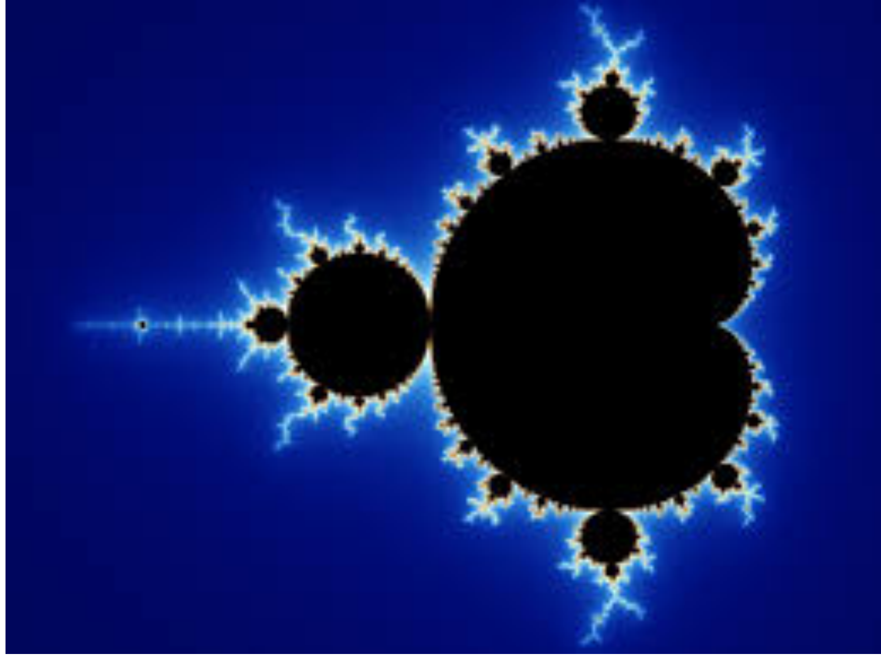
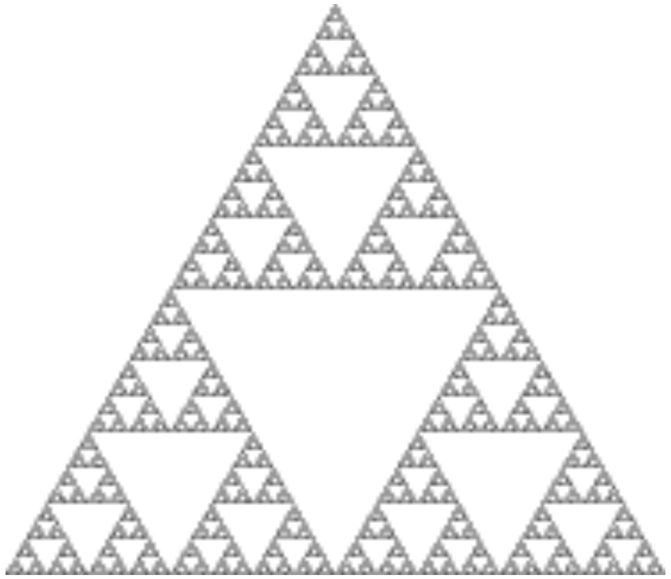
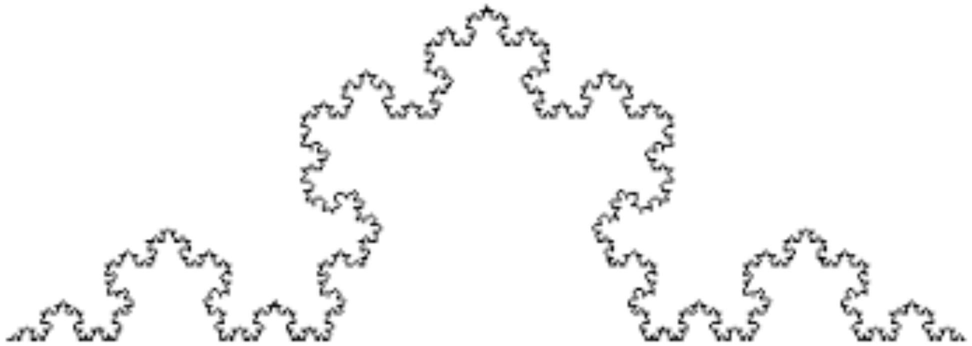
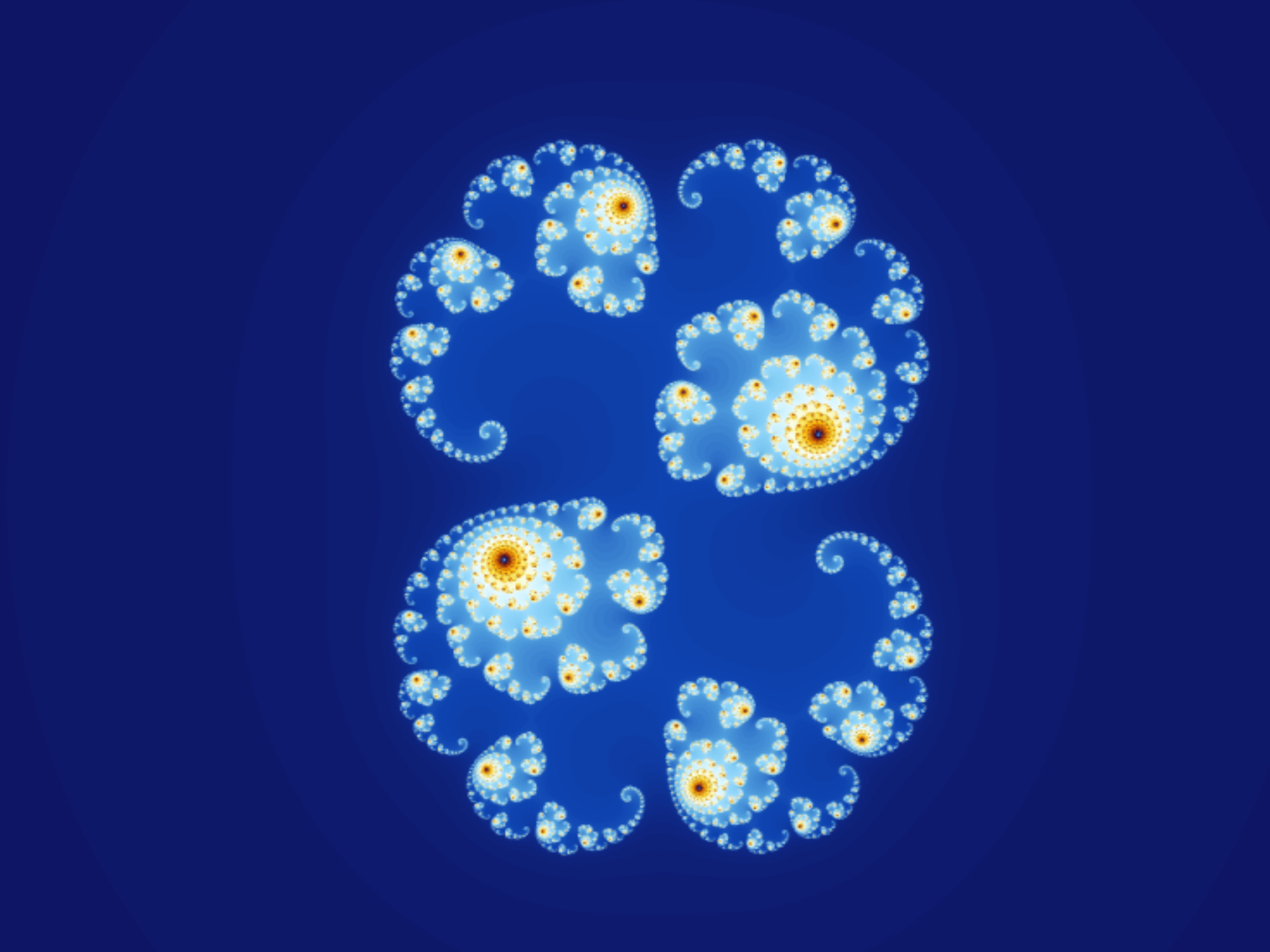


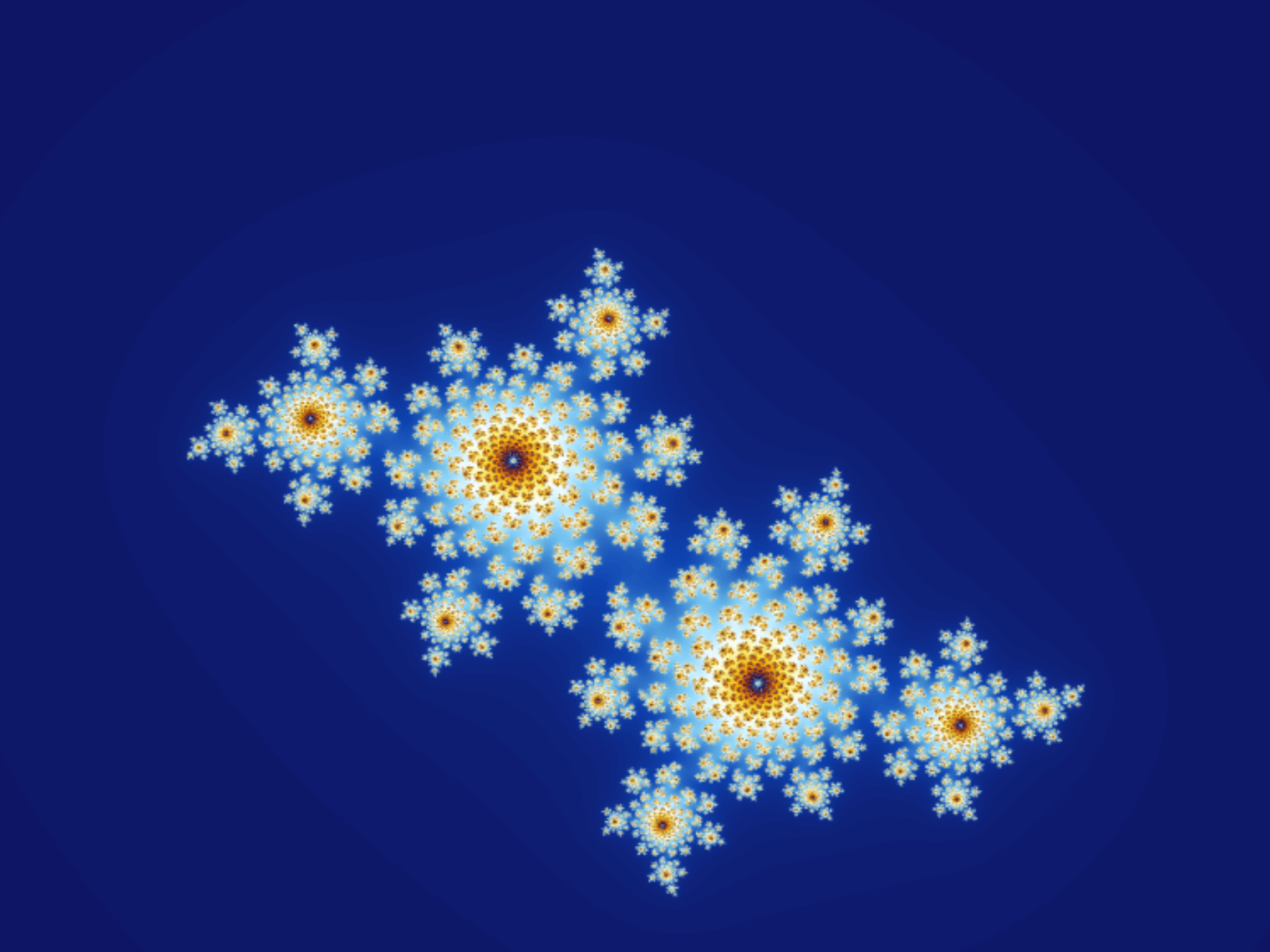
Fractals

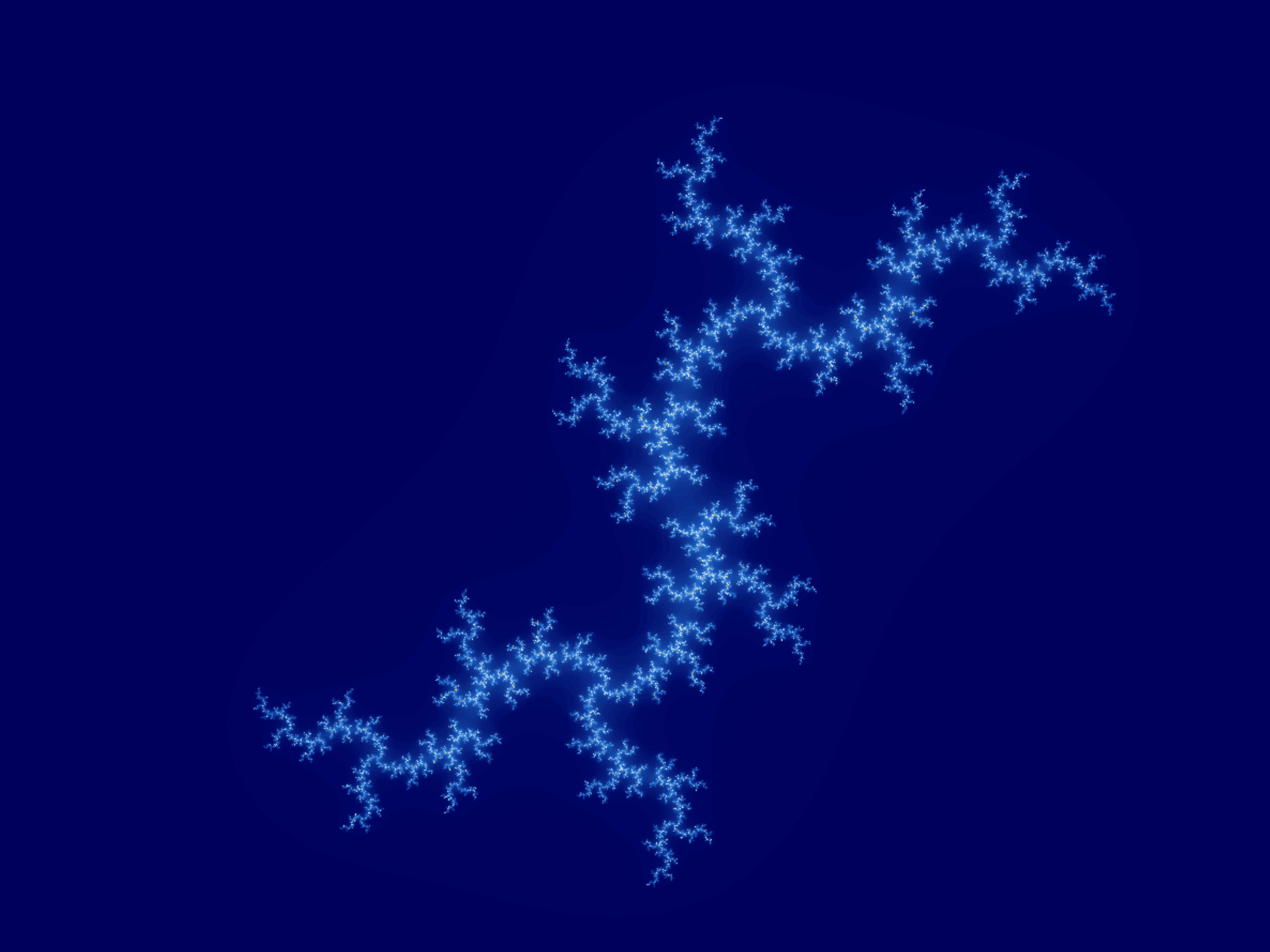
Peter Koroteev
UC Berkeley

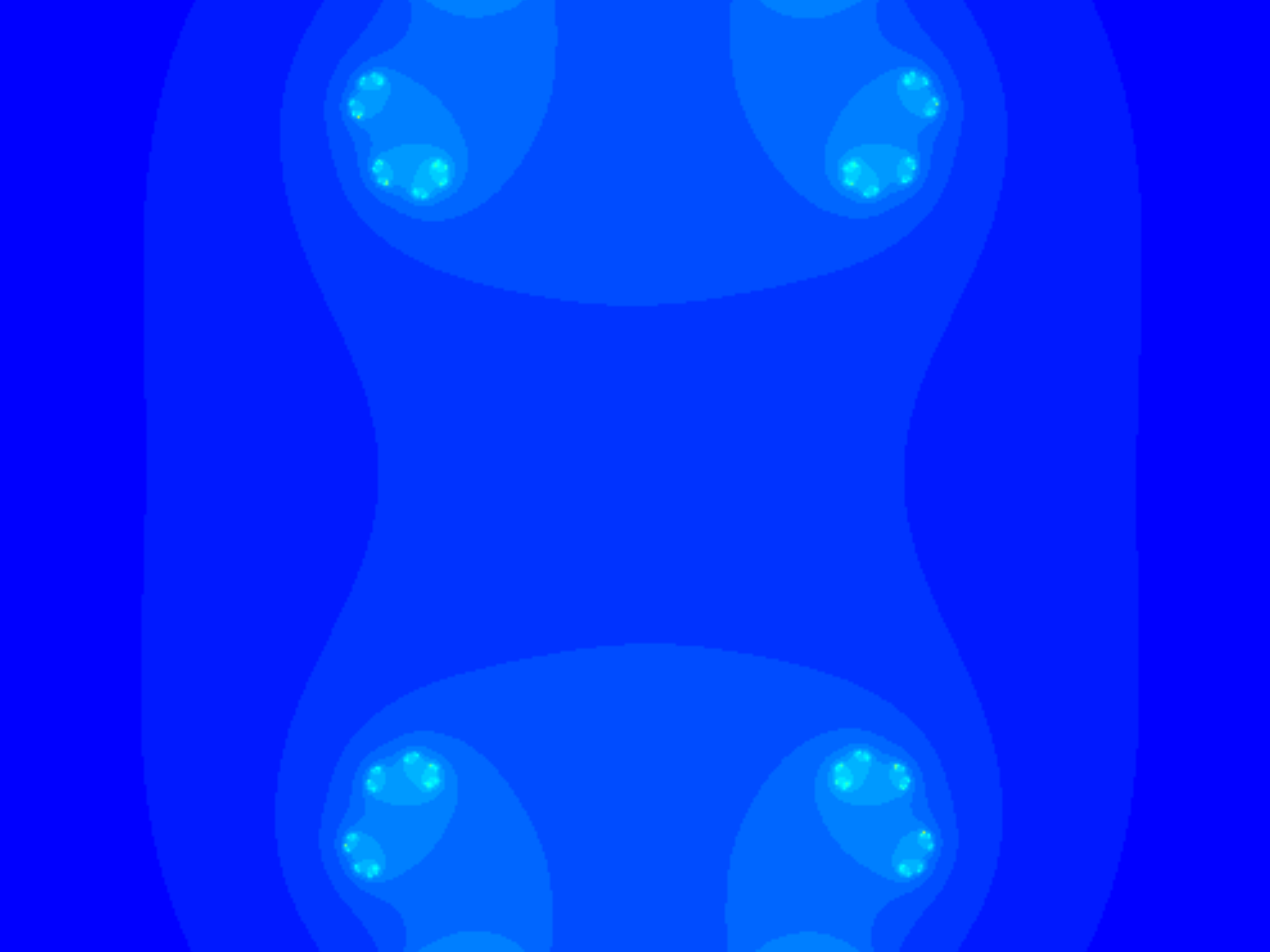


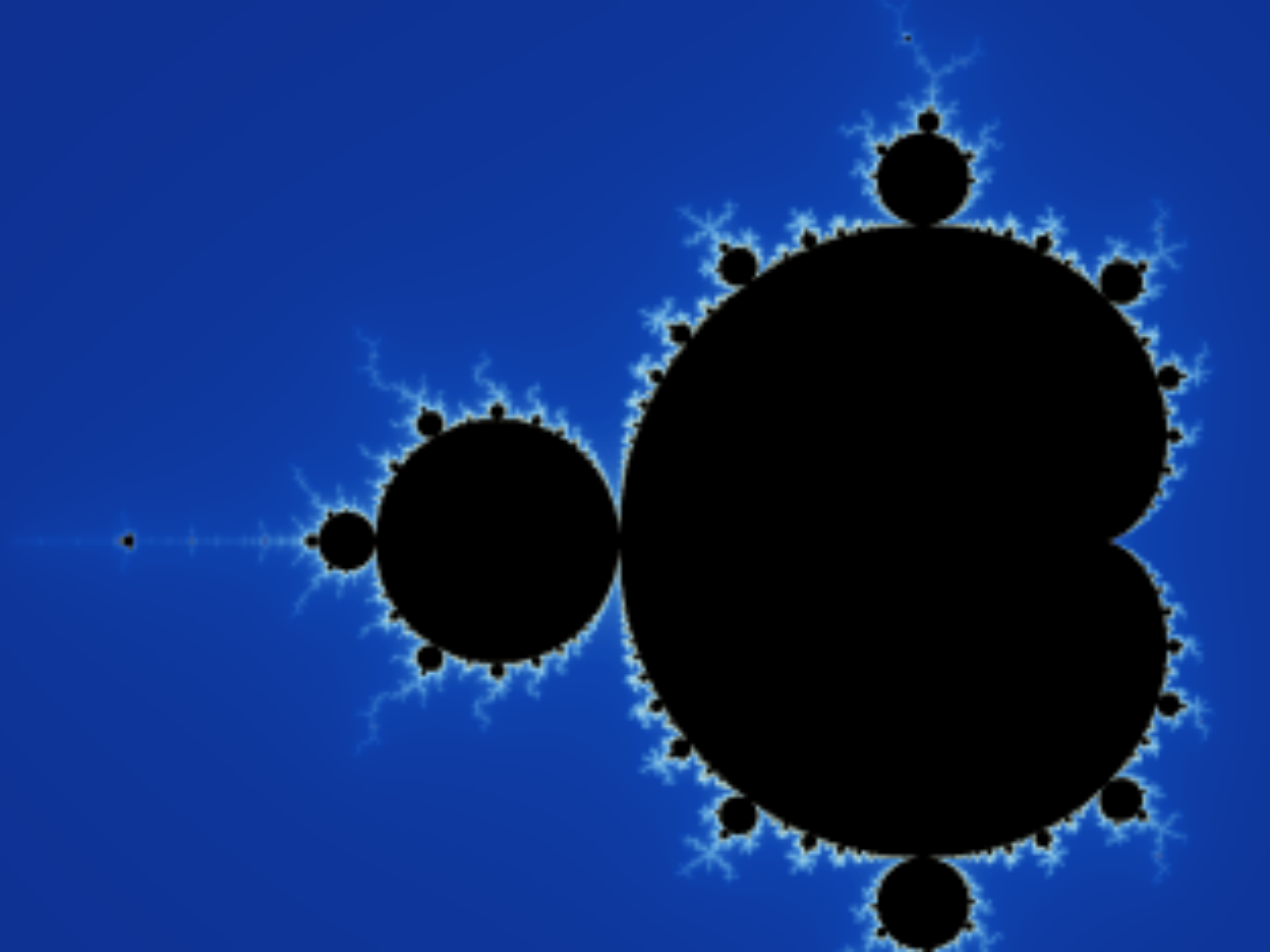




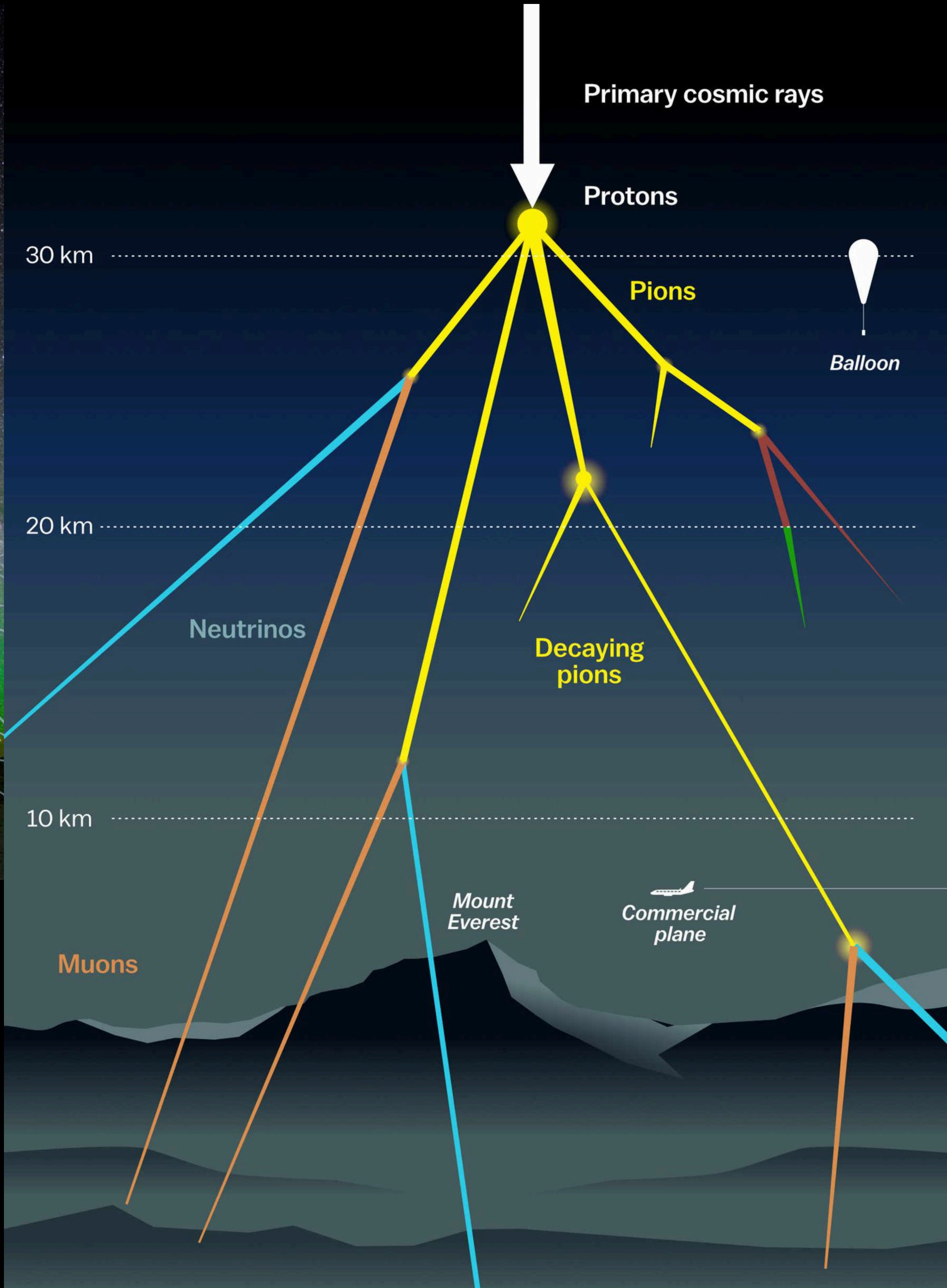




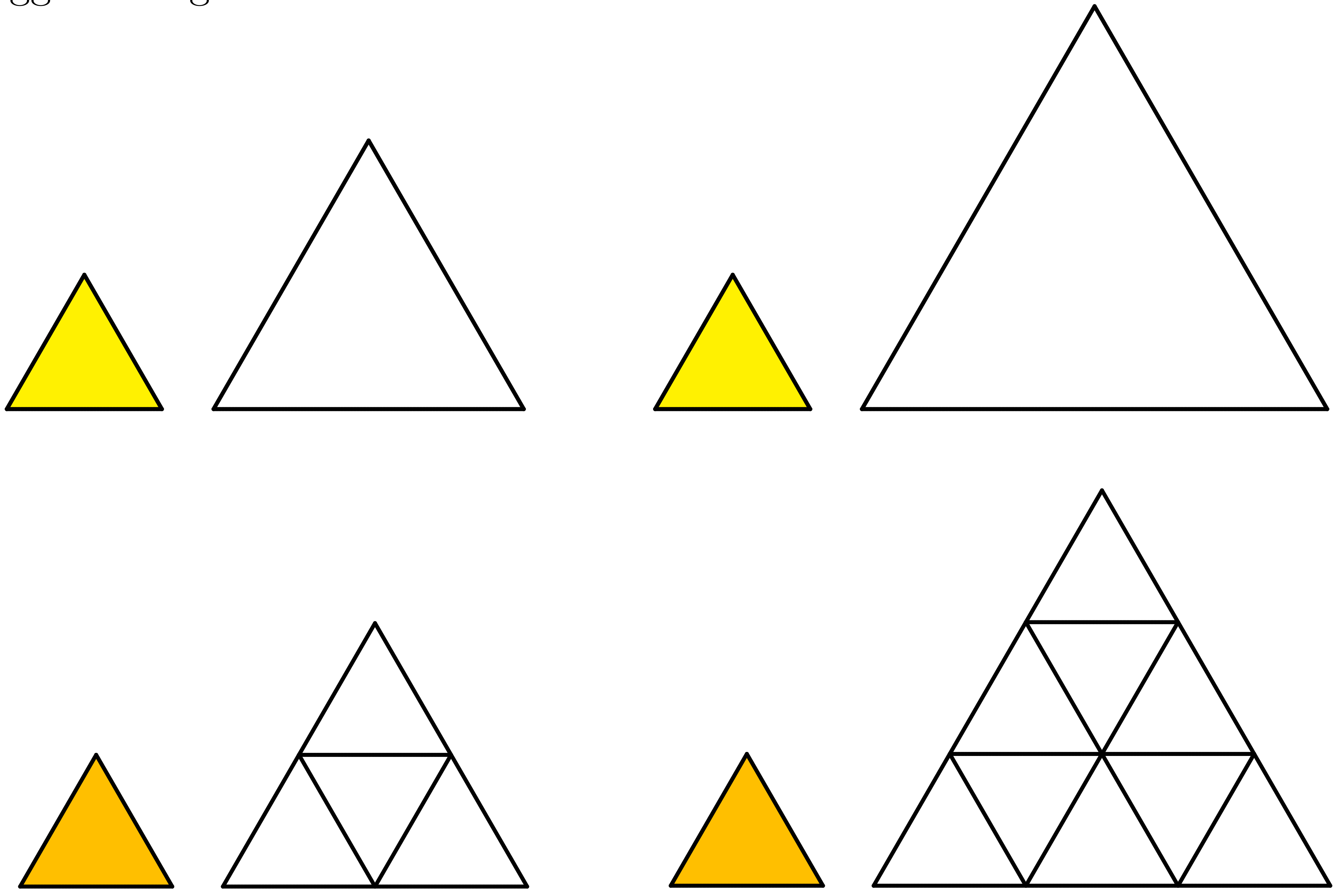




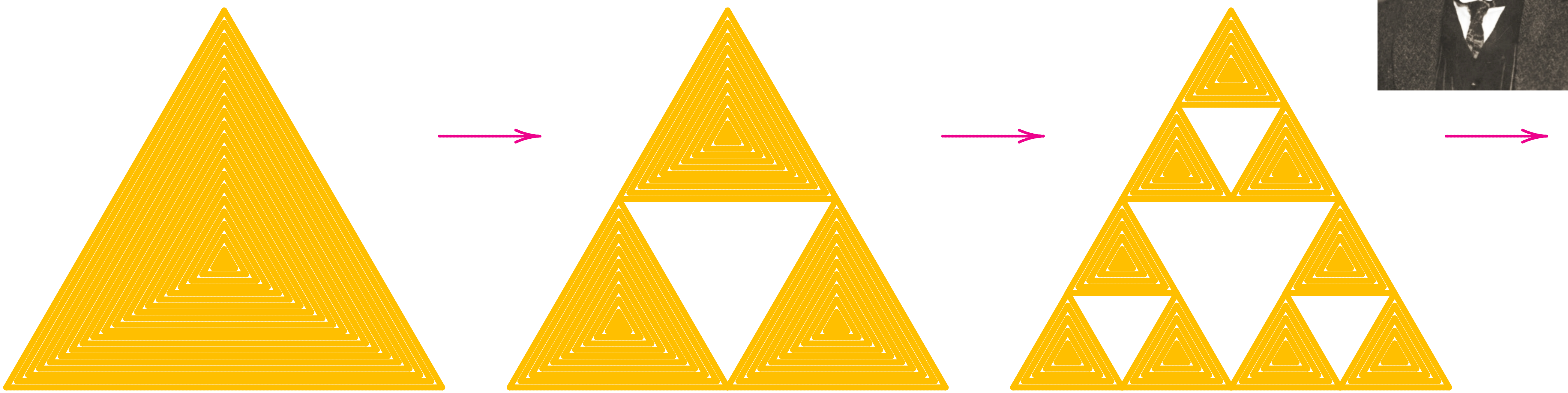
Ultra High-Energy Cosmic Rays



Problem We are to paint two different equilateral triangles; the side of smaller triangle is two times the side of the bigger one, We need one can of paint to paint the small triangle. How many cans of paint do we need to paint the big triangle? Same question if the side of the bigger triangle is three times the side of the smaller one.



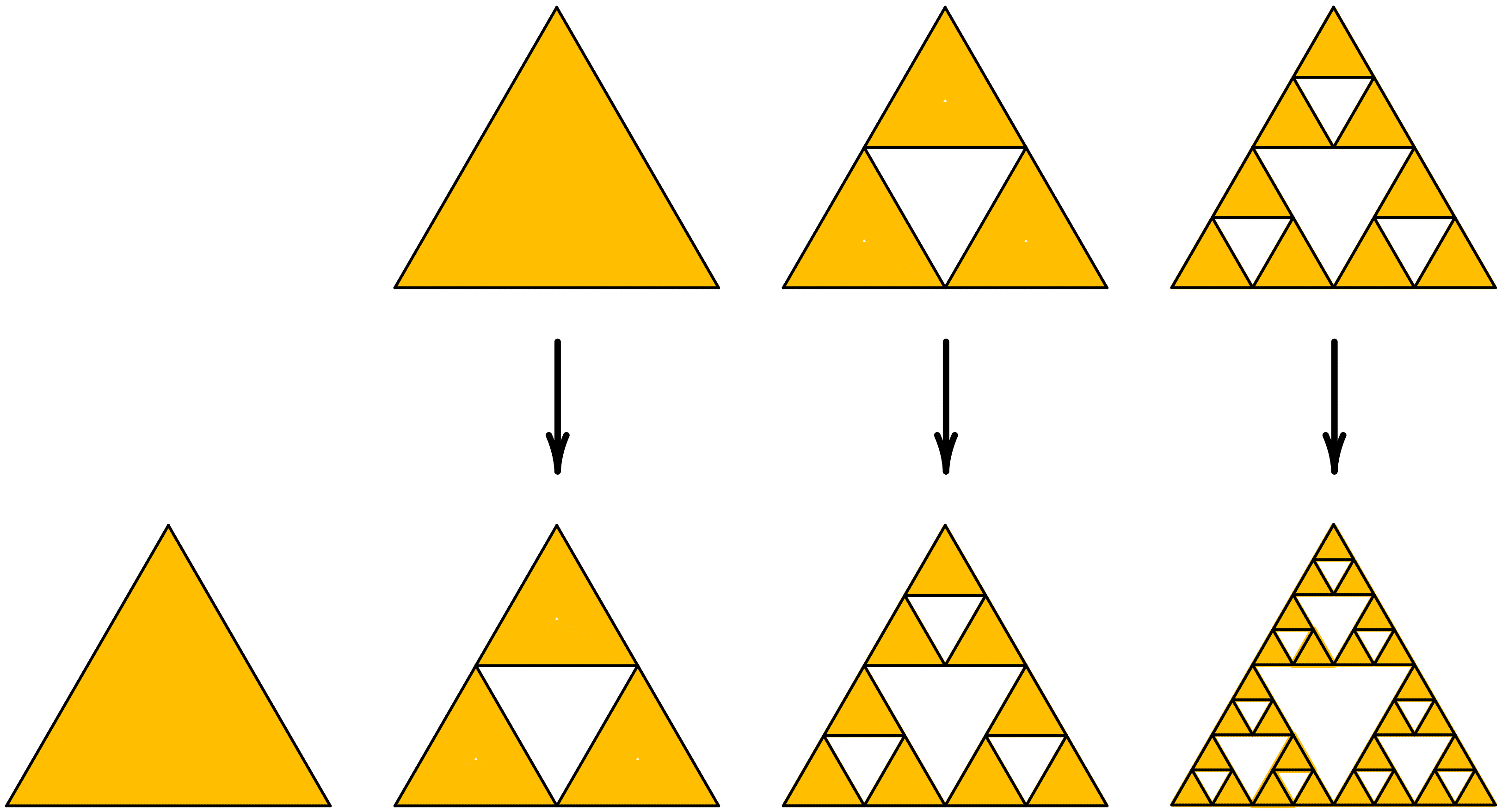
Sierpinski Triangle



Step 1 *Step 2* *Step 3*

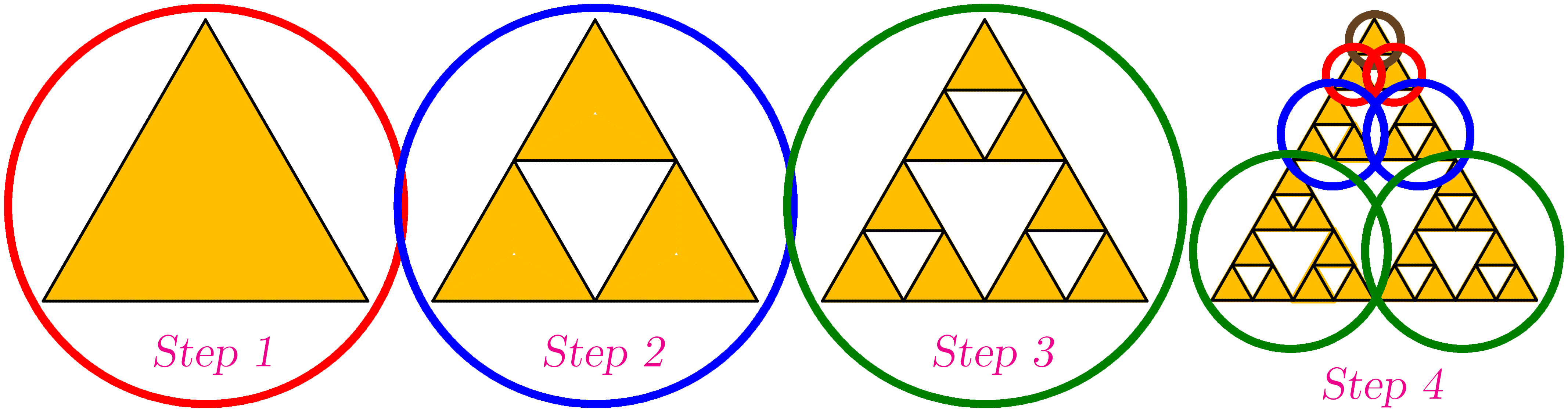
Step	1	2	3	4	5	6	7	8
Number of yellow triangles	1	3	9	27	81	243	729	2187
Number of white triangles	0	1	4	13	40	121	364	1093
Total number of yellow triangles	1	4	13	40	121	364	1093	3280

More Patterns



Step	1	2	3	4	5	6	7	8
Number of yellow triangles	1	3	9	27	81	243	729	2187
Number of white triangles	0	1	4	13	40	121	364	1093
Total number of yellow triangles	1	4	13	40	121	364	1093	3280

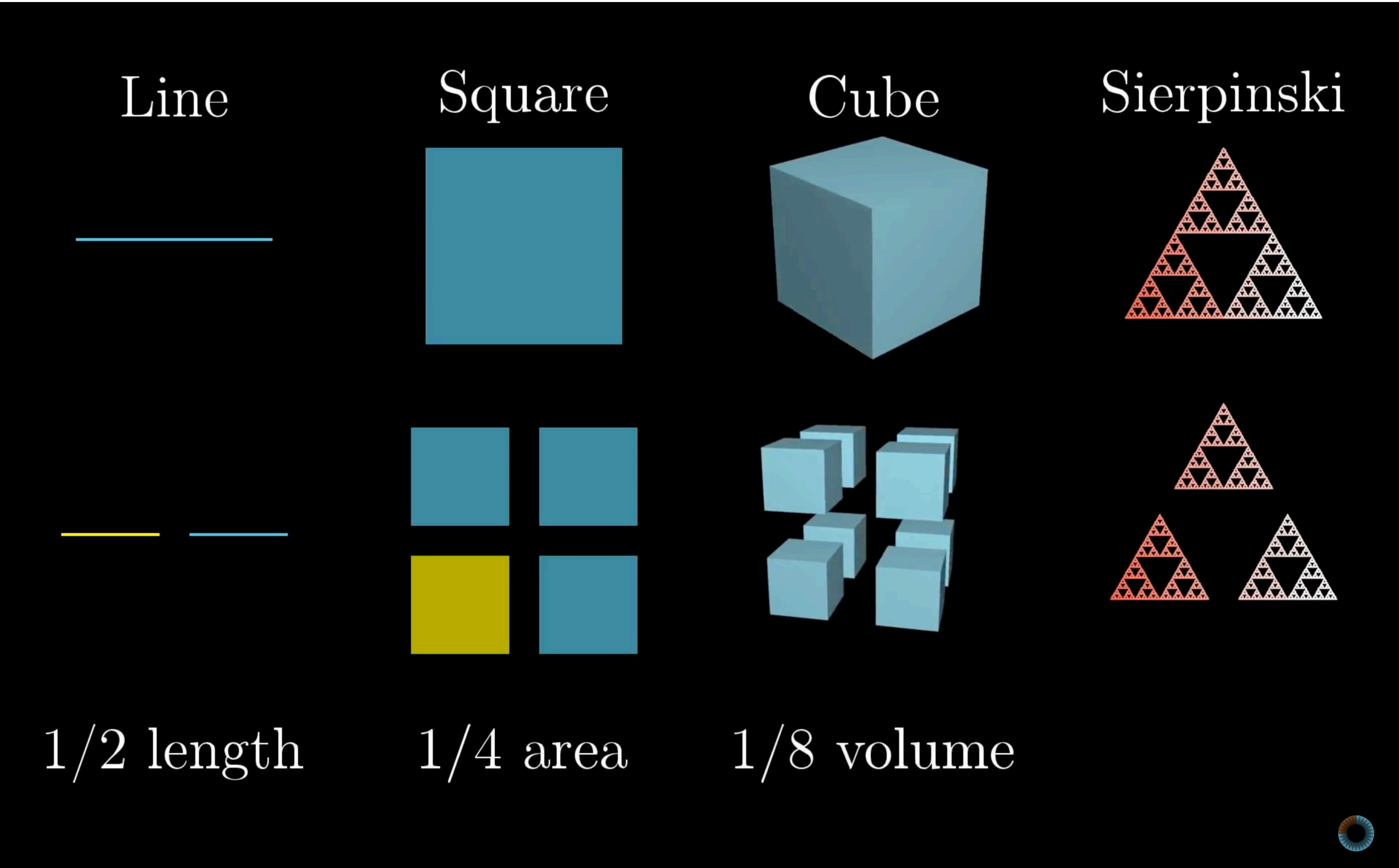
Yet More Patterns!



Powers of 3	1	3	3^2	3^3	3^4	3^5	3^6	3^7
Number of yellow triangles	1	3	9	27	81	243	729	2187
Total number of yellow triangles	1	4	13	40	121	364	1093	3280

Scaling Dimensions

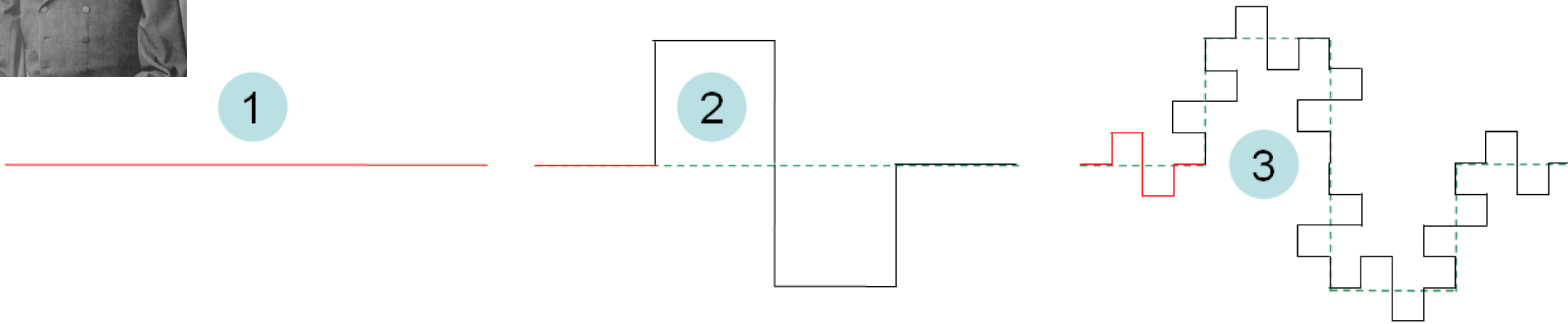
https://www.youtube.com/watch?v=gB9n2gHsHN4



Fractals

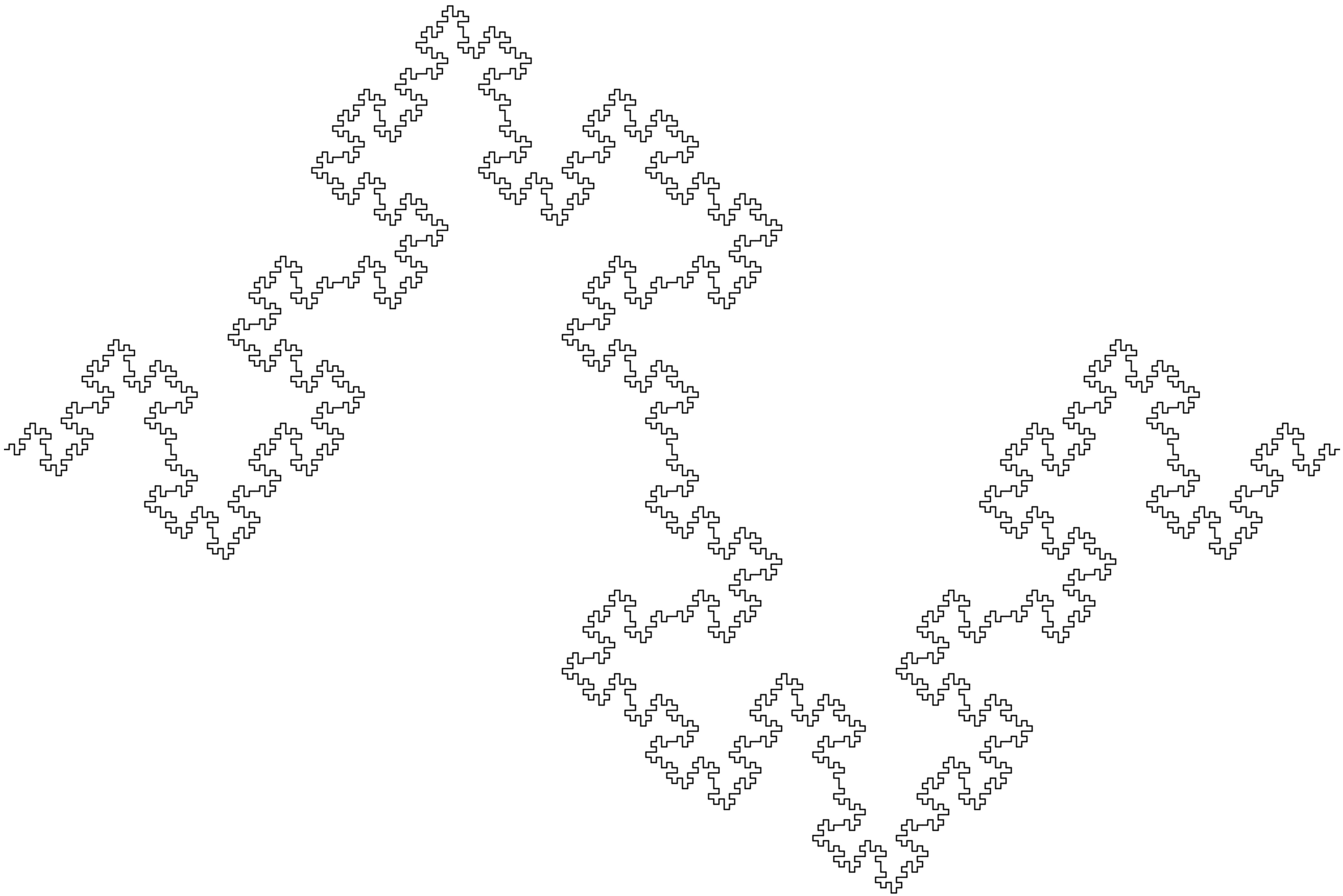
- What is a dimension?
- *Scaling (fractal, Hausdorff)* dimension vs. *topological* dimension

Minkowski Sausage

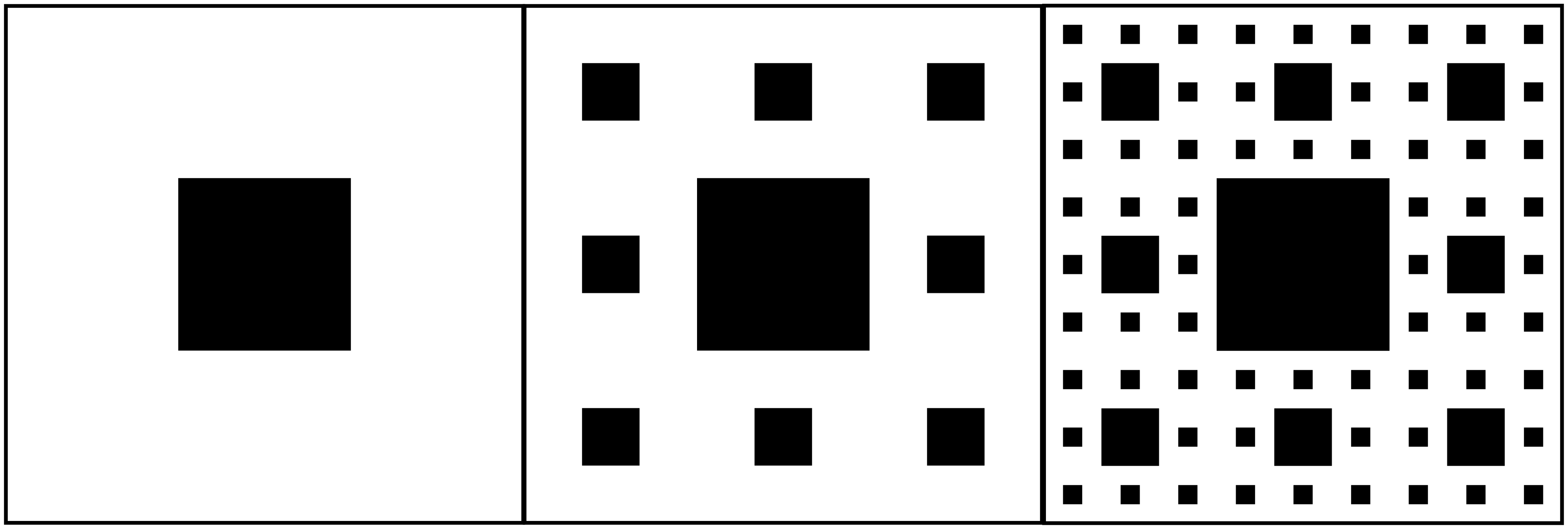


Step	1	2	3	4	5
Scaling Factor	1				
# Segments	1	8			

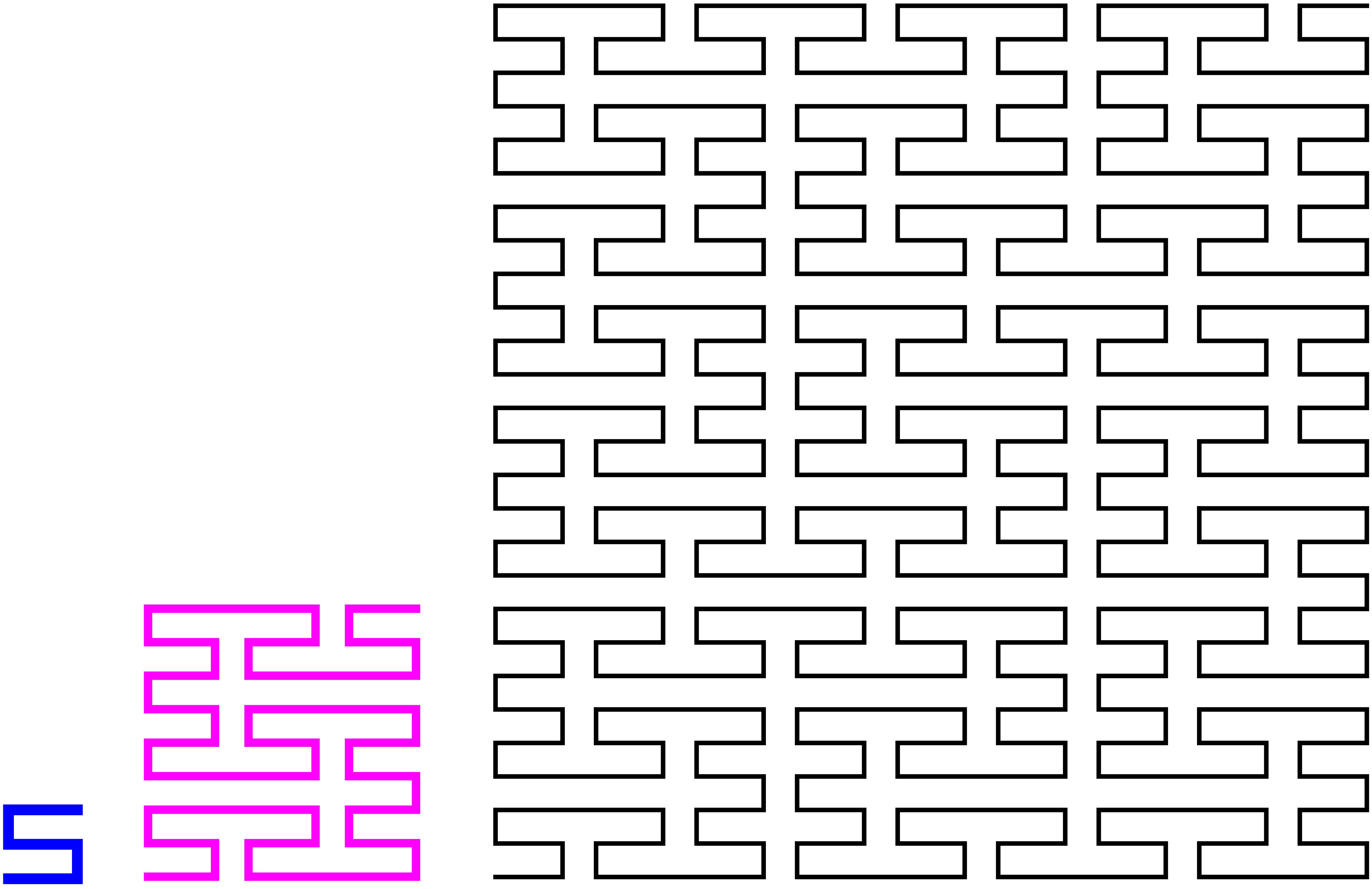
In a few steps..

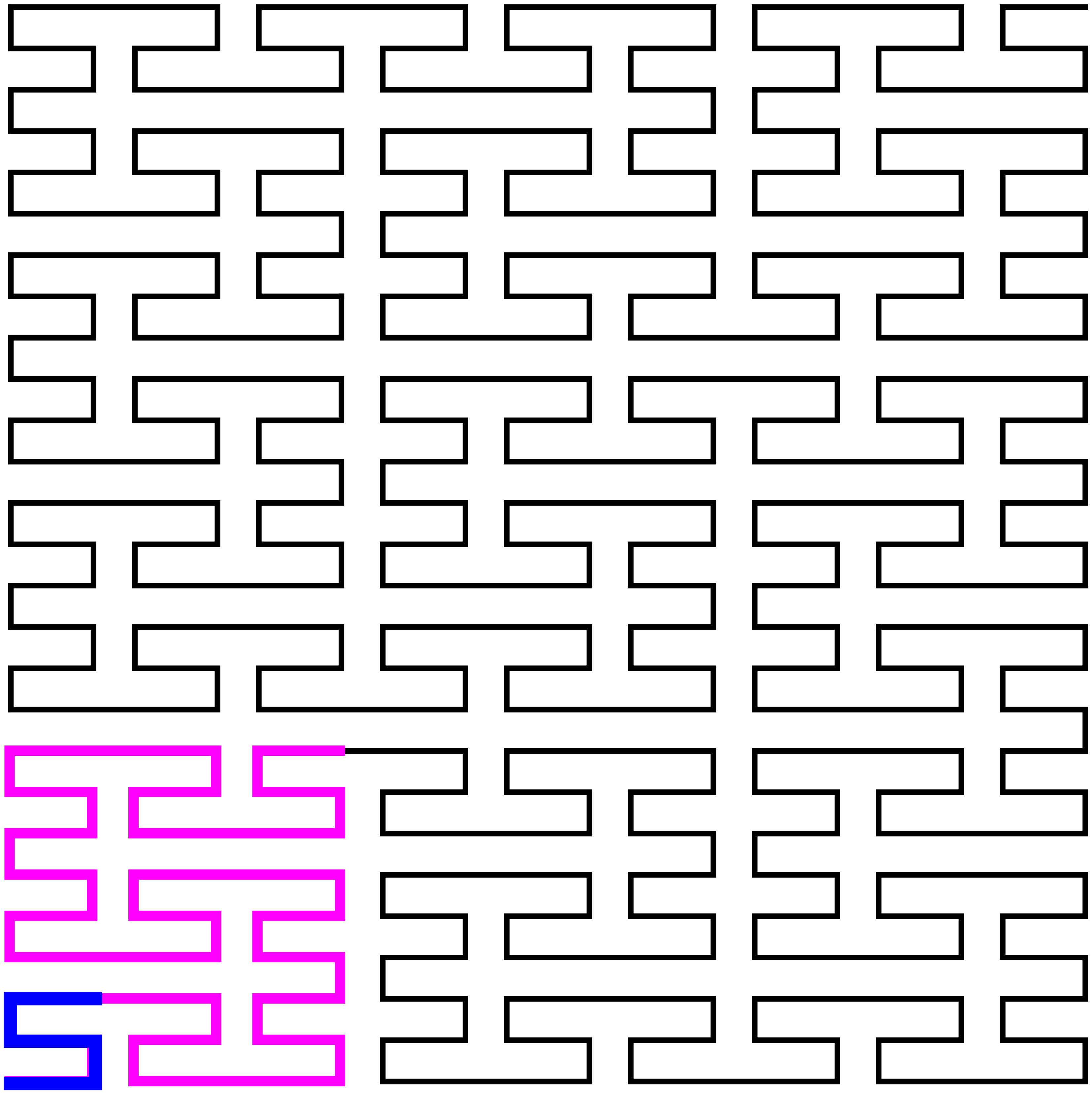


Sierpinski Carpet



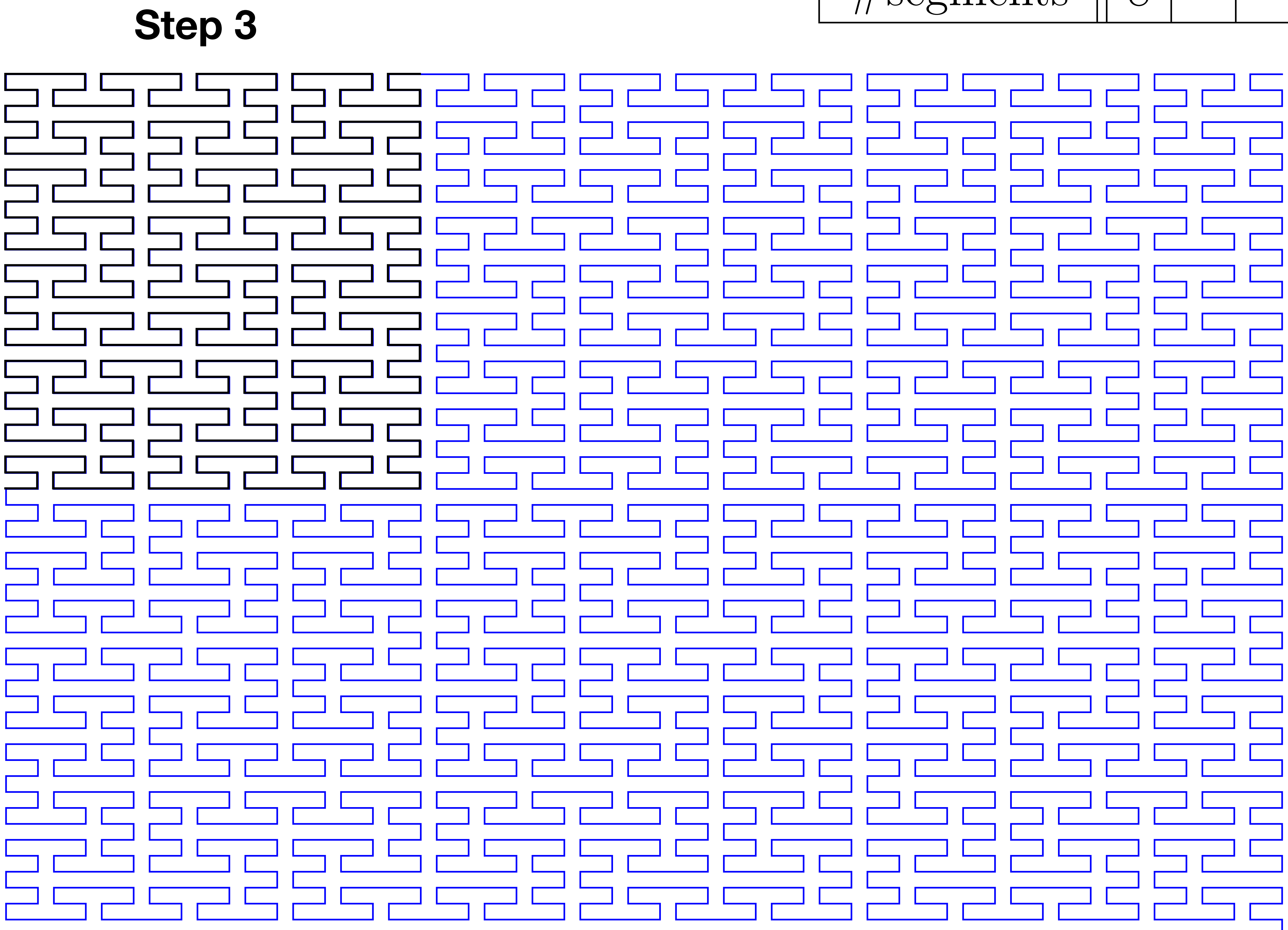
Peano Curve





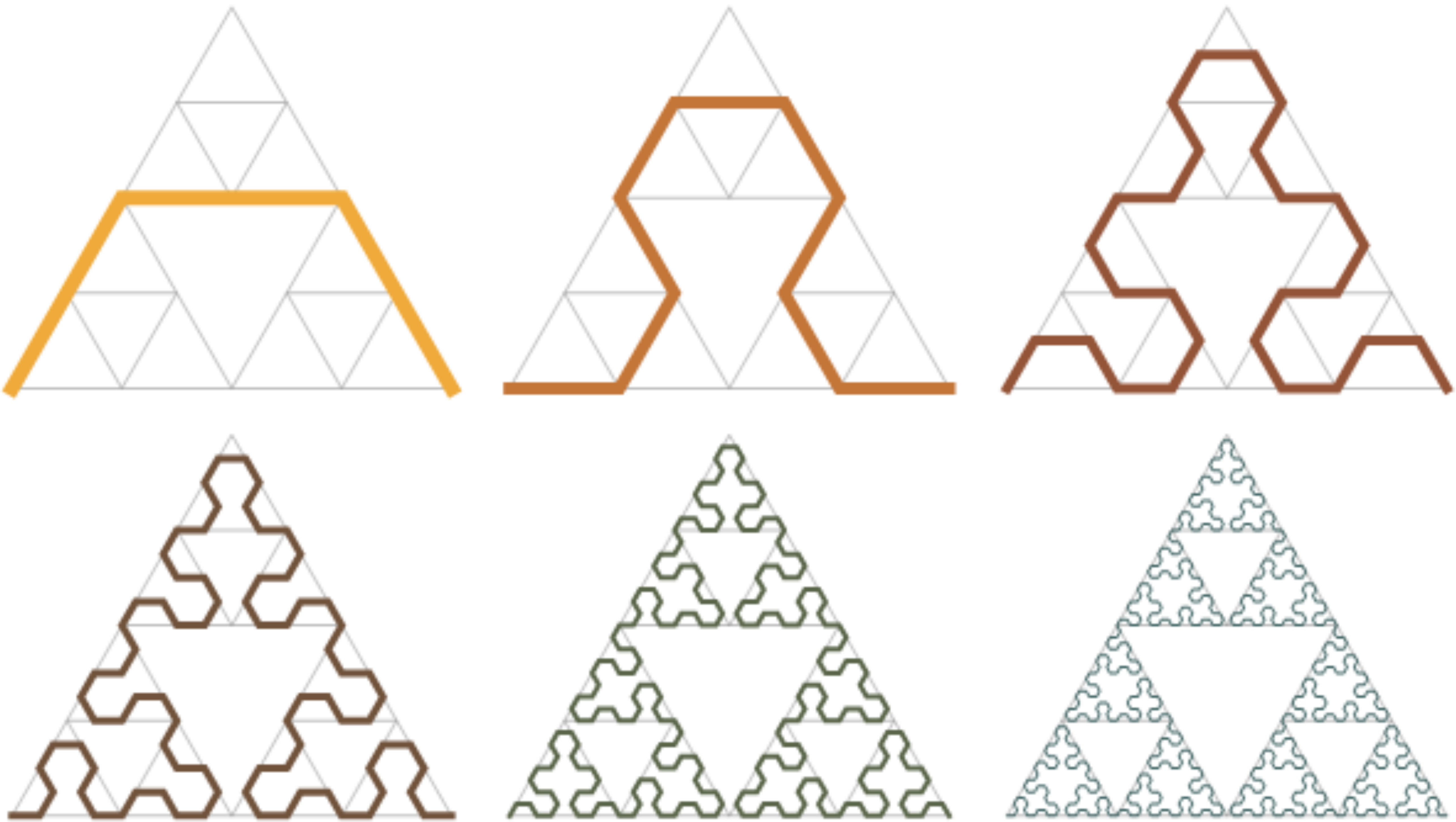
Peano Curve Step 4

Step	1	2	3	4
Scale factor	1			
#segments	8			

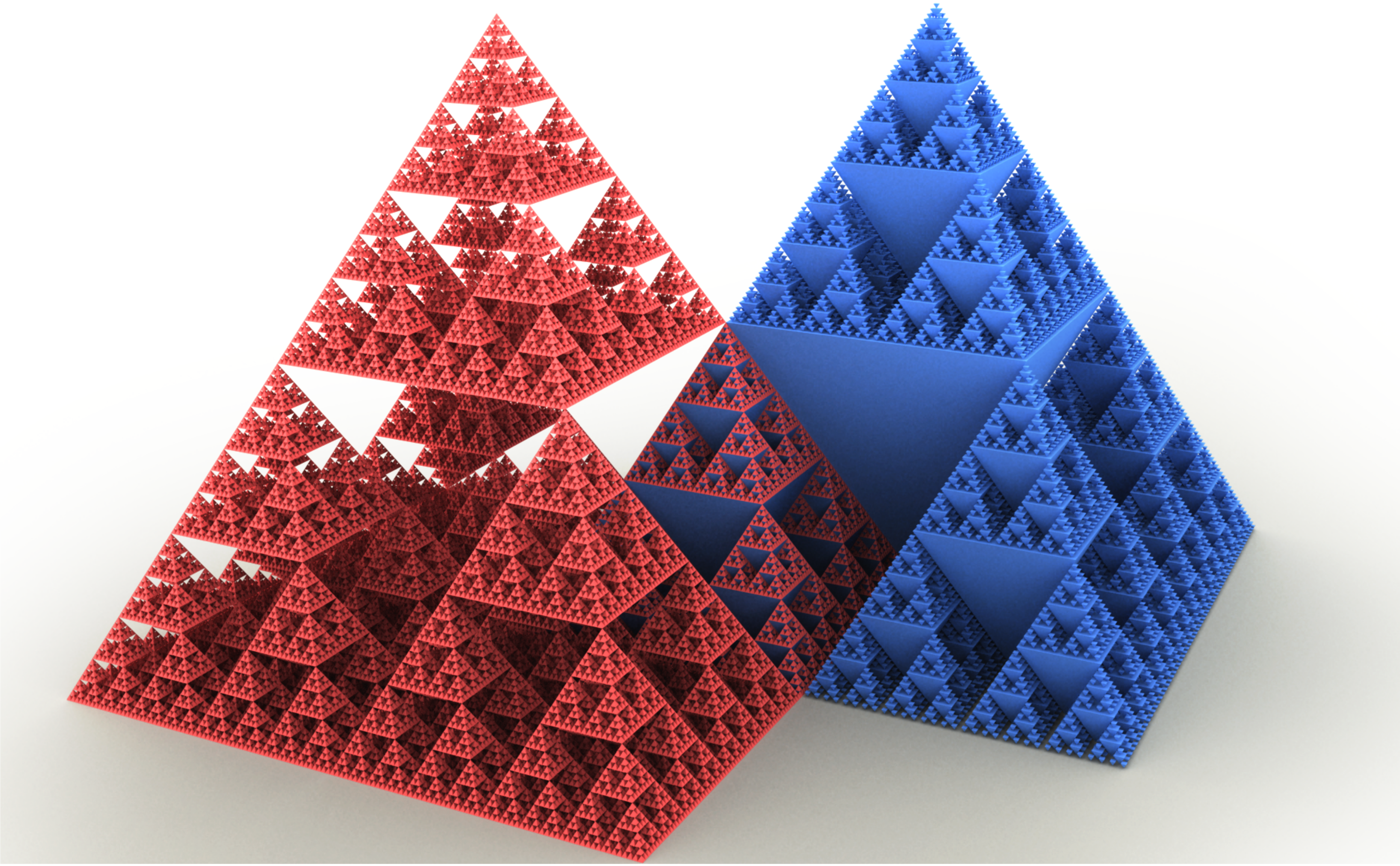


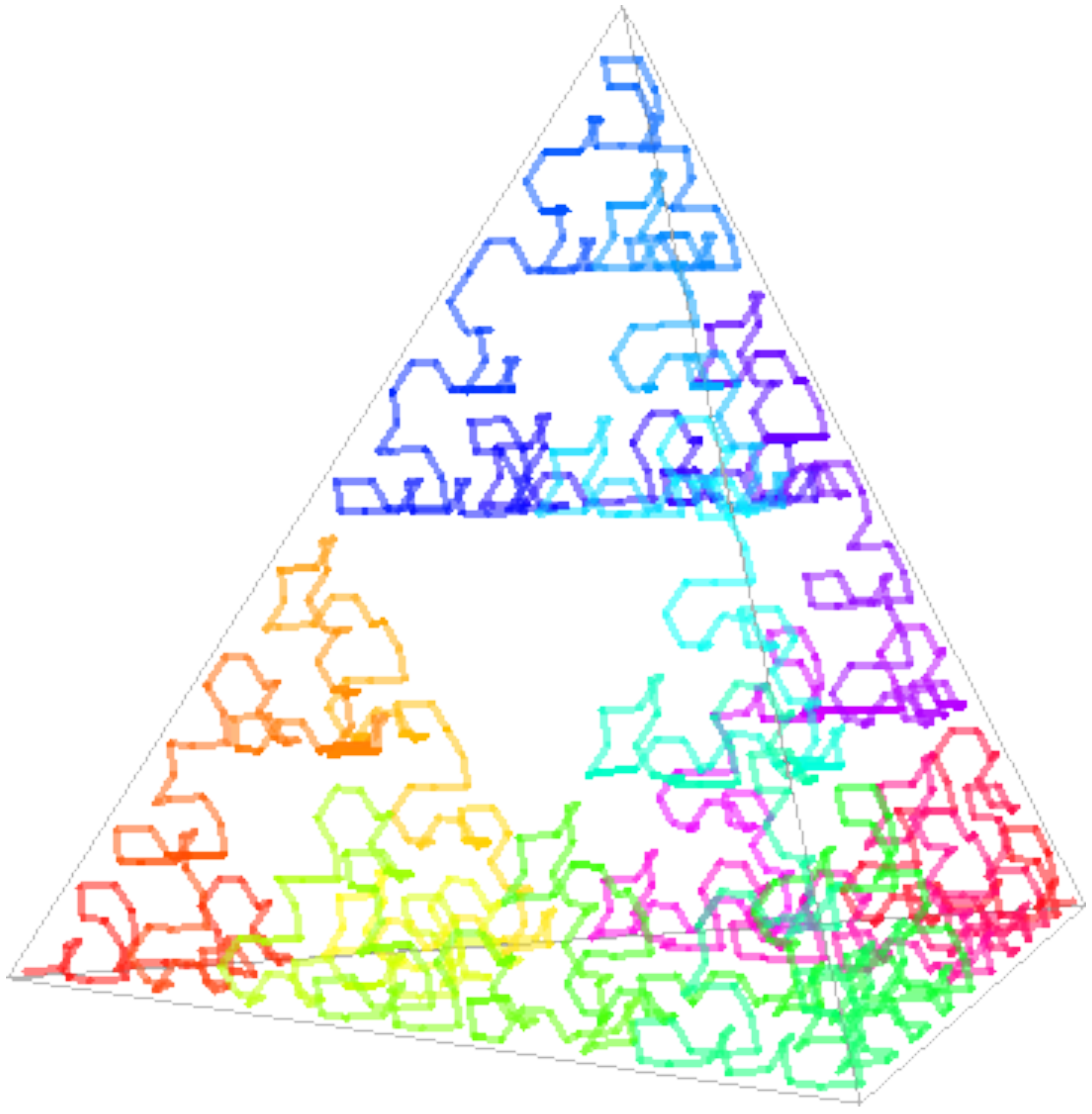
Peano curve

Step	1	2	3	4	5	6
Width W_n	2	8	26	80	$\dots$	$\dots$
#segments S_n	8	80	728	6560	$\dots$	$\dots$



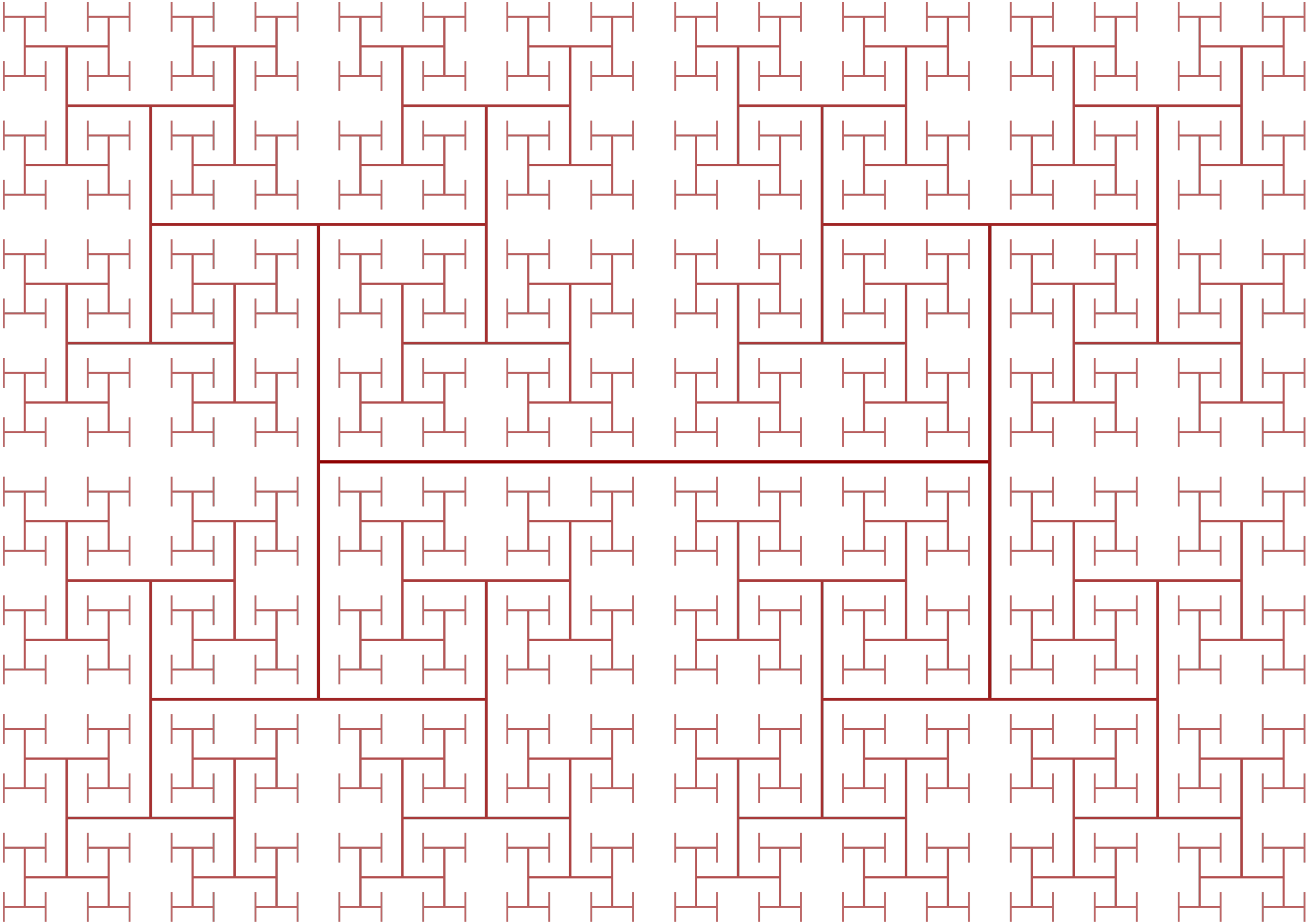
Sierpinski Pyramid



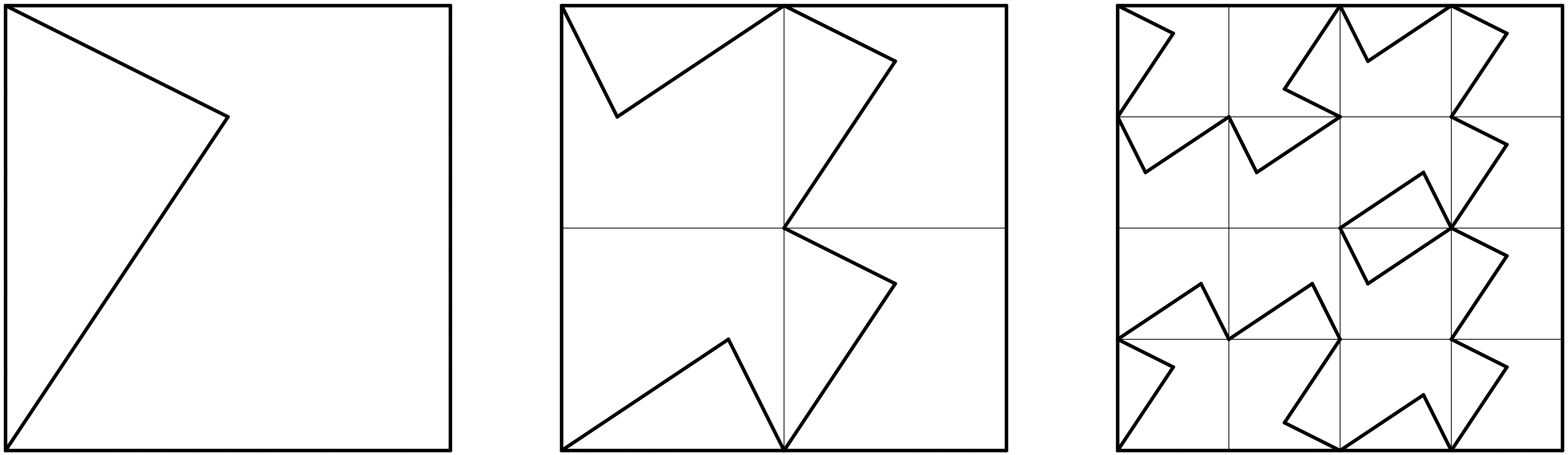


Homework

- Invent your own Peano-type (space-filling) curve!



Another Space-Filling Curve



$P_1(0) = (0,0), P_1(\frac{1}{4}) = (\frac{1}{2}, 0), P_1(\frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}), P_1(\frac{3}{4}) = (\frac{1}{2}, 1),$

$P_1(1) = (0,1)$

$P_2(0) = (0,0), P_2(\frac{1}{16}) = (0, \frac{1}{4}), P_2(\frac{1}{8}) = (\frac{1}{4}, \frac{1}{4}),$

$P_2(\frac{3}{8}) = (\frac{1}{2}, \frac{1}{4}), P_2(\frac{1}{4}) = (\frac{1}{2}, 0)$

Space Filling Curve

THEOREM. (1) *For every* $t \in [0,1]$, *the sequence* $P_0(t), P_1(t), P_2(t), P_3(t), \ldots$ *approaches a certain point* $P(t)$ *in the square* S *(in mathematical language, the sequence* $P_n(t)$ *has a* limit $P(t)$). Thus, there arises a map P from $[0,1]$ into S.

(2) *This map* P *is continuous curve in* S *from* $(0,0)$ *to* $(0,1)$.

(3) *The map* P *is* **onto**, *that is, every point of the square* S *is* $P(t)$ *for some* $t \in [0,1]$.

Proof

Let us divide the interval $[0,1]$ into:

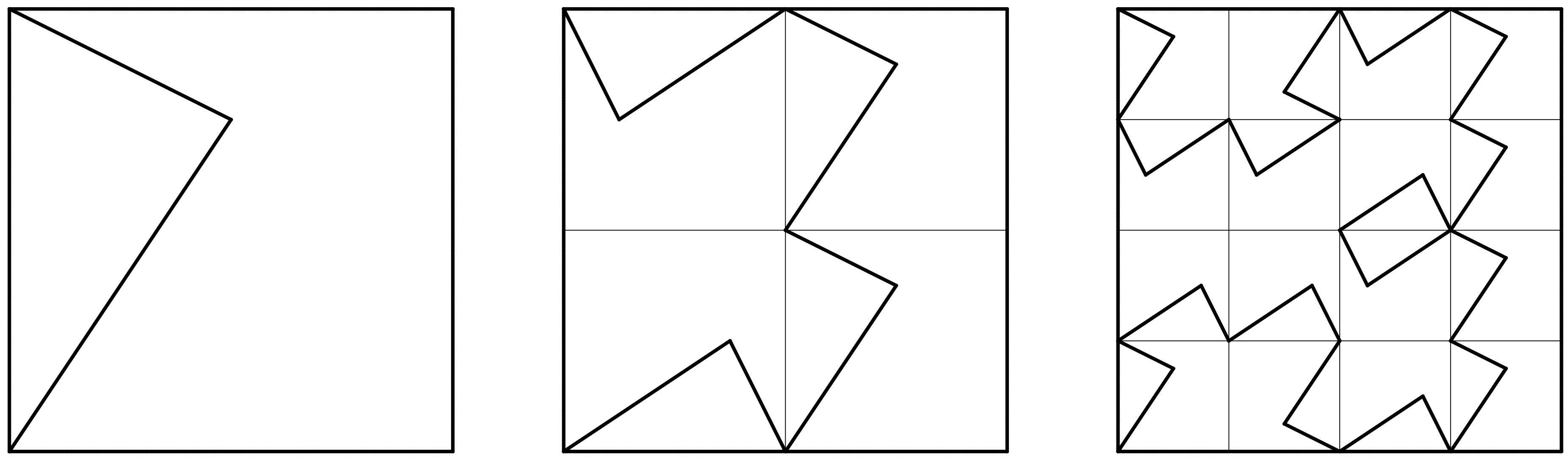
- four intervals of the "first generation," $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{2}{4}\right]$, $\left[\frac{2}{4}, \frac{3}{4}\right]$, $\left[\frac{3}{4}, 1\right]$;
- sixteen intervals of the "second generation," $\left[0, \frac{1}{16}\right]$, $\left[\frac{1}{16}, \frac{2}{16}\right]$, $\ldots$, $\left[\frac{15}{16}, 1\right]$;

  and so on;

thus, for every n, we have 4^n intervals of the n-th generation, $\left[\frac{m-1}{4^n}, \frac{m}{4^n}\right]$, $1 \le m \le 4^n$.

Similarly, we divide the square Q into four squares of the first generation (shown in Figure 1, middle), 16 squares of the second generation (shown in Figure 1, right), ..., 4^n

Claims



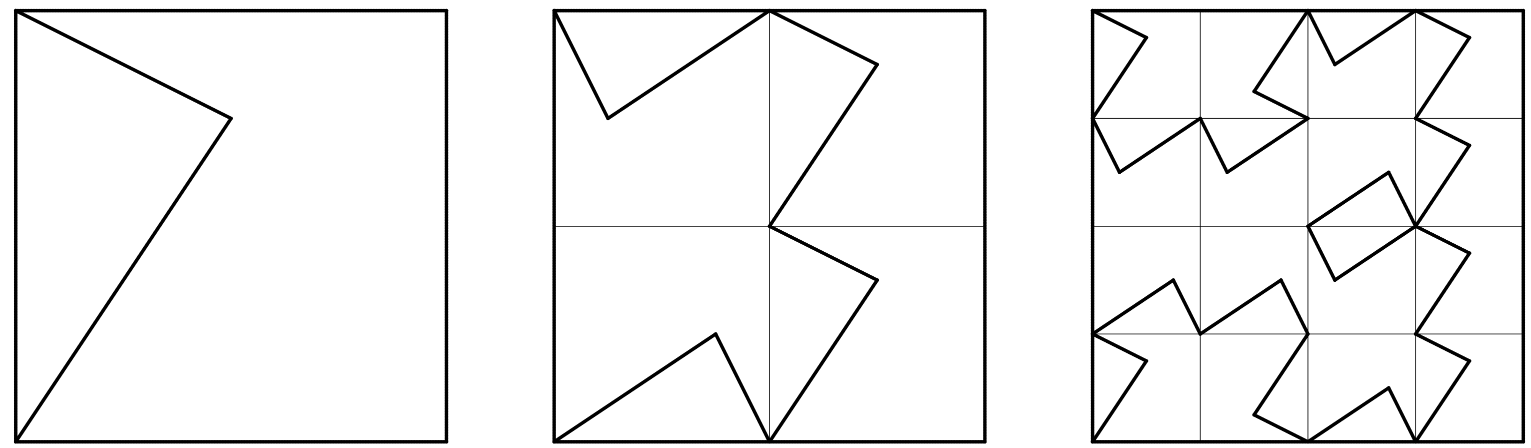
(A) The image of every interval of the n-th generation with respect to the map P_n is contained in some (unique) square of the n-th generation. Moreover, this creates a one-to-one correspondence between the interval of the n-th generation and the squares of n-th gereration.

(B) The images of every interval of the n-th generation with respect to the map P_N with any $N \geq n$ are contained in the same square of the n-th generation.

(C) In addition to this, remark that the images of two adjacent intervals of the n-th generation with respect to P_n (and hence to P_N for any $N \geq n$)are contained in adjacent squares of the n-th generation.

Values at rational points

What is $\mathbf{P(\frac{1}{3})}$? What is $\mathbf{P^{-1}(1,0)}$?



notice that $P(0) = (0,0), P(\frac{1}{4}) = (\frac{1}{2}, 0), P(\frac{1}{4} + \frac{1}{16}) = (\frac{3}{4}, 0)$

$$P(\tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64}) = (\tfrac{7}{8}, 0), \ldots, P(\tfrac{1}{4} + \tfrac{1}{16} + \ldots + \tfrac{1}{64}) = (1 - \tfrac{1}{2^n}, 0), \ldots$$

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots =$$

$$P(\tfrac{1}{3}) = (1,0)$$

Square Without a Corner