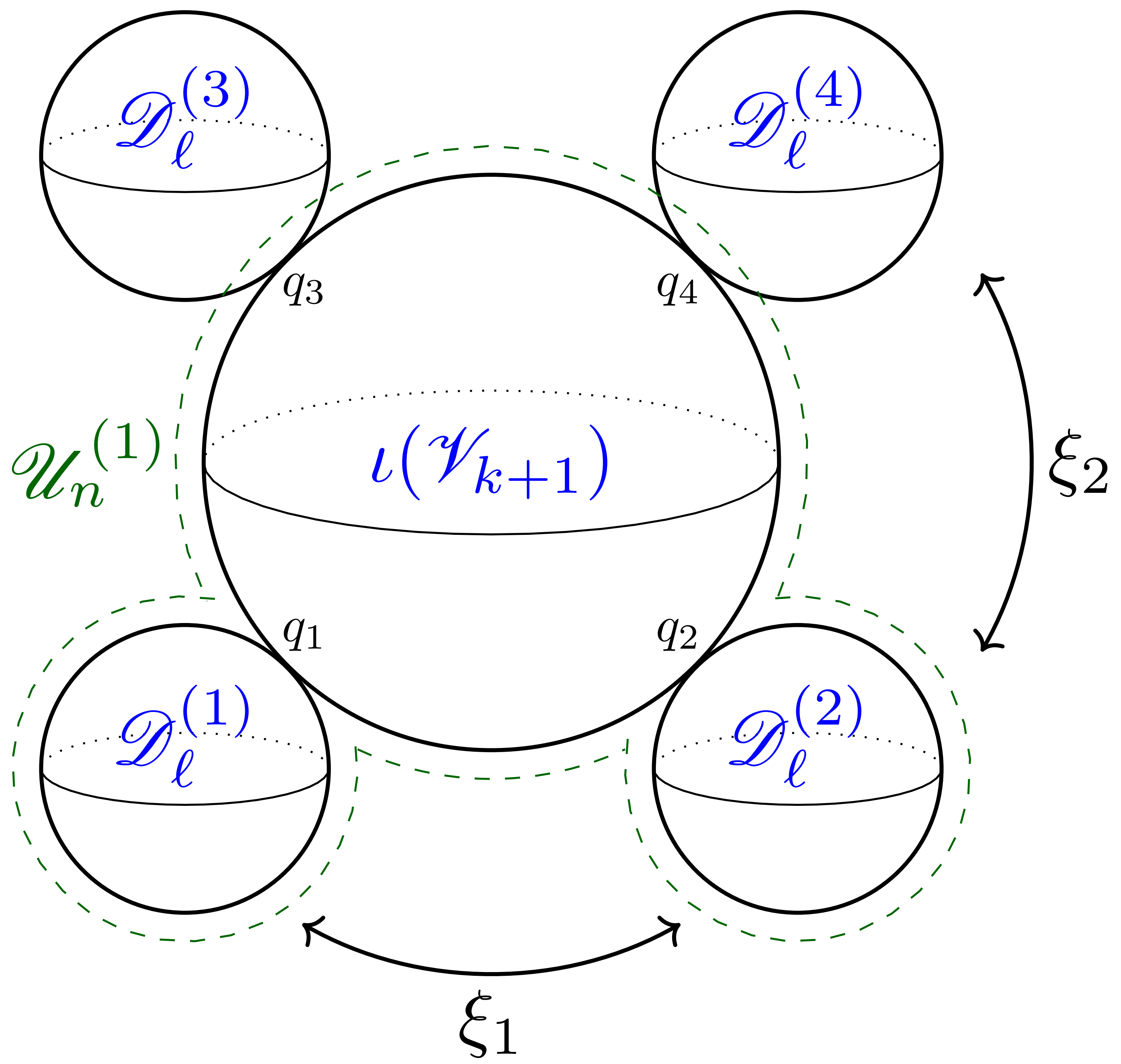
2ℓ-dim representation $$\mathscr{D}_{2\ell} := \mathscr{P}/(P_{2\ell})$$

Splits intro two modules $$\mathscr{D}_{2\ell} = \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)}$$

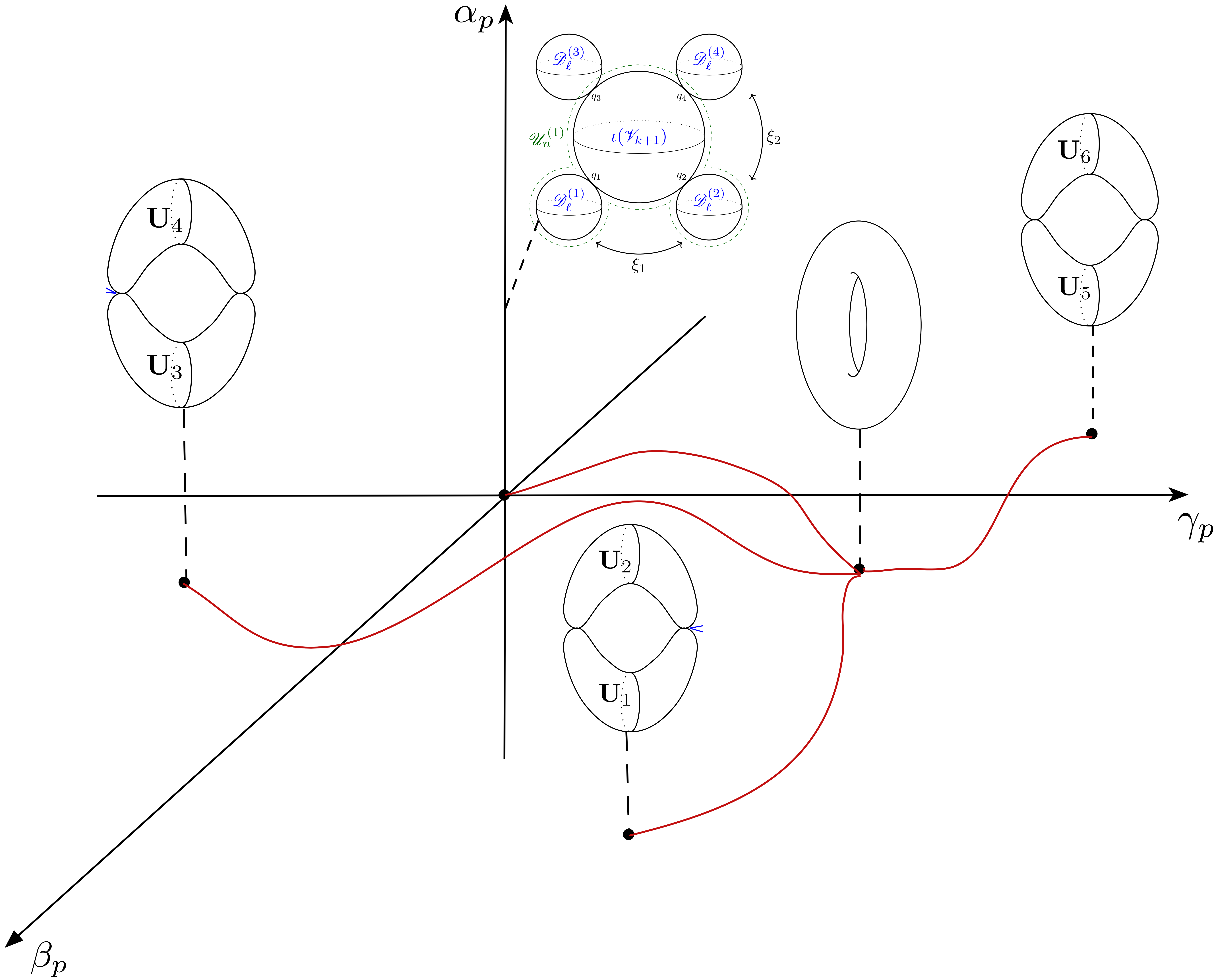
P_j and $P_{2\ell-j-1}$ have the same eigenvalue

$$\mathscr{D}_\ell^{(1)} = \bigoplus_{j=0}^{\ell-1} \mathbb{C}_{q,t}\Big[\frac{P_j(X)}{P_j(t^{-1})} + \frac{P_{2\ell-j-1}(X)}{P_{2\ell-j-1}(t^{-1})}\Big]\ , \quad \mathscr{D}_\ell^{(2)} = \bigoplus_{j=0}^{\ell-1} \mathbb{C}_{q,t}\Big[\frac{P_j(X)}{P_j(t^{-1})} - \frac{P_{2\ell-j-1}(X)}{P_{2\ell-j-1}(t^{-1})}\Big]$$

Summary



finite-dim rep	shortening condition	A-brane condition
$\mathscr{F}_m^{(x_m,y_m)}$	$q^m = 1$	$m = \frac{1}{\hbar}$
$\mathscr{U}_n$	$q^{2n} = 1$	$n = \frac{1}{2\hbar}$
$\mathscr{V}_{k+1}$	$t^2 = -q^{-k}$	$k = \frac{1}{2\hbar} + \frac{\gamma_p + i\alpha_p}{i\hbar}$
$\mathscr{D}_\ell$	$t^2 = q^{-\ell+1/2}$	$\ell = \frac{\gamma_p + i\alpha_p}{2i\hbar}$

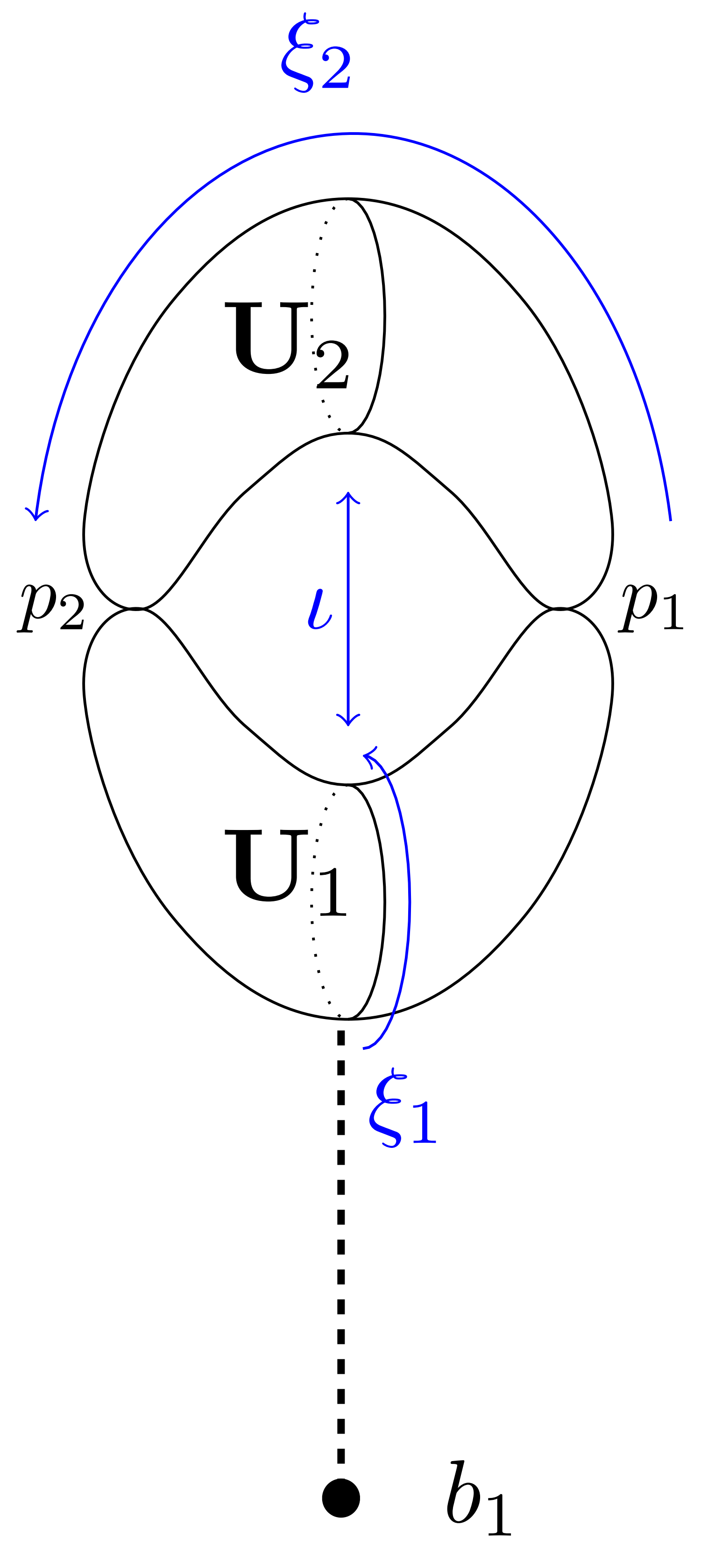


α_p
β_p
γ_p
$\mathscr{D}_\ell^{(1)}$
$\mathscr{D}_\ell^{(2)}$
$\mathscr{D}_\ell^{(3)}$
$\mathscr{D}_\ell^{(4)}$
$\iota(\mathscr{V}_{k+1})$
$\mathscr{U}_n^{(1)}$
q_1
q_2
q_3
q_4
ξ_1
ξ_2
$\mathbf{U}_1$
$\mathbf{U}_2$
$\mathbf{U}_3$
$\mathbf{U}_4$
$\mathbf{U}_5$
$\mathbf{U}_6$

Extensions

Compact Lagrangians $\mathfrak{B}_{\mathsf{F}}$ and $\mathfrak{B}_{\mathsf{U}_i}$ can exist when q is a root of unity and t generic

Irreducible components $\mathbf{U}_1$ and $\mathbf{U}_2$ intersect at two double points



Floer complex $\quad \mathrm{Hom}^*(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2}) := CF^*(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2}) \cong \mathbb{C}\langle p_1 \rangle \oplus \mathbb{C}\langle p_2 \rangle$

Generic fiber **F** over b_1 may split into **$\mathbf{U}_1$** and **$\mathbf{U}_2$**

Corresponding representation of DAHA — τ_--invariant module $\quad \mathscr{F}_{2n}^{(-,+)} \cong \mathscr{P}/(P_{2n})$

$$P_{2n} = (P_n)^2 \qquad\qquad P_{2n} = X^{2n} + X^{-2n} + 2$$

Short exact sequence $\quad \mathfrak{B}_{\mathbf{F}}^{(-,+)} \in \mathrm{Hom}^1(\mathfrak{B}_{\mathbf{U}_1}, \mathfrak{B}_{\mathbf{U}_2})$

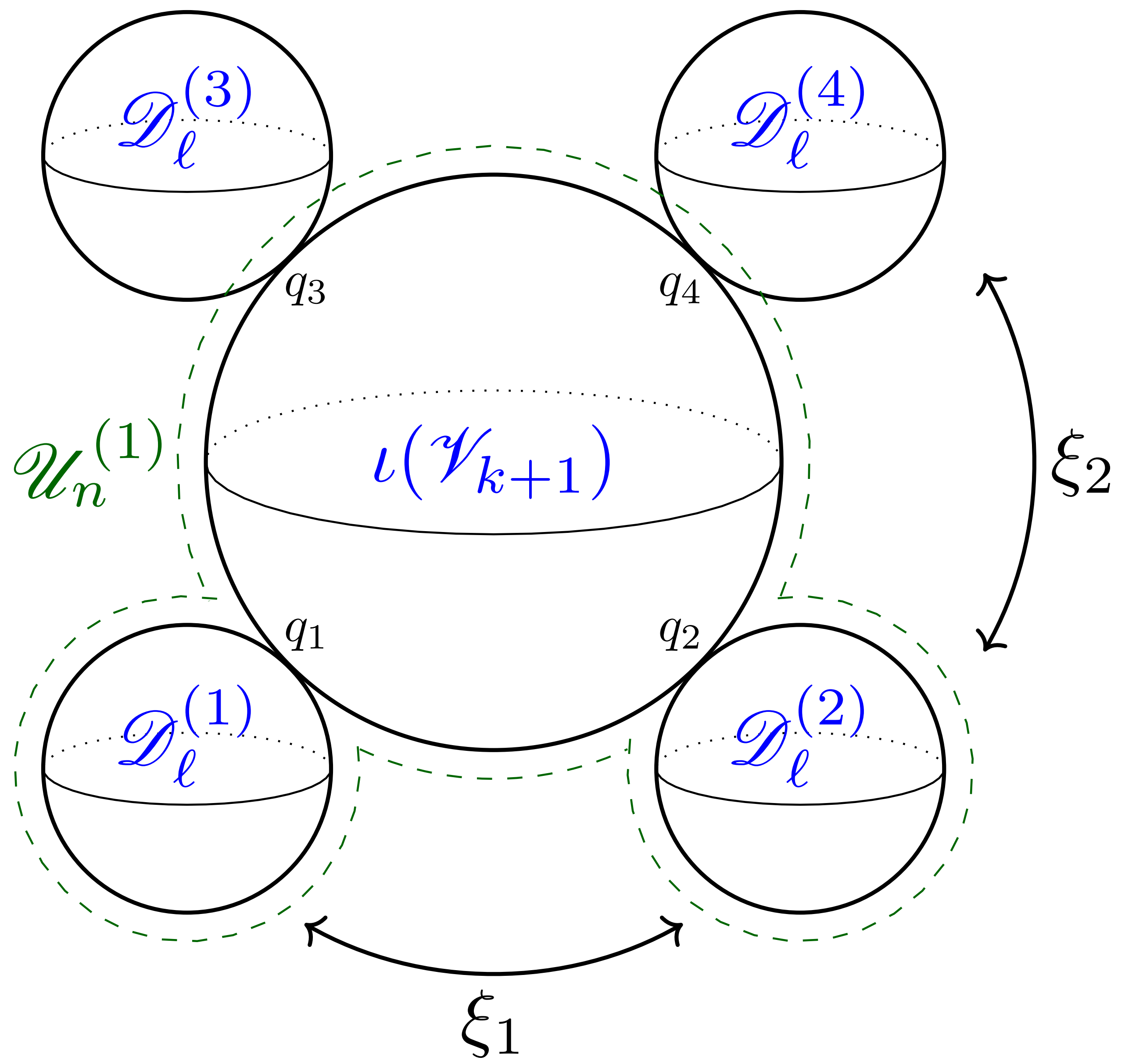
$$0 \to \mathscr{U}_n^{(2)} \to \mathscr{F}_{2n}^{(-,+)} \to \mathscr{U}_n^{(1)} \to 0$$

Global Nilpotent Cone I_0^*

In order to $\mathfrak{B}_{\mathsf{V}}$ and $\mathfrak{B}_{\mathsf{D}_i}$ be Lagrangian two conditions must be satisfied at the same time

$$\mathrm{Im}\,\frac{\left(\frac{1}{2}-\alpha_p\right)+i\gamma_p}{\hbar} = 0 \qquad \mathrm{Im}\,\frac{\gamma_p+i\alpha_p}{2i\hbar} = 0$$

This implies $\gamma_p = 0$, $\hbar$ is real, and α_p, γ_p are arbitrary $\qquad \omega_{\mathfrak{X}} = \omega_K/\hbar$



Quantization conditions $\qquad -c+\frac{1}{2}=\ell\,, \qquad \frac{1}{2\hbar}+2c-1=k+1$

Entail $\qquad 1/2\hbar = 2\ell+k+1$

$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow \mathscr{U}_n^{(1)} \xrightarrow{f} \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \longrightarrow 0$$

Morphism matching $\qquad \mathrm{Hom}^1(\mathfrak{B}_{\mathbf{D}_1}, \mathfrak{B}_{\mathbf{V}}) \cong \mathbb{C}\langle q_1\rangle$

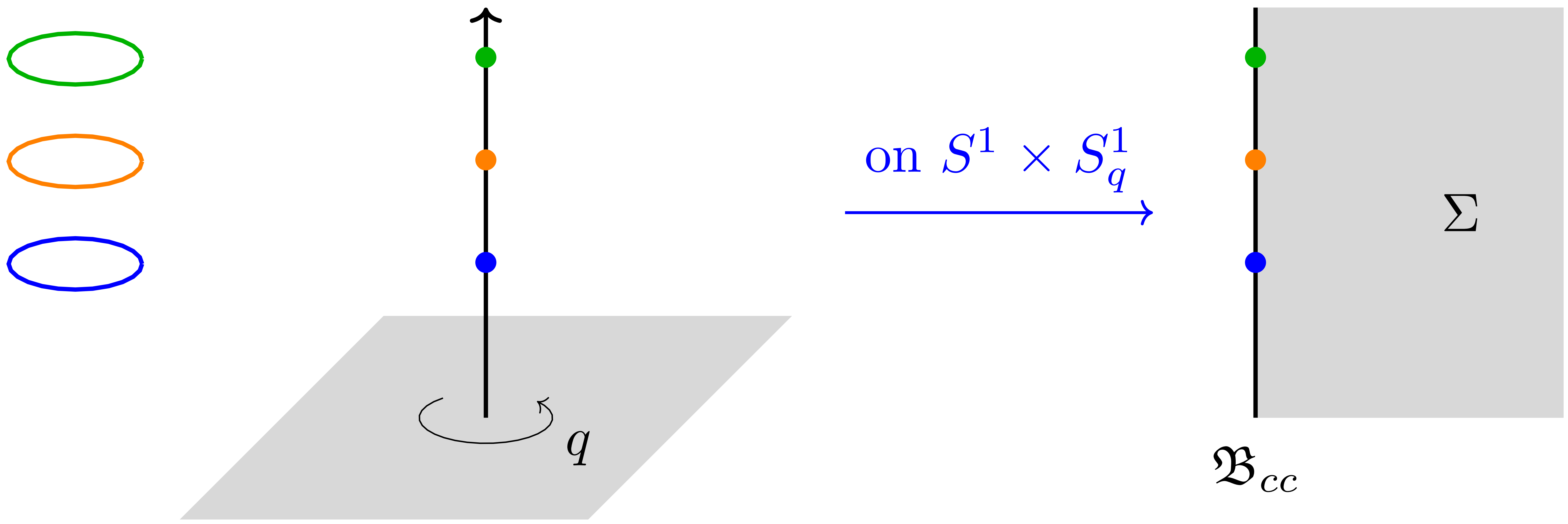
$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow f^{-1}(\mathscr{D}_\ell^{(1)}) \longrightarrow \mathscr{D}_\ell^{(1)} \longrightarrow 0$$

$$\mathrm{Hom}^1(\mathfrak{B}_{\mathbf{D}_1} \oplus \mathfrak{B}_{\mathbf{D}_2}, \mathfrak{B}_{\mathbf{V}}) \cong \mathbb{C}\langle q_1\rangle \oplus \mathbb{C}\langle q_2\rangle$$

$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow f^{-1}(\mathscr{D}_\ell^{(1)}) \oplus \mathscr{D}_\ell^{(2)} \longrightarrow \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \longrightarrow 0$$

$$0 \longrightarrow \iota(\mathscr{V}_{k+1}) \longrightarrow f^{-1}(\mathscr{D}_\ell^{(2)}) \oplus \mathscr{D}_\ell^{(1)} \longrightarrow \mathscr{D}_\ell^{(1)} \oplus \mathscr{D}_\ell^{(2)} \longrightarrow 0$$

Line operators and DAHA



$S^1 \times$ $\mathbb{R} \times_q \mathbb{R}^2$

Invariant under isotropic lattice $\mathsf{L} \subset H^1(C, Z(G))$

Algebra of line operators yields L-invariant subalgebra of DAHA

Coulomb branch

$$\begin{tikzcd} \Sigma \arrow[r, hook] & T^*C \arrow[d] \\ & C \end{tikzcd}$$

Elliptic fibration over the base

$$\pi : \mathcal{M}_C(C, G, \mathsf{L}) \to \mathcal{B}_u$$

$$\mathcal{M}_C(C, G, \mathsf{L}) = \mathcal{M}_H(C, G)/\mathsf{L}$$

L is compatible with Dirac quantization condition

Coulomb Branches

Dirac quantization

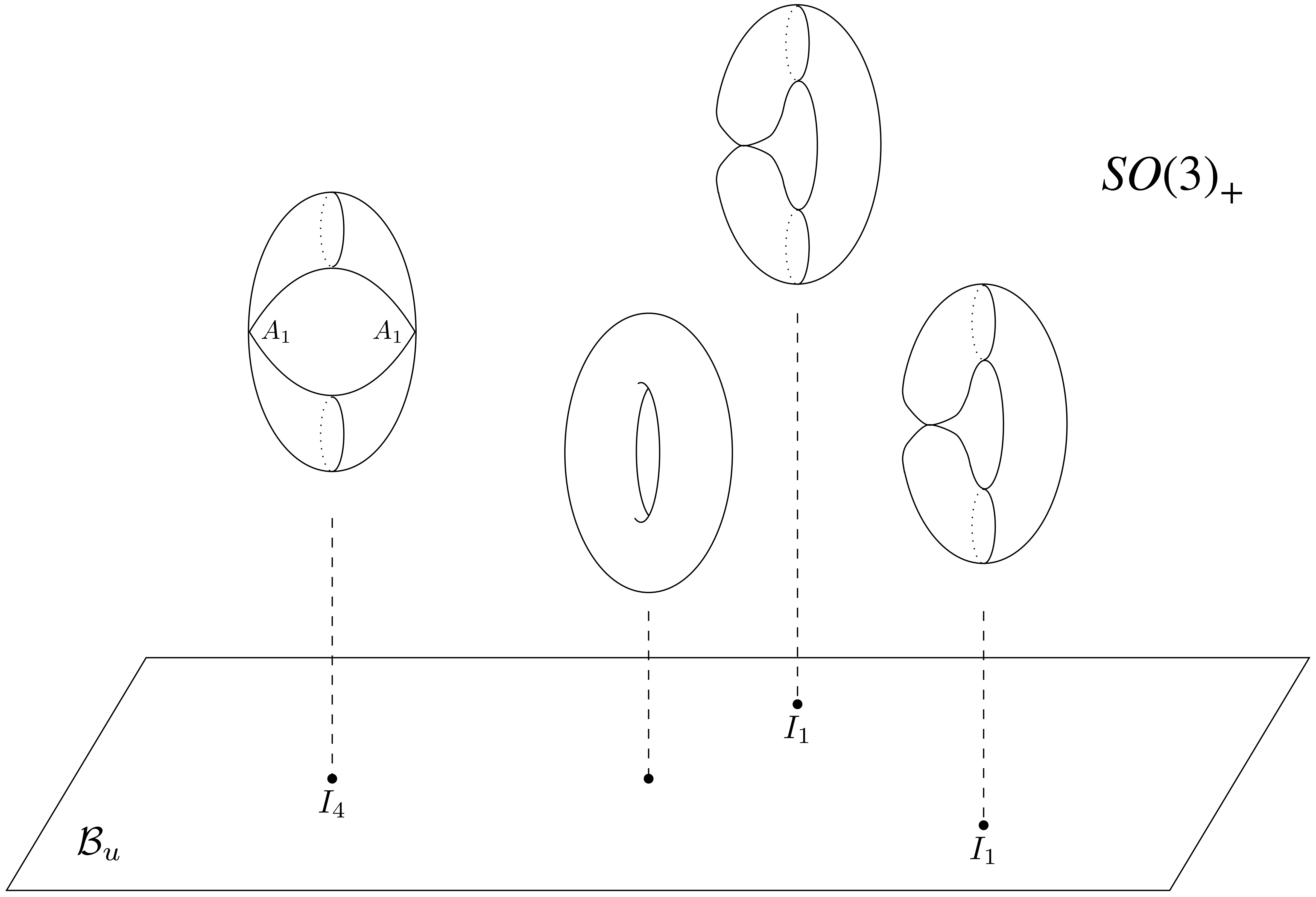
$$\omega(\lambda, \nu) = \lambda_e \nu_m - \lambda_m \nu_e \in 2\mathbb{Z}$$

$$\begin{tikzcd}
 & \mathrm{SU}(2) \arrow[loop above, "T"] \arrow[dl, leftrightarrow, "S"'] \arrow[dr, leftrightarrow, "TST"] & \\
\mathrm{SO}(3)_+ \arrow[loop left, "TST"] \arrow[rr, leftrightarrow, "T"'] & & \mathrm{SO}(3)_- \arrow[loop right, "S"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{M}_H(C_p, \mathrm{SU}(2)) \arrow[dl, "\xi_2"'] \arrow[d, "\xi_1"] \arrow[dr, "\xi_3"] & \\
\mathcal{M}_C(C_p, \mathrm{SU}(2)) \arrow[dr, "\xi_1"'] & \mathcal{M}_C(C_p, \mathrm{SO}(3)_+) \arrow[d, "\xi_2"] & \mathcal{M}_C(C_p, \mathrm{SO}(3)_-) \arrow[dl, "\xi_1"] \\
 & \mathcal{M}_H(C_p, \mathrm{SO}(3)) &
\end{tikzcd}$$

Three ways to pick maximal isotropic lattice L

(0,1) (1,0) and (1,1) $\in H^1(C_p, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

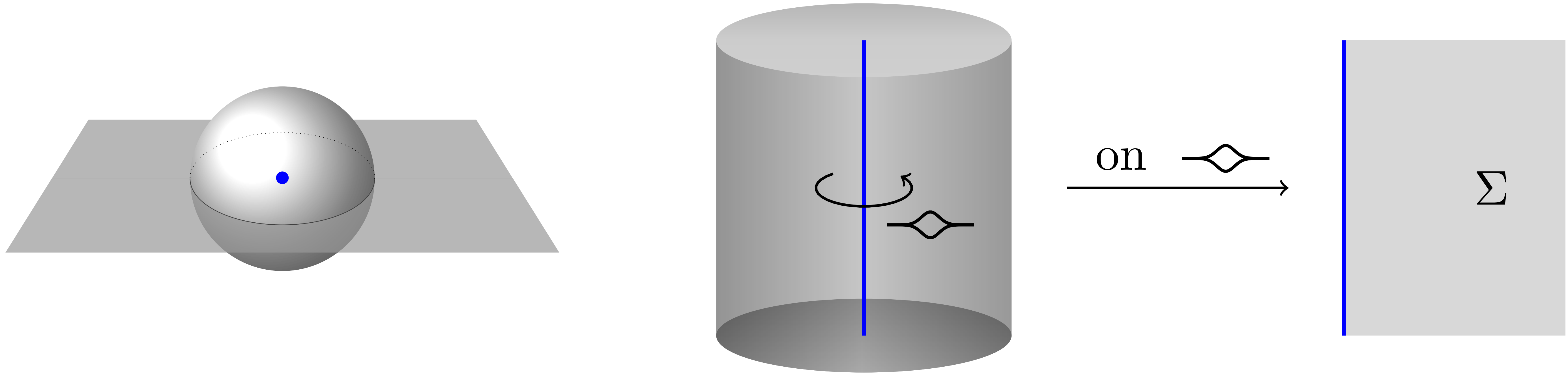


Algebra of Line Operators

Reduce $\mathscr{N}=4$ SYM on the raviolo configuration to get a 2d sigma model on $\mathcal{M}_H(-\!\diamond\!-, G)$

Hecke modifications [KW]

Raviolo $-\!\diamond\!- = \mathbb{C} \cup_{\mathbb{C}^\times} \mathbb{C}$ [BDG]



Line operator — worldsheet boundary condition

Wilson (B,B,B)

’t Hooft (B,A,A)

Left quotient of the affine Grassmannian $\mathrm{Bun}_G(-\!\diamond\!-) = G_{\mathbb{C}}^{\mathscr{O}} \backslash G_{\mathbb{C}}^{\mathscr{K}} / G_{\mathbb{C}}^{\mathscr{O}}$

Affine Steinberg variety $\mathcal{R} = \{(x,[g]) \in \mathfrak{g}_{\mathbb{C}}^{\mathscr{O}} \times \mathcal{G}r(G_{\mathbb{C}}) \mid \mathrm{Ad}_{g^{-1}}(x) \in \mathfrak{g}_{\mathbb{C}}^{\mathscr{O}}\}$

Category of line operators $\mathsf{Line}\left[\mathcal{T}[C,G,\mathsf{L}]\right] \cong D^b\mathsf{Coh}^{G_{\mathbb{C}}^{\mathscr{O}}}(\mathcal{R})$

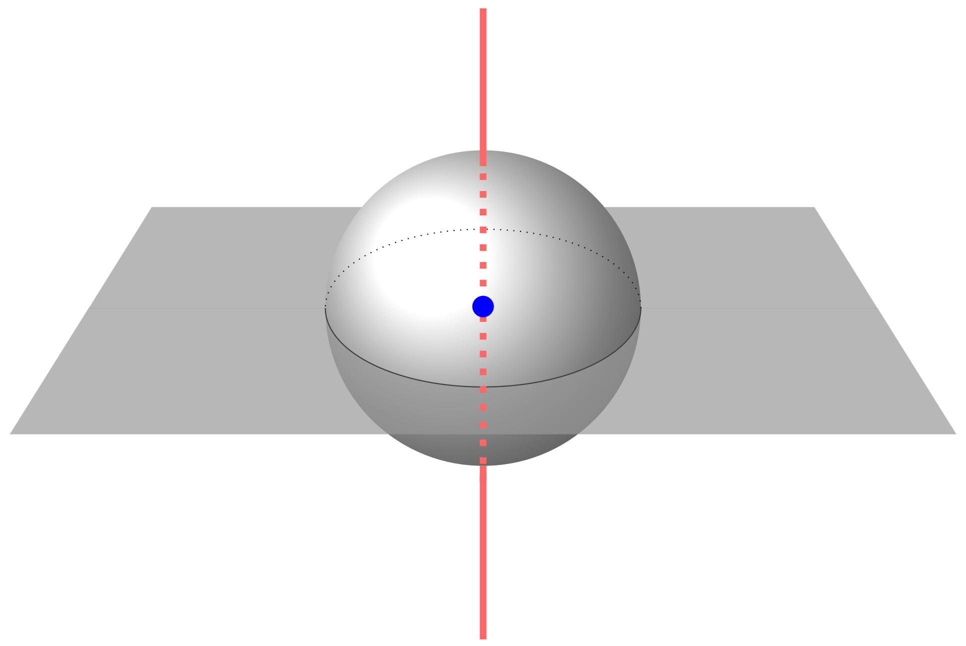
Algebra of line operators from Grothendieck ring $K^{G_{\mathbb{C}}^{\mathscr{O}}}(\mathcal{R}) \cong \mathbb{C}[T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee}]^W$ [BFM]

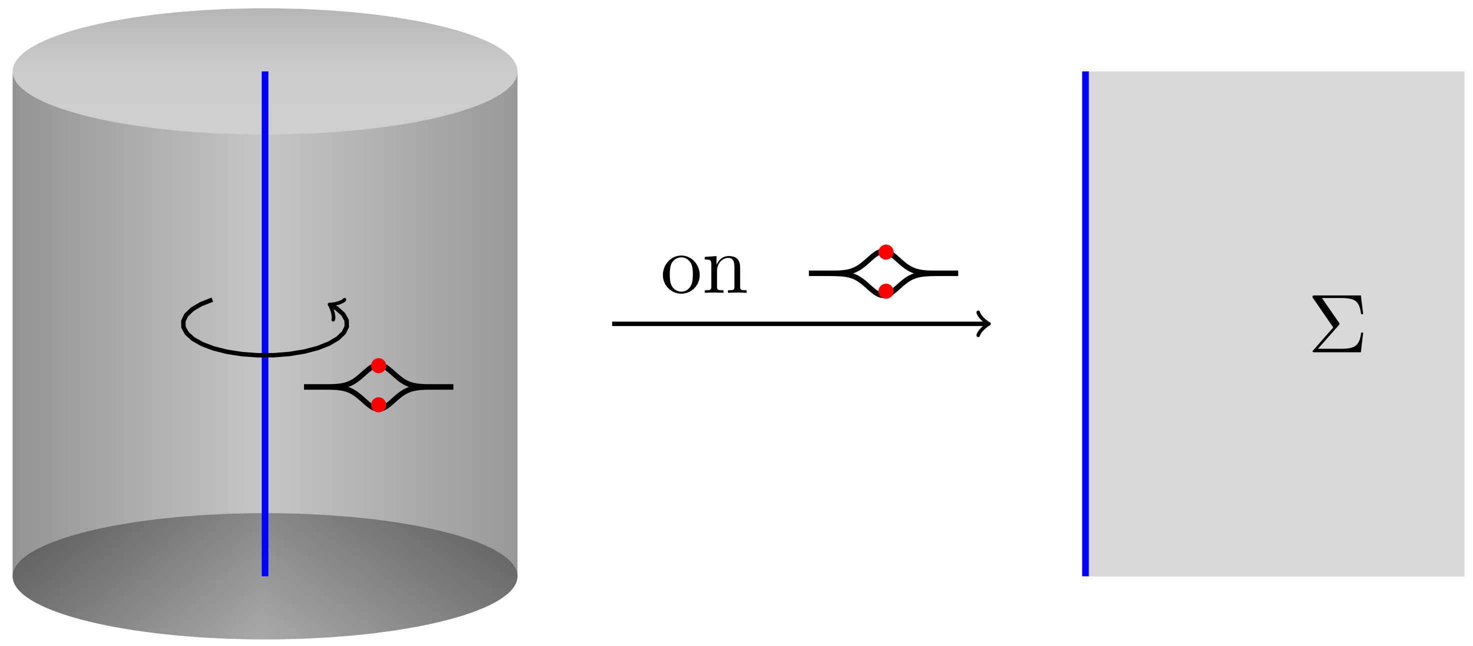
Quantized Coulomb branch $K^{(G_{\mathbb{C}}^{\mathscr{O}} \times \mathbb{C}_t^{\times}) \rtimes \mathbb{C}_q^{\times}}(\mathcal{R}) \cong S\ddot{H}(W)^{\mathsf{L}}$

Full DAHA

	space-time:	$\mathbb{R}^4$	$\times$	T^*C	$\times$	$\mathbb{R}^3$
	N M5-branes:	$\mathbb{R}^4$	$\times$	C	$\times$	pt
(surface operator)	M5'-brane:	$\mathbb{R}^2$	$\times$	C	$\times$	$\mathbb{R}^2$

$$A = \alpha d\vartheta + \ldots \qquad D_{\bar{z}}\Phi = (\beta + i\gamma)\delta^{(2)}(z,\bar{z})$$





Full defect breaks gauge symmetry on the disk to the Borel subgroup (Iwahori)

$$\mathscr{I} := \{a_0 + a_1 z + a_2 z^2 + \cdots \in G_{\mathbb{C}}^{\mathscr{O}} | a_0 \in B\}$$

Now Bun(G) is the left quotient of the affine flag variety

$$\mathrm{Bun}_G(\quad) = \mathscr{I}\backslash G_{\mathbb{C}}^{\mathscr{K}}/\mathscr{I}$$

Introduce affine Steinberg flag variety

$$\mathcal{Z} = \{(x,[g]) \in \mathrm{Lie}(\mathscr{I}) \times \mathcal{F}l(G_{\mathbb{C}}) \mid \mathrm{Ad}_{g^{-1}}(x) \in \mathrm{Lie}(\mathscr{I})\}$$

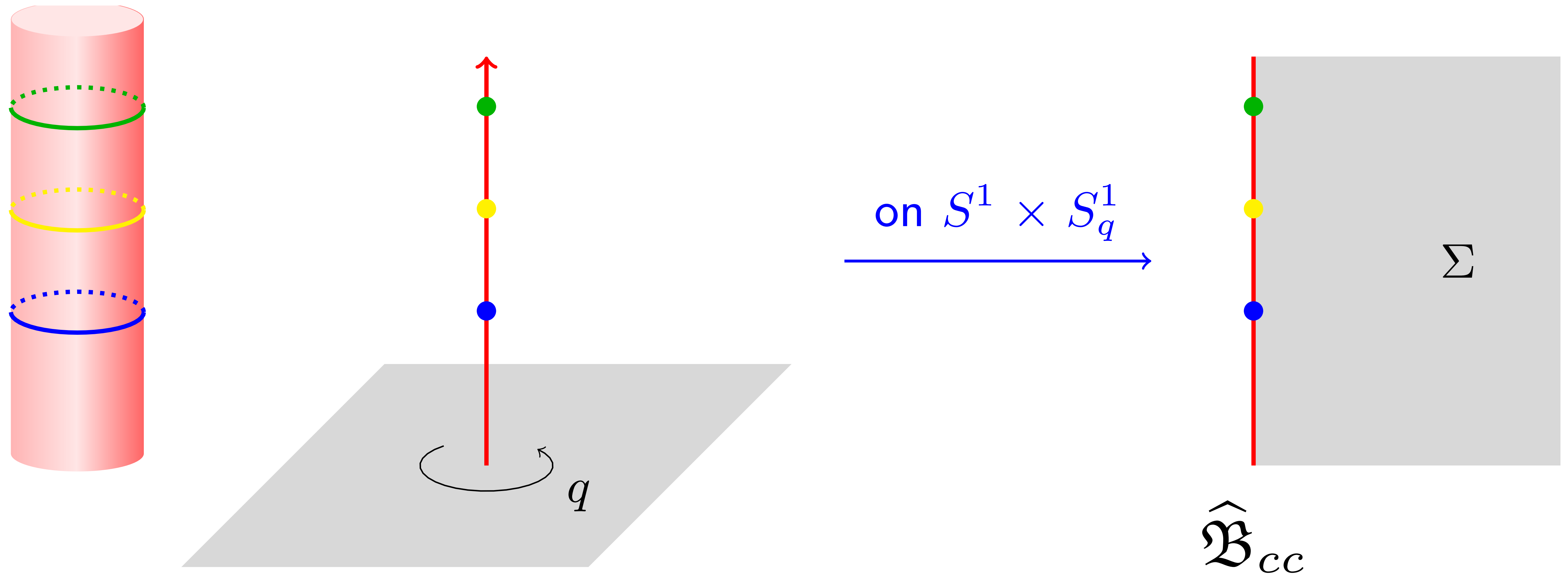
Category, algebra

$$\mathsf{Line}[\mathcal{T}[C,G,\mathsf{L}],T] \cong D^b\mathsf{Coh}^{\mathscr{I}}(\mathcal{Z}) \qquad K^{\mathscr{I}}(\mathcal{Z}) = \mathbb{C}[T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee}] \rtimes \mathbb{C}[W]$$

In Omega background

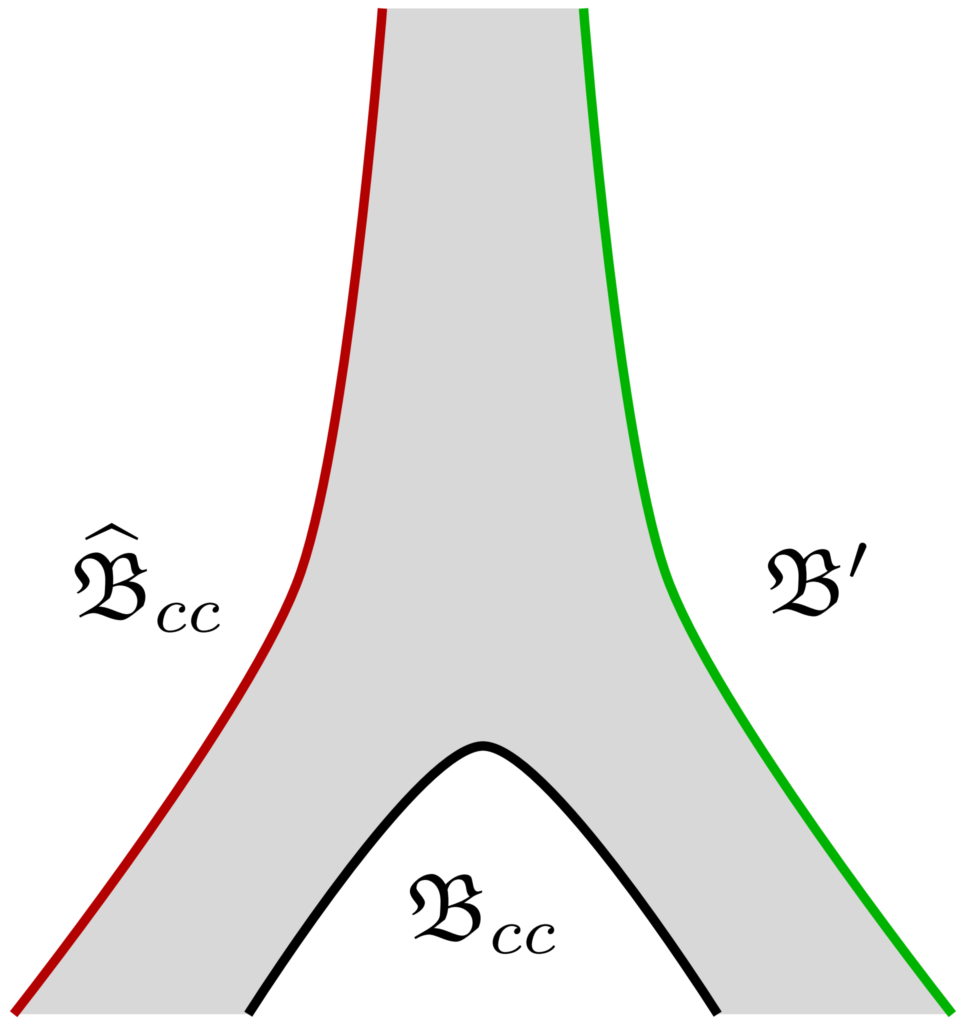
$$K^{(\mathscr{I}\times\mathbb{C}_t^{\times})\rtimes\mathbb{C}_q^{\times}}(\mathcal{Z}) \cong \ddot{H}(W)^{\mathsf{L}}$$

Morita Equivalence



Full DAHA

$$\mathrm{Hom}(\widehat{\mathfrak{B}}_{cc}, \widehat{\mathfrak{B}}_{cc}) \cong \ddot{H}(W)^{\mathsf{L}}$$



Morita equivalence

$$\mathrm{Hom}(\widehat{\mathfrak{B}}_{cc}, \mathfrak{B}_{cc}) : \mathsf{Rep}(S\ddot{H}(W)) \xrightarrow{\sim} \mathsf{Rep}(\ddot{H}(W))$$