Geometric Aspects of Integrable Systems

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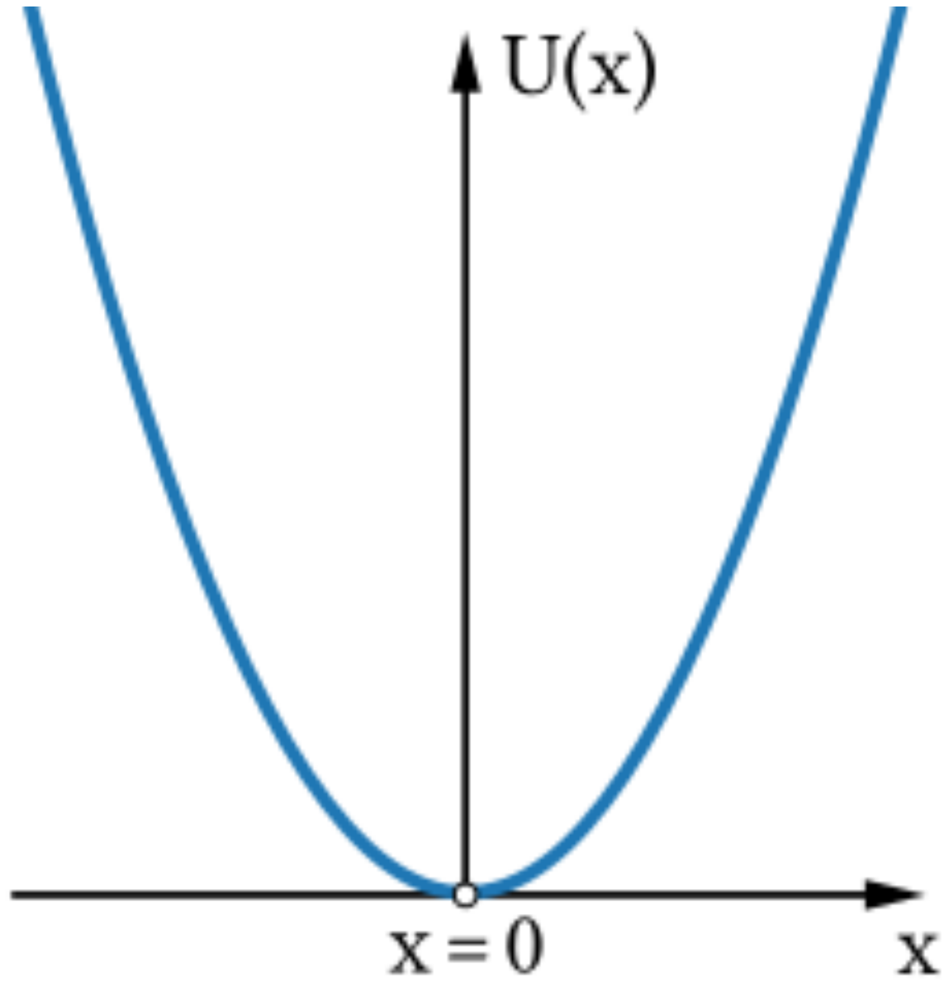
Special Colloquium
@ University of Arizona
2/3/2025

Harmonic Oscillator

Harmonic oscillator $$H = \frac{p^2}{2} + \frac{x^2}{2}$$



Hamilton equations

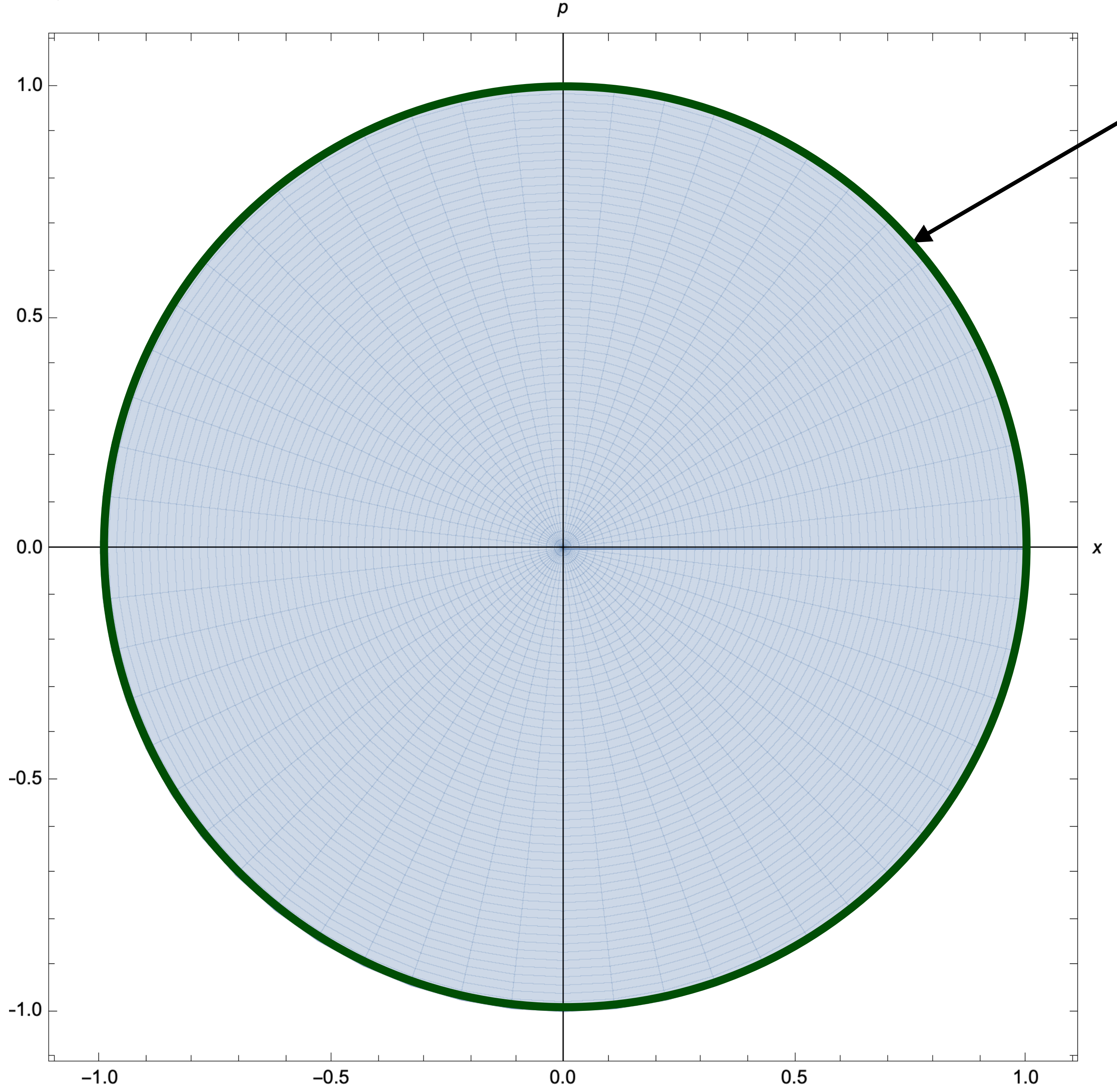
$$\dot{x} = p$$
$$\dot{p} = -x$$

Combining

$$\ddot{x} + x = 0$$

Phase space — symplectic manifold $\mathscr{M}$

Symplectic form $\omega = dp \wedge dx$



$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$

Lagrangian $\mathscr{L} \subset \mathscr{M}$ is a middle-dimensional submanifold and such that the restriction of the symplectic form on $\mathscr{L}$ vanishes $\omega|_{\mathcal{L}} = 0$

Classical Integrability

Equations of motion

$$\frac{df}{dt} = \{H, f\} = \sum_a \frac{\partial H}{\partial p_a}\frac{\partial f}{\partial x_a} - \frac{\partial H}{\partial x_a}\frac{\partial f}{\partial p_a}$$

Integrability — family of n conserved quantities that Poisson commute with each other

$$\{H_i, H_j\} = 0 \quad i, j = 1, \ldots, n$$

Poisson bracket is induced by the symplectic form

Liouville-Arnold Theorem

Compact Lagrangians $\mathscr{L}$: $\{H_i = E_i\}$ are isomorphic to tori

Evolution in the neighborhood of $\mathscr{L}$ is linearized in action/angle variables $\{I_i, \varphi_i\}_{i=1}^n$

$$\frac{d\varphi_i}{dt} = \omega_i, \qquad \frac{dI_i}{dt} = 0$$

Action/angle variables are hard to find

Examples

- Simple models from grade school/undergraduate — oscillator, Kepler problem, etc.

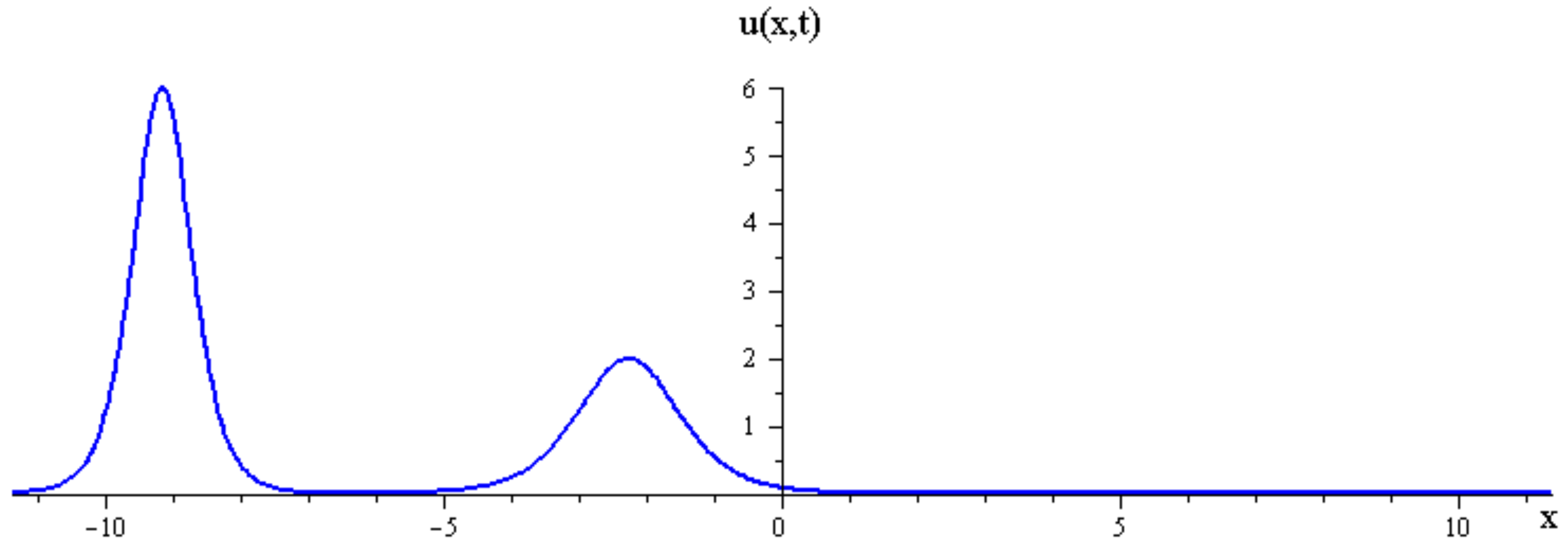
- Many-body integrable systems — Calogero, Toda, Ruijsenaars

- Continuous integrable models in (1+1)-dimensions: Korteweg-de-Vries, Intermediate Long-Wave, etc.

$$u_t = 6uu_x - u_{xxx}$$

- They admit soliton solutions. Sectors with N solitons are described by finite N-body integrable systems

[UofA faculty: Newell, Gabitov, Chertkov
Moloney, Zakharov, Izosimov,...]



Quantization

Coordinates and momenta become operators

$$p, x \mapsto \hat{p}, \hat{x}$$

Poisson brackets associated to ω become commutators

$$\{A, B\}_{P.B.} \mapsto [A, B]$$

Heisenberg algebra

$$[\hat{p}, \hat{x}] = -i\hbar$$

$$\hat{x} f(x) = x f(x)$$
$$\hat{p} f(x) = -i\hbar f'(x)$$

Lagrangian constraint

$$\frac{p^2}{2} + \frac{x^2}{2} - E = 0$$

Replaced by operator

$$\left(\frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2} - E\right) Z(x) = 0$$

Integrability

$$[H_i, H_j] = 0$$

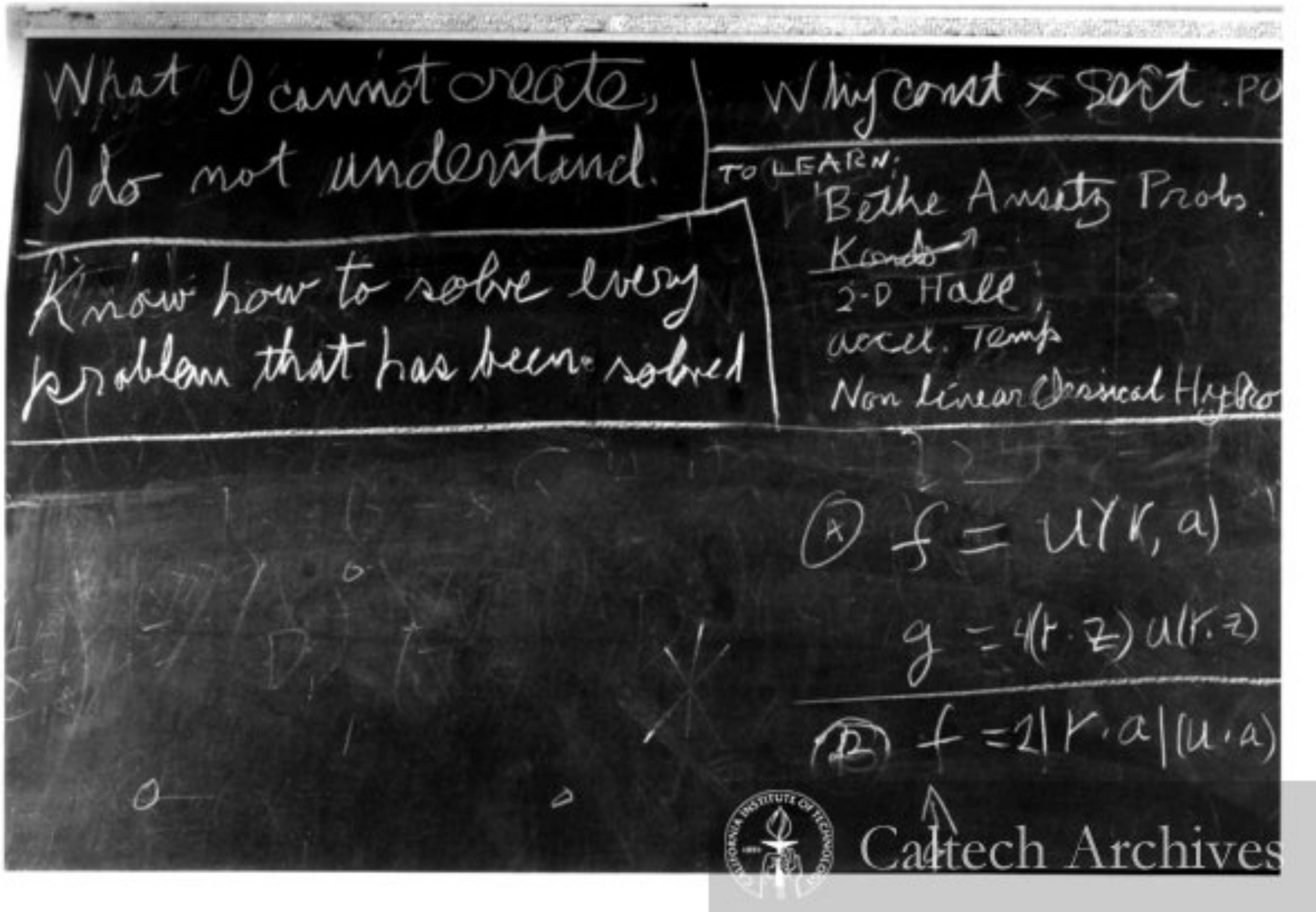
$$H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables -> simultaneous diagonalization of H_i

Some models like spin chains are intrinsically quantum

Quantization is as much art as it is science

[Gukov, PK, Nawata, Pei, Saberi
Monograph **SpringerBriefs**]



What I cannot create,
I do not understand.

Why const × sort .PO

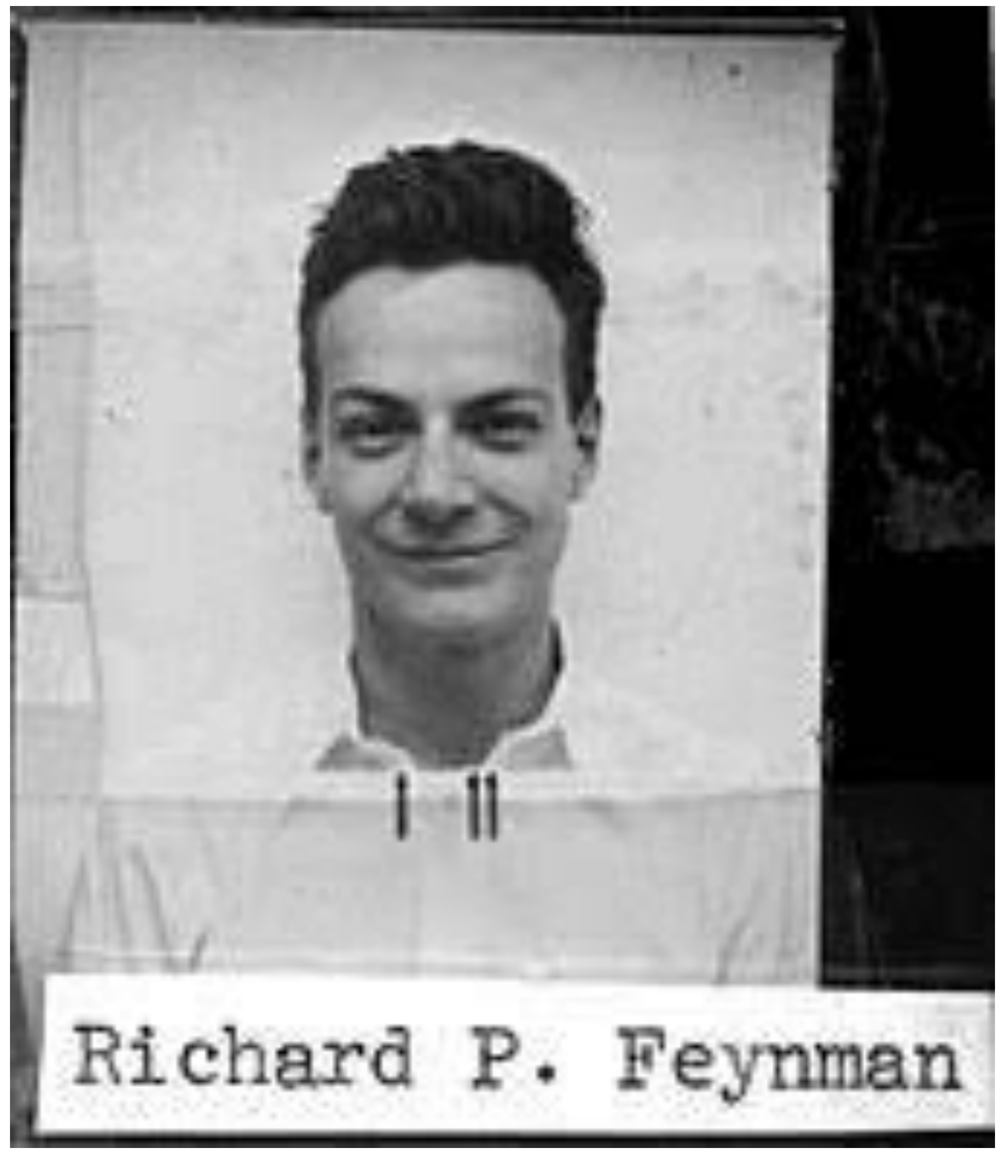
TO LEARN:
Bethe Ansatz Probs.
~~Kondo~~
2-D Hall
accel. Temp
Non Linear Classical Hydro

Know how to solve every
problem that has been solved

Ⓐ $f = u\,r(r, a)$

$g = 4(r \cdot z)\,u(r \cdot z)$

Ⓑ $f = 2|r \cdot a|(u \cdot a)$



I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better.

Physical Mathematics

We will see that geometry and integrability go hand in hand and that both subjects benefit from each other:

i) Geometry provides a universal framework to study integrable systems while integrability helps performing certain curve counting calculations among other things

ii) Geometry helps to prove dualities

Enumerative Algebraic Geometry

[Givental, Kim] [Okounkov] [Givental, Lee]
[Pushkar, Zeitlin, Smirnov] [PK, Pushkar, Smirnov, Zeitlin]

Geometric (q-)Langlands Correspondence

[Frenkel] [Aganagic, Frenkel, Okounkov]
[Frenkel, PK, Sage, Zeitlin]

Dualities between Integrable Systems

[Matsuo, Cherednik][PK, Gaiotto][PK, Zeitlin]
[Bazhanov, Lukyanov, Zamolodchikov][Dorey, Tateo]

Literature

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qKZ/tRS Duality via Quantum K-Theoretic Counts
P. Koroteev, A. M. Zeitlin

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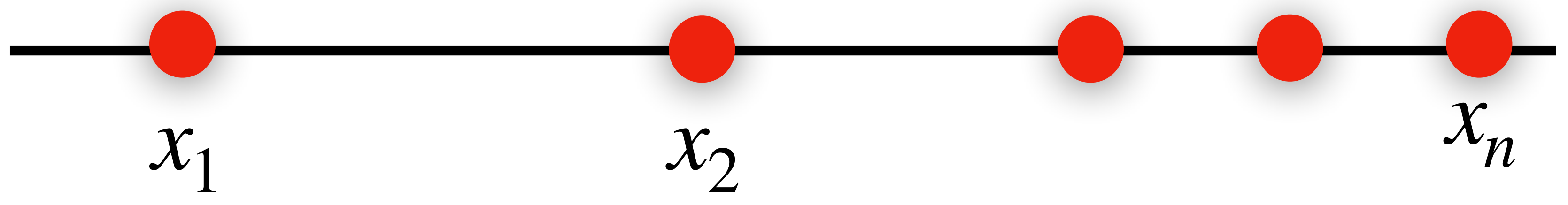
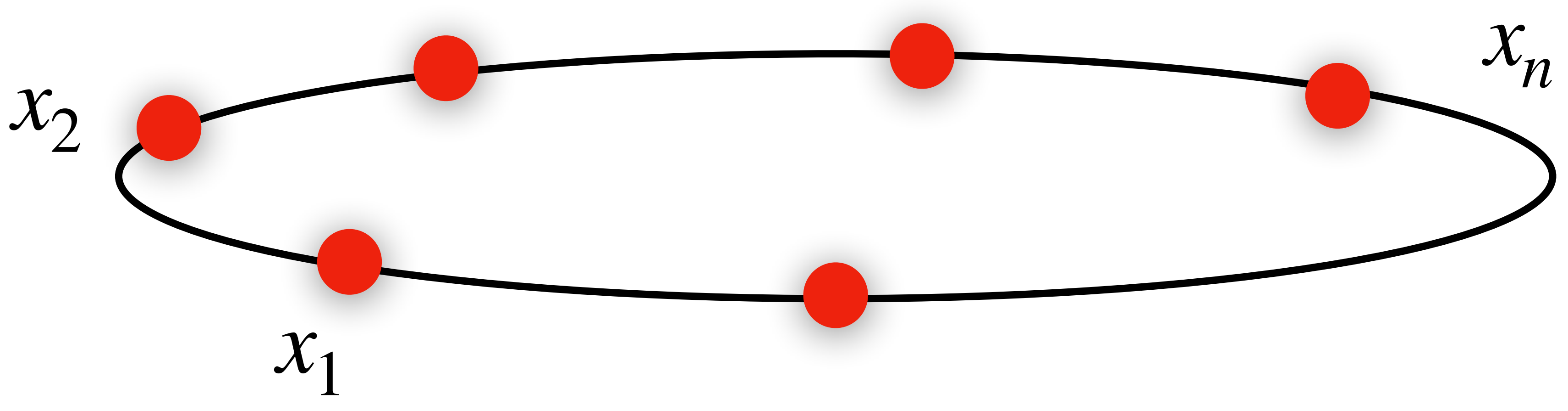
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

I. Many-Body Systems

Calogero in 1971 introduced a new many-body system. Moser in 1975 proved its integrability





$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$

The **Calogero-Moser (CM)** system admits generalizations:

rational CM → trigonometric CM → elliptic CM

Relativistic generalization is called
Ruijsenaars-Schneider (RS) family

rRS → tRS → eRS

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

Example: tRS Model with 2 Particles

Hamiltonians (Macdonald operators)

$$T_1 = \frac{\xi_1 - t\xi_2}{\xi_1 - \xi_2} p_1 + \frac{\xi_2 - t\xi_1}{\xi_2 - \xi_1} p_2$$

$$T_2 = p_1 p_2$$

Log-symplectic form

$$\Omega = \sum_i \frac{dp_i}{p_i} \wedge \frac{d\xi_i}{\xi_i}$$

Integrals of motion

$$T_i = E_i$$

Coordinates ξ_i, momenta p_i

coupling constant t, energies E_i

Quantization

$$p_i \xi_j = \xi_j p_i q^{\delta_{ij}} \qquad q \in \mathbb{C}^\times$$

tRS Momenta are shift operators

$$p_i f(\xi_i) = f(q\xi_i)$$

Eigenvalue Equations

$$T_i V = E_i V$$

II. Quantum Integrability

Let $\mathfrak{g}$ be Lie algebra

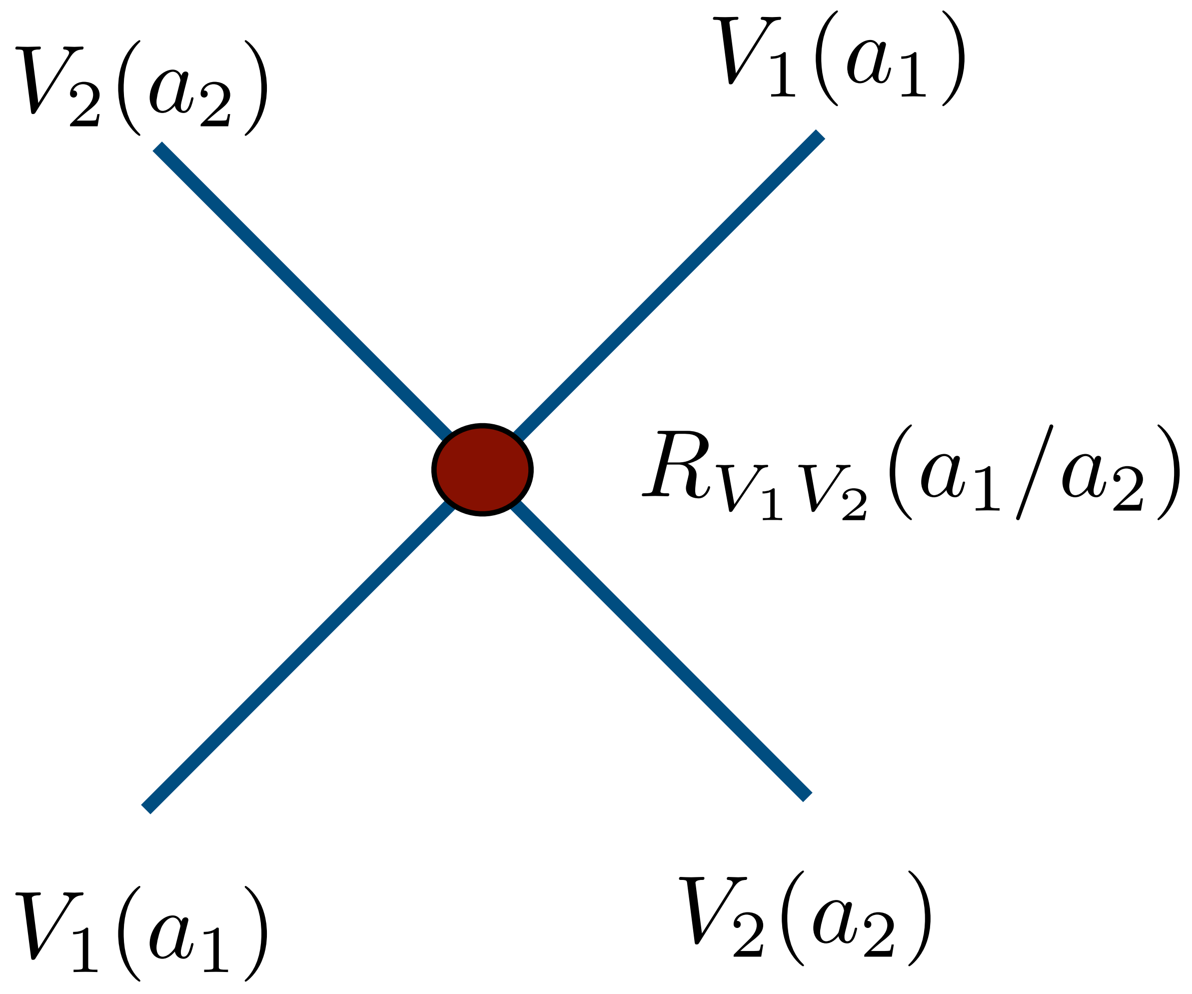
$[a,b] \in \mathfrak{g}$

$\hat{\mathfrak{g}} = \mathfrak{g}(t)$ loop algebra of Laurent polynomials in t valued in $\mathfrak{g}$

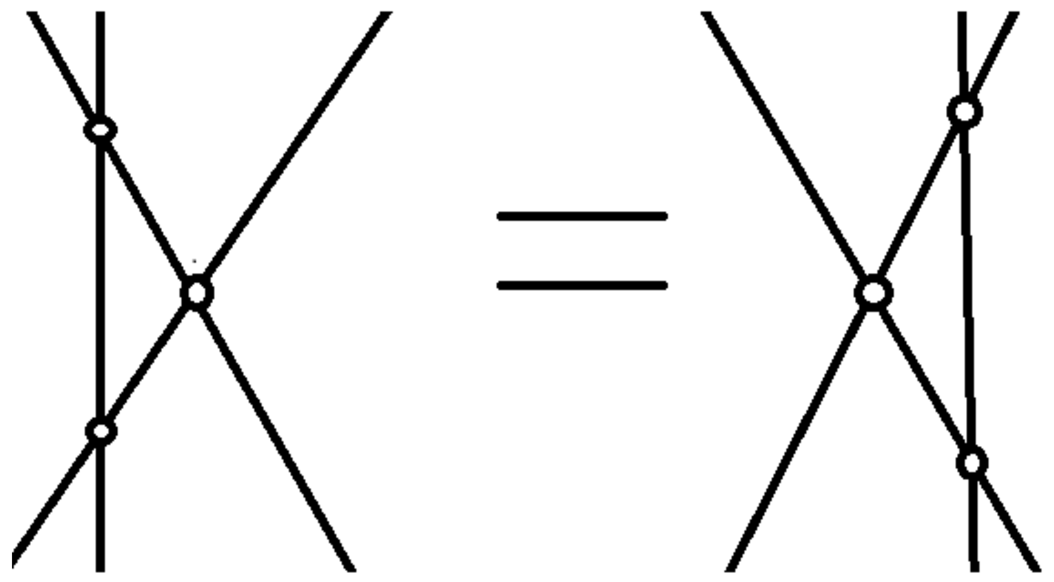
Tensor product of its representations $V_1(a_1) \otimes \cdots \otimes V_n(a_n)$ a_i are values for t

Quantum group is a noncommutative deformation $U_\hbar(\hat{\mathfrak{g}})$

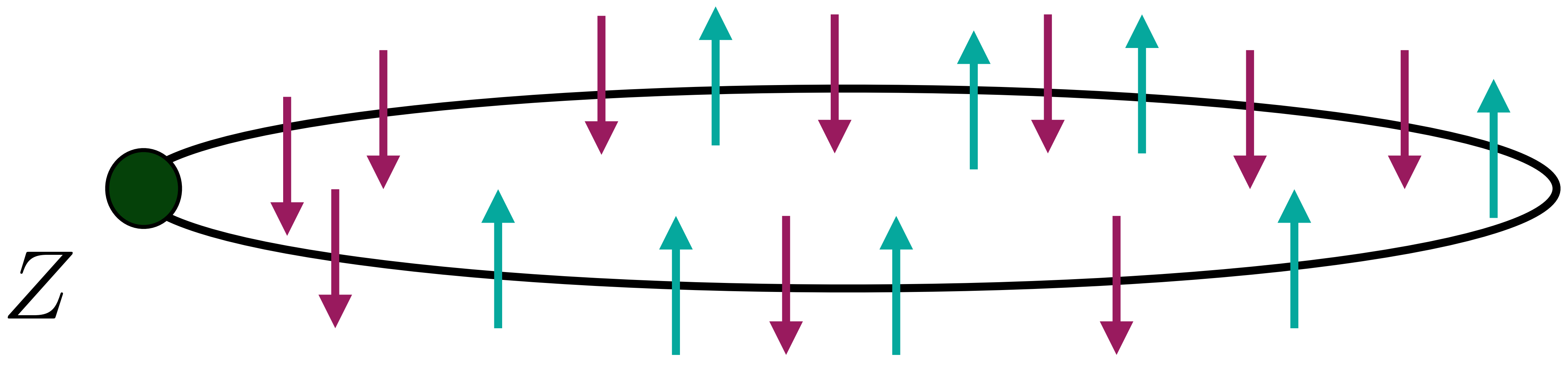
with an intertwiner R-matrix



satisfying Yang-Baxter equation



Heisenberg Spin Chain



$U_\hbar(\mathfrak{sl}_2)$

spin-1/2 XXZ chain on n sites

$V(a) \simeq \mathbb{C}^2(a)$ $\quad \uparrow \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix} \quad \downarrow \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

sector k n-k

$$e = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \qquad [h,e] = 2e$$

$$f = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \qquad [h,f] = -2f$$

$$h = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \qquad [e,f] = h$$

Hamiltonian $\quad [\Delta(g), H] = 0$

$$H = \sum_i e_i \otimes f_{i+1} + f_i \otimes e_{i+1} + \Delta h_i \otimes h_{i+1}$$

Spectrum will depend on twist eigenvalues **z** and on values of spectral parameter **a**

Solved by Bethe Ansatz

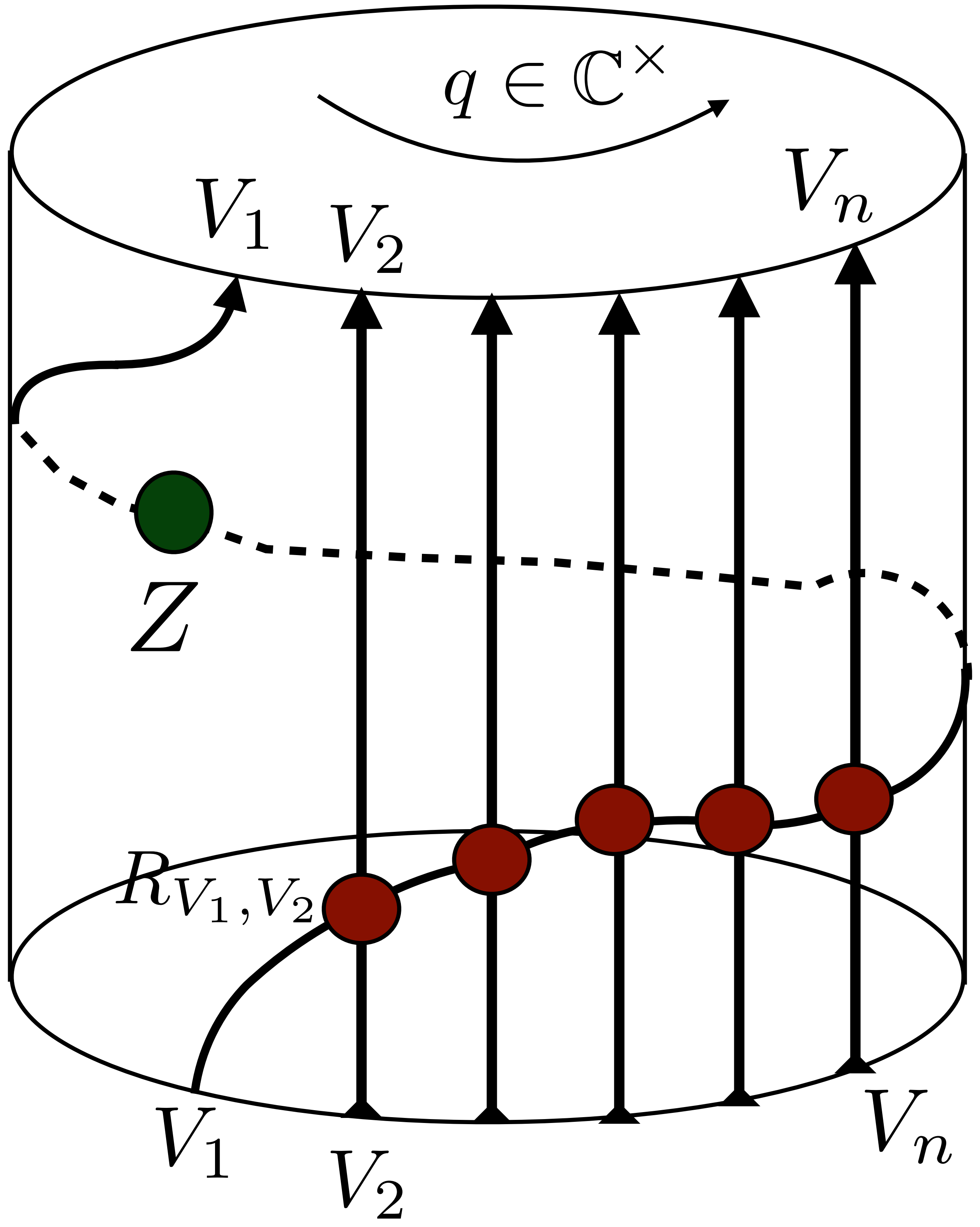
The qKZ Equation

Consider Knizhnik-Zamolodchikov q-difference equation [I. Frenkel Reshetikhin]

Let $\Psi(a_1,\ldots,a_n)\in V_1(a_1)\otimes\cdots\otimes V_n(a_n)$

qKZ equation $\Psi(qa_1,\ldots a_n)=M(z,a)\Psi(a_1,\ldots a_n)$

where $M(z,a)=(Z\otimes 1\otimes\cdots\otimes 1)R_{V_1V_n}\cdots R_{V_1V_2}$



In the limit $q\to 1$
qKZ becomes an eigenvalue problem for $M(z,a)$

Integrability

Compose q-shifts

$$\Psi(qa_1, qa_2) = M_{12}(a_1, a_2)\Psi(a_1, a_2) = M_1(a_1, qa_2)M_2(a_1, a_2)\Psi(a_1, a_2)$$

$$= M_2(qa_1, a_2)M_1(a_1, a_2)\Psi(a_1, a_2)$$

so $$M_1(a_1, qa_2)M_2(a_1, a_2) = M_2(qa_1, a_2)M_1(a_1, a_2)$$

Taking $q \to 1$ limit we get $$[M_1(a_1, a_2), M_2(a_1, a_2)] = 0$$

Thus we get a set of commuting quantum operators → Integrability!

Operators M yield quantum Hamiltonians for the XXZ spin chain

What does it mean geometrically?

Solutions of qKZ

[Aganagic Okounkov]

Schematic solution

$$\Psi^i_\alpha = \int_{\gamma_i} \frac{d\mathbf{s}}{\mathbf{s}} f_\alpha(\mathbf{s},a)\mathcal{K}(\mathbf{s},z,a,q)$$

Matrix of fundamental solutions

component

representation

universal kernel

$$\log\mathcal{K}(\mathbf{s},z,a,q) \underset{q\to 1}{\sim} \frac{S(\mathbf{s},z,a)}{\log q}$$

$$\frac{\partial S(\mathbf{s},z,a)}{\partial \mathbf{s}} = 0$$

Bethe equations for Bethe roots **s**

$$a_i \frac{\partial S}{\partial a_i} = \Lambda_i$$

Eigenvalues of qKZ operators $M(z,a)$

The map $\alpha \mapsto f_\alpha(\mathbf{s}^*)$ provides diagonalization

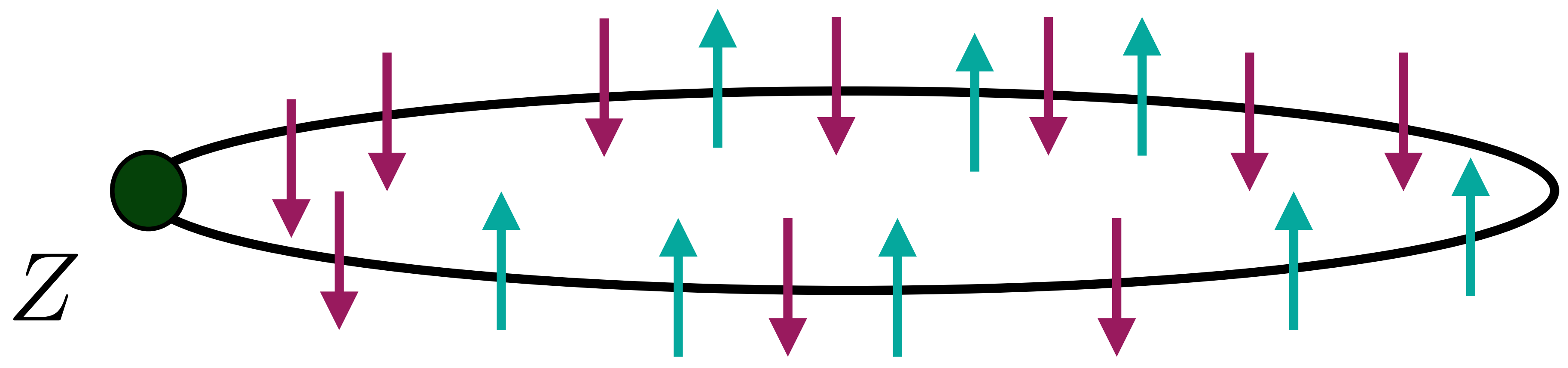
So we need to find `off shell' Bethe eigenfunctions $f_\alpha(\mathbf{s}^*,a)$

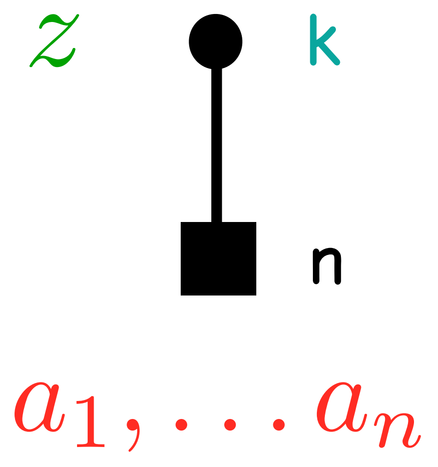
III. From Spin Chains to Geometry

The solution comes from enumerative algebraic geometry inspired by physics

Hilbert space of states of quantum spin chain

Enumerative counts into quiver variety X





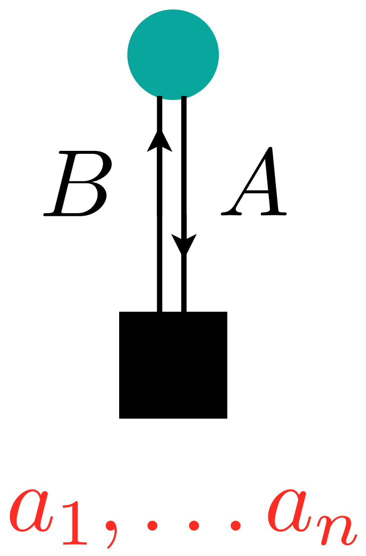
Gauge group $G = U(k)$ encodes the sector with k spins up

choice of k planes in n-dimensional space — Grassmannian

Flavor group (framing) $U(n)$ encodes the number of sites, its maximal torus gives parameters **a**

Integration variables **s** (Bethe roots) live in the maximal torus of G. By integrating we project down on certain symmetric functions of **s**

Equivariant K-theory of $X = T^*Gr_{k,n}$



$V \simeq \{(p, v) \in Gr_{k,n} \times W | v \in p\}$ tautological vector bundle over $Gr_{k,n}$ of rank k

$W \simeq \mathbb{C}^n$ trivial bundle

Torus T acting on X $\left(\mathbb{C}^\times\right)^n \times \mathbb{C}^\times_\hbar$ $(y_1, \dots, y_n) \mapsto (a_1 y_1, \dots, a_n y_n)$

$\mathbb{C}^\times_\hbar$ dilates cotangent fibers

$\binom{n}{k}$ fixed points are labelled by subsets $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$

Nakajima quiver variety

$$X = \mu^{-1}(0)_s / GL(V)$$

$\mu(A, B) = BA$ - moment map

Stability condition: map A is injective

Tensor polynomials of tautological bundles V, W and their duals generate classical T-equivariant K-theory ring of X

$R = Hom(V, W)$ acted by $GL(V)$

$\mu : T^*R \to \mathfrak{gl}(V)^*$

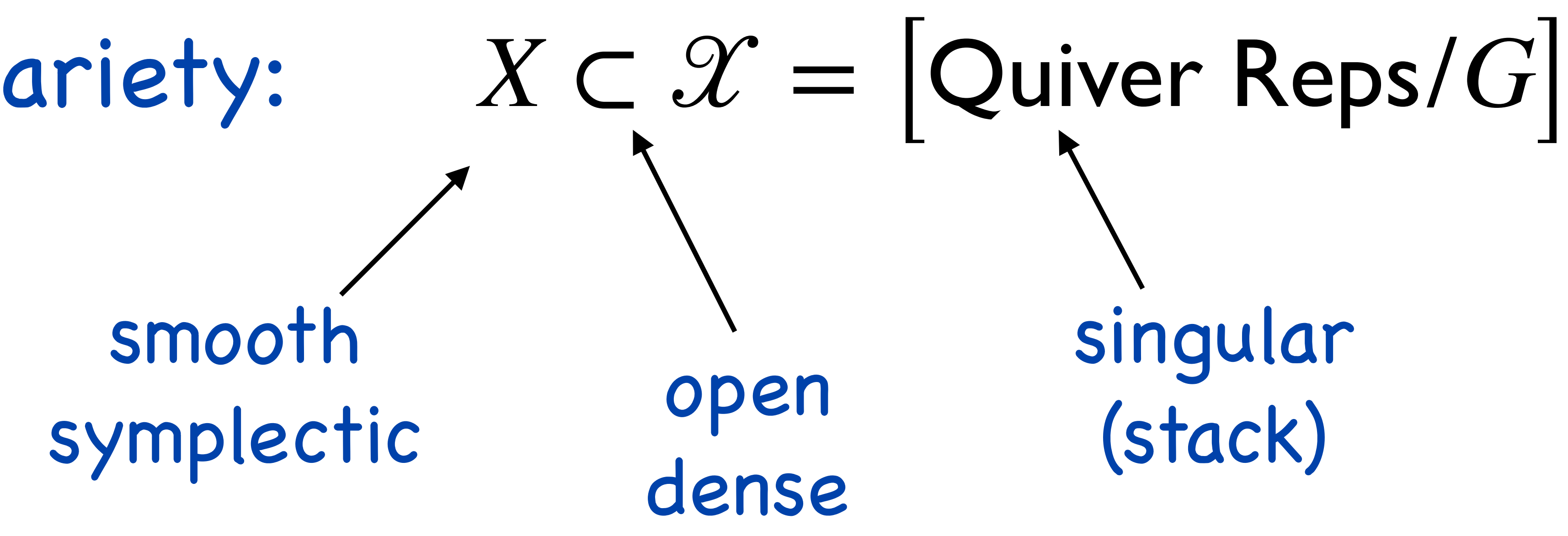
$$\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$$

$$\tau(s_1, \cdots, s_k) = (s_1 + \cdots + s_k)^2 - \sum_{1 \le i_1 < i_2 < i_3 \le k} s_{i_1}^{-1} s_{i_2}^{-1} s_{i_3}^{-1}$$

Quantum K-theory of X

[Okounkov]
[Pushkar Smirnov Zeitlin]
[PK Pushkar Smirnov Zeitlin]

The quiver variety:

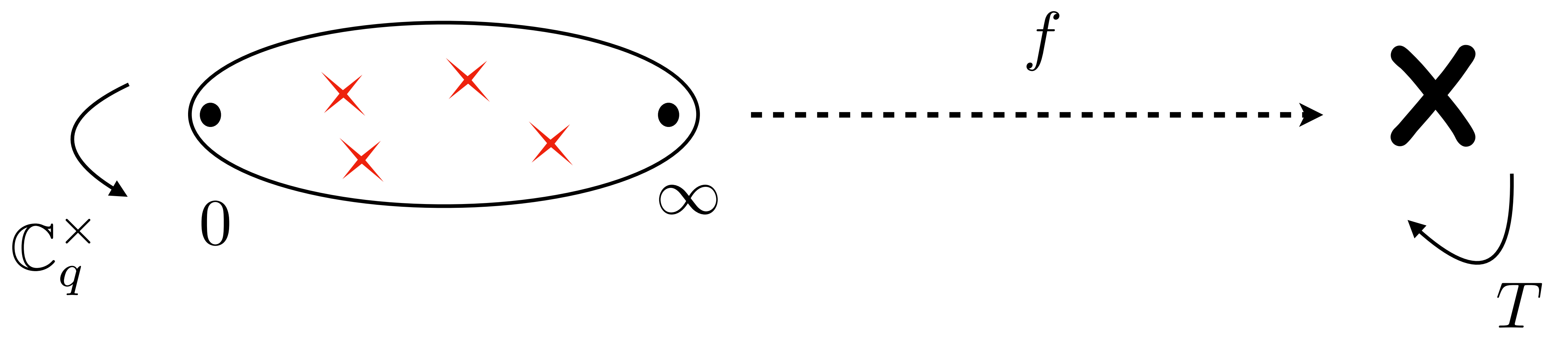


$X \subset \mathscr{X} = \left[\text{Quiver Reps}/G\right]$

smooth symplectic

open dense

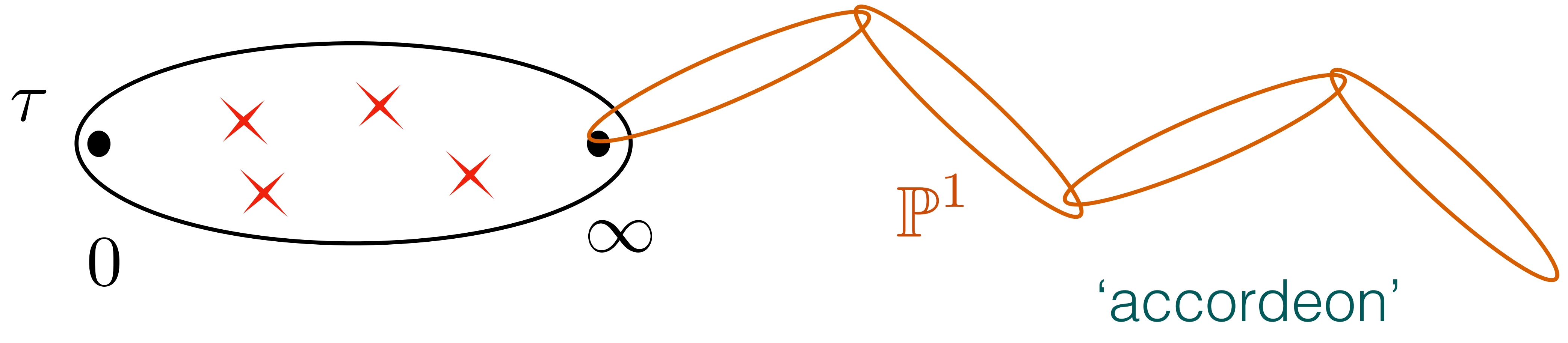
singular (stack)



Space of quasimaps of degree $d \in H_2(X, \mathbb{Z})$ $\mathrm{QM}^{\mathbf{d}} = \{f : \mathbb{P}^1 \to \mathcal{X} | f(x) \in X\}$ for all but finitely many points

The idea is to pull back K-theory classes $\tau(\mathbf{s})$ to the moduli space and then 'integrate'

Need to compactify — relative quasimaps



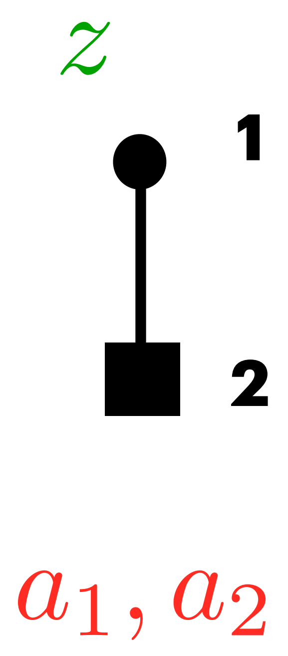
Quantum deformation parameters **z** — Kähler parameters of X (and twists of spin chain)

$$\widehat{\tau} = \tau + \sum_{d \in H_2(X;\mathbb{Z})} \tau_d(z, a, q) z^d$$

Vertex Functions

Vertex function — trivial quantum class $\tau = 1$

$X = T^*\mathbb{P}^1$



fixed points

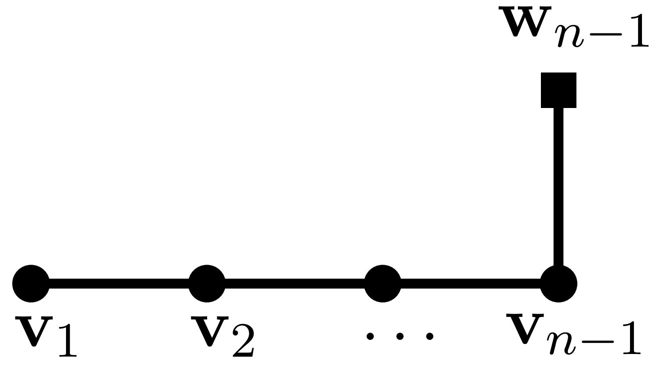
$\mathbf{p} = \{a_1\}$ and $\mathbf{p} = \{a_2\}$.

$$V^{(1)} = 1 + \frac{(\hbar - 1)(a_2\hbar - a_1)}{(q-1)(a_2 q - a_1)} z + \frac{(1-\hbar)\left(1 - \frac{a_2\hbar}{a_1}\right)(1 - q\hbar)\left(1 - \frac{a_2 q\hbar}{a_1}\right)}{(1-q)(1-q^2)\left(1 - \frac{a_2 q}{a_1}\right)\left(1 - \frac{a_2 q^2}{a_1}\right)} z^2 + \ldots$$

Truncation $a_2/a_1 = q^{-n}\hbar^{-1}$ leads to Macdonald Polynomials $V^{(1)} = P_n(z|q,\hbar)$

Theorem: Vertex functions are eigenfunctions of quantum tRS (Macdonald) difference operators for Nakajima quiver varieties of type A

[PK Zeitlin]

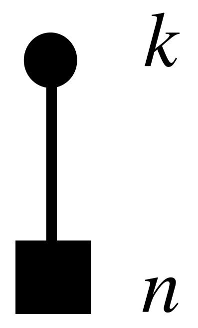


$$\left(\frac{z-\hbar}{z-1} p + \frac{1-\hbar z}{1-z} p^{-1}\right) V^{(1)} = (a_1 + a_2) V^{(1)}$$

Integrability: tRS class does not receive quantum corrections

Bethe Equations for $T^*Gr_{k,n}$

[Pushkar Smirnov Zeitlin]



The operator of quantum multiplication
by class $\hat{\tau}(z)\circledast$

$$\tau_{\boldsymbol{p}}(z) = \lim_{q\to 1} \frac{V^{(\tau)}_{\boldsymbol{p}}(z)}{V^{(1)}_{\boldsymbol{p}}(z)}$$

$$\prod_{i=0}^{\infty} \frac{1-qwq^i}{1-\hbar wq^i} \quad \to \quad \exp\left(-\frac{\mathrm{Li}_2(w)-\mathrm{Li}_2(\hbar w)}{1-q}\right)$$

Theorem: Quantum multiplication by $\hat{\tau}(z)$ is given by $\tau(s_1,\ldots,s_k)$ evaluated at solutions of Bethe equations

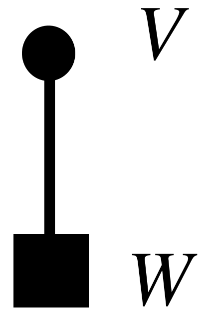
$$\prod_{l=1}^{n} \frac{s_i-\hbar a_l}{s_i-a_l} = z\hbar^{n/2} \prod_{j=1}^{k} \frac{\hbar s_i - s_j}{s_i-\hbar s_j}$$

Equivariant parameters a_i,
twist z, Planck constant $\hbar$

Baxter Q-operator $\quad Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z)\circledast \quad$ has eigenvalue $\quad Q(u) = \prod_{i=1}^{k}(u-s_i)$

The SL(2) QQ-System

Short exact sequence of bundles



$$0 \to V \to W \to V^{\vee} \to 0$$

Eigenvalues of Q-operators

$$Q(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V)(z) \circledast$$

$$\widetilde{Q}(u) = \sum_{i=1}^{k} (-1)^k u^{k-i} (\Lambda^i V^{\vee})(z) \circledast$$

Satisfy the QQ-relation

$$z\, \widetilde{Q}(\hbar u) Q(u) - \widetilde{Q}(u) Q(\hbar u) = \prod_{i=1}^{n} (u - a_i)$$

equivalent to the XXZ Bethe equations

The Ubiquitous QQ-System

Bethe Ansatz equations for XXX, XXZ models — eigenvalues of Baxter operators

[Mukhin, Varchenko] ....

Relations in equivariant cohomology/K-theory of Nakajima quiver varieties

[Nekrasov-Shatashvili] [Pushkar, Smirnov, Zeitlin] [PK, Pushkar, Smirnov, Zeitlin] ....

Relations in the extended Grothendieck ring for finite-dimensional representations of $U_\hbar(\hat{g})$

[Frenkel, Hernandez] ....

Spectral determinants in the QDE/IM Correspondence

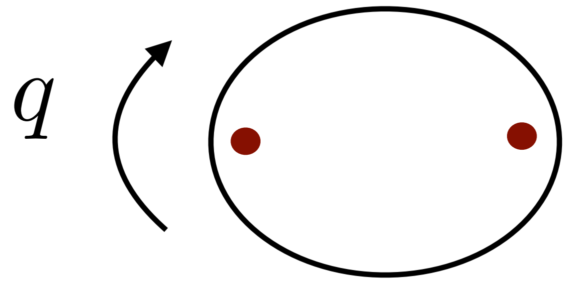
[Bazhanov, Lukyanov, Zamolodchikov] [Masoero, Raimondo, Valeri] ....

Opers

IV. (SL(2),q)-Opers

Riemann sphere with multiplication

$$\begin{aligned} M_q : \mathbb{P}^1 &\to \mathbb{P}^1 \\ u &\mapsto qu \end{aligned}$$



Section $s(u)$

Connection $A(u) : E \to E^q$

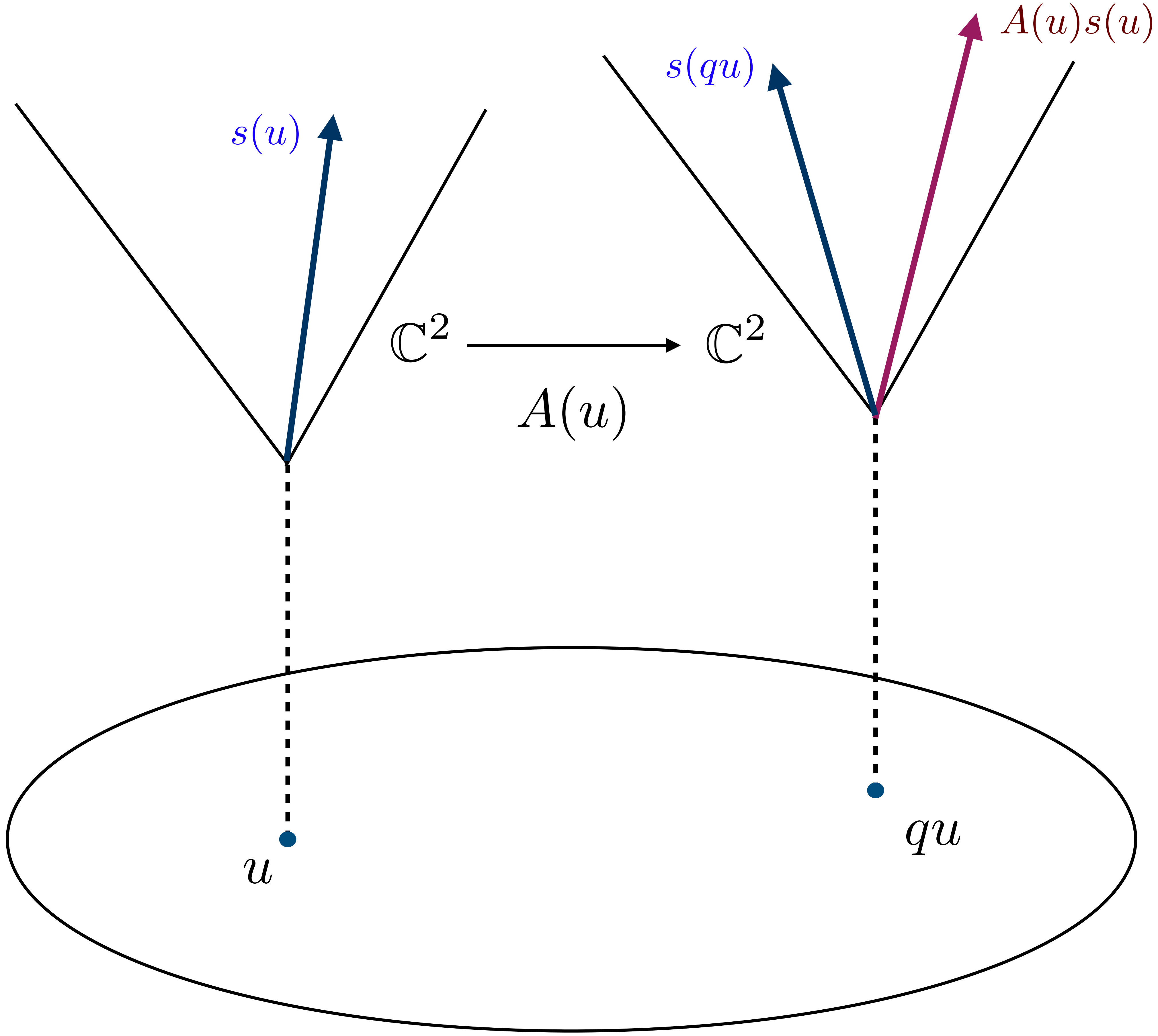
q-gauge transformation

$$A(u) \mapsto g(qu)A(u)g(u)^{-1}$$

(SL(2),q)-oper condition

$$s(qu) \wedge A(u)s(u) \neq 0$$

Vector bundle E of rank 2



(SL(2),q)-Opers

The $(SL(2), q)$-oper definition can be formulated as follows

Triple $(E, A, \mathscr{L})$

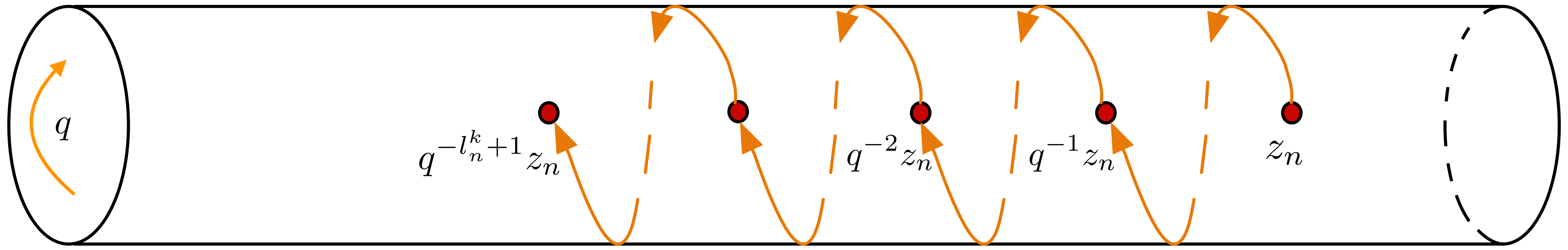
(E, A) is the $(SL(2), q)$ connection

$\mathscr{L} \subset E$ is a line subbundle

The induced map $\bar{A} : \mathscr{L} \to (E/\mathscr{L})^q$ is an isomorphism in a trivialization $\mathscr{L} =$ Span(s)

$$s(qu) \wedge A(u)s(u) \neq 0$$

Allow singularities $\quad s(qu) \wedge A(u)s(u) = \Lambda(u)$



$$\Lambda(u) = \prod_{l, j_l} (u - q^{j_l} a_l)$$

Add Twists $\quad Z = g(qu)A(u)g(u)^{-1}$

$$Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

q-Opers, QQ-System & Bethe Ansatz

Choose trivialization of $\mathcal{L}$ $\quad s(u) = \begin{pmatrix} Q_+(u) \\ Q_-(u) \end{pmatrix}$ $\qquad$ Twist element $\quad Z = \mathrm{diag}(\zeta, \zeta^{-1})$

q-Oper condition — SL(2) **QQ-system**

$$s(qu) \wedge A(u) s(u) = \Lambda(u) \longrightarrow \det \begin{pmatrix} Q_+(u) & \zeta Q_+(qu) \\ Q_-(u) & \zeta^{-1} Q_-(qu) \end{pmatrix} = \Lambda(u)$$

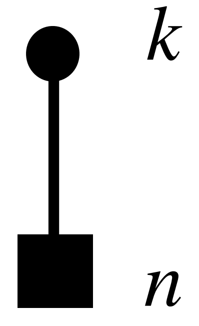
$$\zeta^{-1} Q_-(qu) Q_+(u) - \zeta\, Q_-(u) Q_+(qu) = \Lambda(u)$$

QQ-system to XXZ Bethe equations

$$Q_+(u) = \prod_{k=1}^{m} (u - s_k)$$

$$\prod_{l=1}^{n} \frac{s_i - q^{r_l} a_l}{s_i - a_l} = \zeta^2 q^k \prod_{j=1}^{k} \frac{q s_i - s_j}{s_i - q s_j}$$

$$i = 1, \dots, k$$



$\hbar = q$

q-Miura Transformation

Miura q-oper: $(E, A, \mathscr{L}, \hat{\mathscr{L}})$, where $(E, A, \mathscr{L})$ is a q-oper and $\hat{\mathscr{L}}$ is preserved by q-connection A

$$A(u) = \begin{pmatrix} g(u) & \Lambda(u) \\ 0 & g(u)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition $A(u) = v(qu) Z v(u)^{-1}$ $Z = \mathrm{diag}(\zeta, \zeta^{-1})$

$$g(u) = \zeta \frac{Q_+(qu)}{Q_+(u)}$$

$$v(u) = \begin{pmatrix} Q_+(u) & \zeta Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(u) Q_+(qu) \\ 0 & Q_+(u) \end{pmatrix} \in B_+(u)$$

The q-oper condition becomes the $SL(2)$ **QQ-system** $\zeta\, Q_-(u) Q_+(qu) - \zeta^{-1} Q_-(qu) Q_+(u) = \Lambda(u)$

Difference Equation $D_q(s) = As$

Scalar difference operator $\left(D_q^2 - T(qu) D_q - \frac{\Lambda(qu)}{\Lambda(u)} \right) s_1 = 0$

Back to tRS/Macdonald

Recover 2-particle tRS Hamiltonian from an $(SL(2),q)$-Oper — relation on q-Wronskian

$$\det\begin{pmatrix} Q_+(u) & \zeta Q_+(qu) \\ Q_-(u) & \zeta^{-1}Q_-(qu) \end{pmatrix} = \Lambda(u)$$

Let $Q_+(u) = u - p_+$ $Q_-(u) = u - p_-$

$$u^2 - u\left[\frac{\zeta - q\zeta^{-1}}{\zeta - \zeta^{-1}}p_+ + \frac{q\zeta - q\zeta^{-1}}{\zeta^{-1} - \zeta}p_-\right] + p_+p_- = (u-a_+)(u-a_-)$$

T_1 T_2

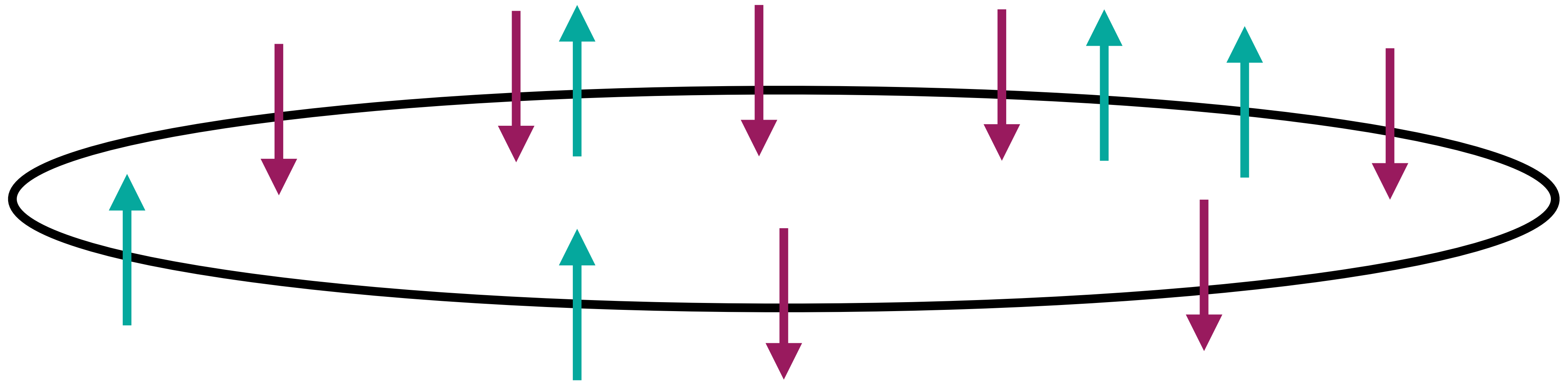
qOper condition yields tRS/Macdonald Hamiltonians!

$$\det(u - T) = (u-a_+)(u-a_-)$$

tRS Lax Matrix

Quantum

QQ-Systems



SU(n) XXZ spin chain

Planck's constant $\hbar$

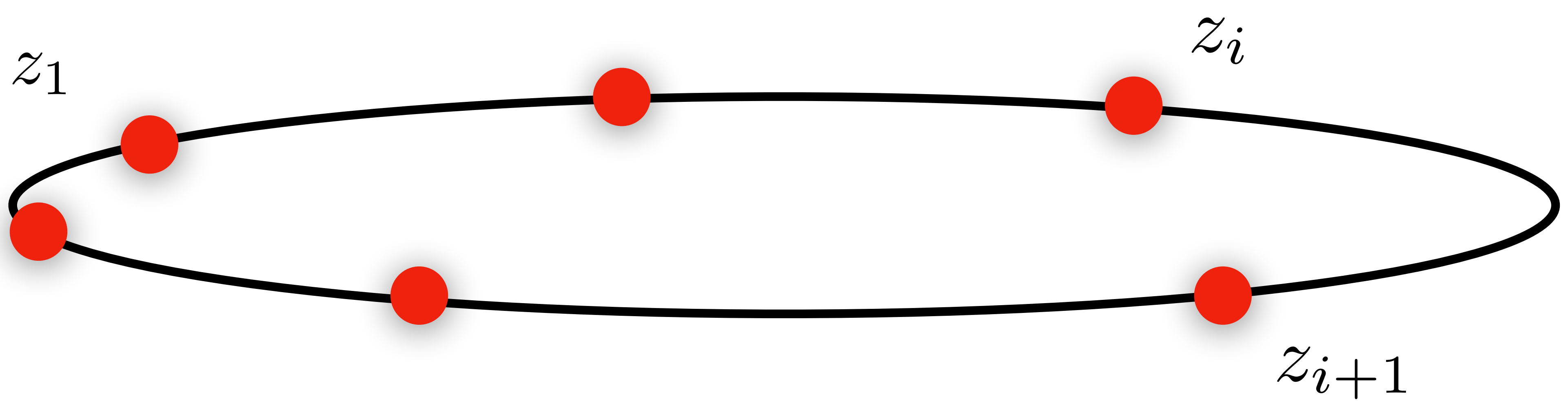
twist eigenvalues z_i

equivariant parameters (anisotropies) a_i

Bethe Ansatz Equations: $\exp \frac{\partial S}{\partial s_i} = 1$

Classical

q-Opers



n-particle trigonometric Ruijsenaars-Schneider model

Coupling constant $\hbar$

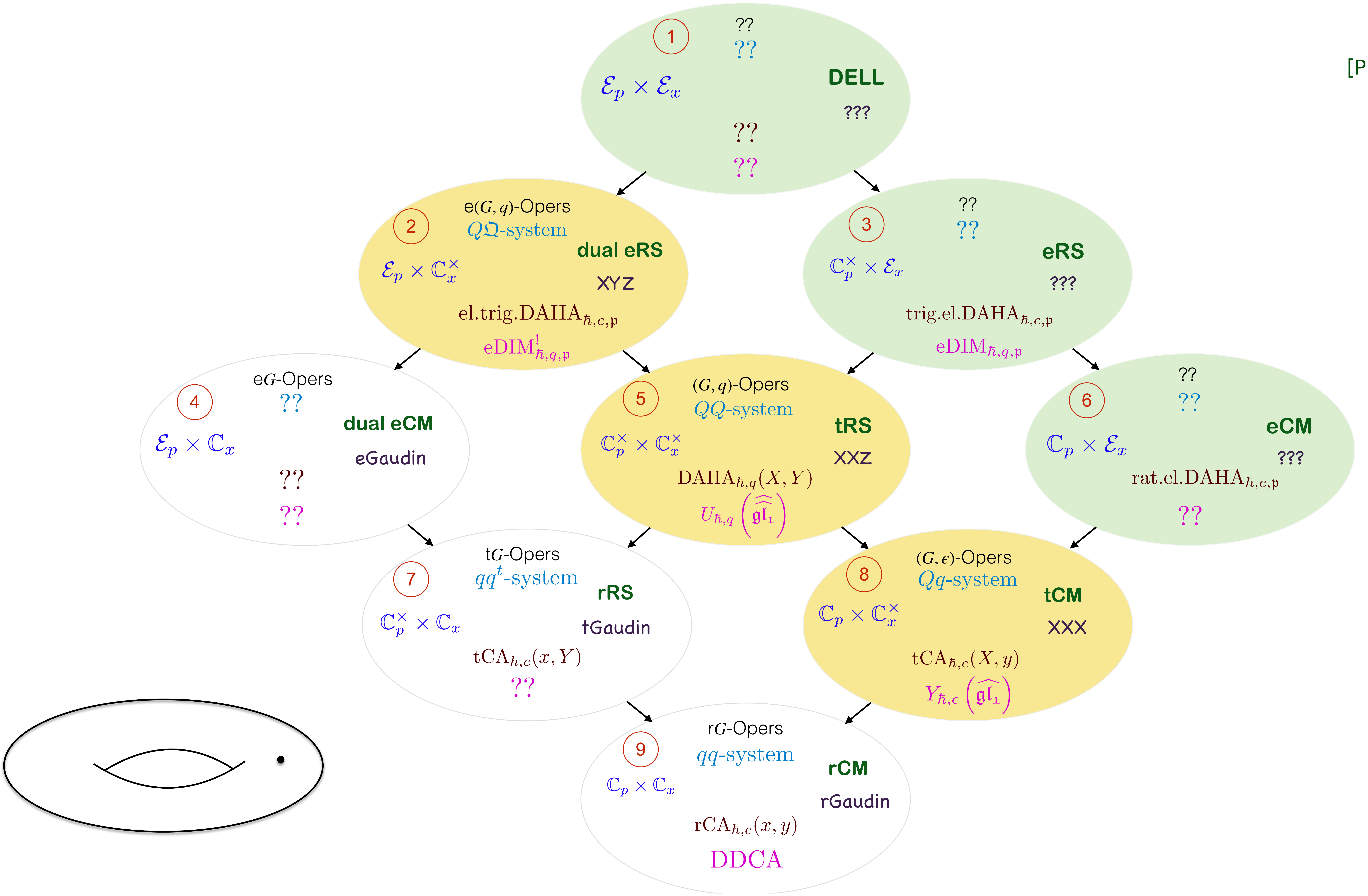
coordinates z_i

energy (eigenvalues of Hamiltonians) $e_i(a_i)$

Energy level equations

$$T_i(\mathbf{z}, \hbar) = e_i(\mathbf{a}), \qquad i = 1, \dots, n$$

[PK 2312.17500]



① ?? ?? $\mathcal{E}_p \times \mathcal{E}_x$ **DELL** ??? ?? ??

② e(G,q)-Opers $Q\mathfrak{Q}$-system $\mathcal{E}_p \times \mathbb{C}_x^\times$ **dual eRS** XYZ el.trig.DAHA$_{\hbar,c,\mathfrak{p}}$ eDIM$^!_{\hbar,q,\mathfrak{p}}$

③ ?? ?? $\mathbb{C}_p^\times \times \mathcal{E}_x$ **eRS** ??? trig.el.DAHA$_{\hbar,c,\mathfrak{p}}$ eDIM$_{\hbar,q,\mathfrak{p}}$

④ eG-Opers ?? $\mathcal{E}_p \times \mathbb{C}_x$ **dual eCM** eGaudin ?? ??

⑤ (G,q)-Opers QQ-system $\mathbb{C}_p^\times \times \mathbb{C}_x^\times$ **tRS** XXZ DAHA$_{\hbar,q}(X,Y)$ $U_{\hbar,q}\left(\widehat{\widehat{\mathfrak{gl}}}_1\right)$

⑥ ?? ?? $\mathbb{C}_p \times \mathcal{E}_x$ **eCM** ??? rat.el.DAHA$_{\hbar,c,\mathfrak{p}}$??

⑦ tG-Opers qq^t-system $\mathbb{C}_p^\times \times \mathbb{C}_x$ **rRS** tGaudin tCA$_{\hbar,c}(x,Y)$??

⑧ (G,ϵ)-Opers Qq-system $\mathbb{C}_p \times \mathbb{C}_x^\times$ **tCM** XXX tCA$_{\hbar,c}(X,y)$ $Y_{\hbar,\epsilon}\left(\widehat{\widehat{\mathfrak{gl}}}_1\right)$

⑨ rG-Opers qq-system $\mathbb{C}_p \times \mathbb{C}_x$ **rCM** rGaudin rCA$_{\hbar,c}(x,y)$ DDCA

(G,q)-Opers

G – simple, simply connected complex Lie group

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathcal{F}_{B_-}$ is a reduction of $\mathcal{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathcal{F}_G \longrightarrow \mathcal{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell for Coxeter element $c = \prod_i s_i$

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Locally $A(u) = n'(u) \prod_i (\phi_i(u)_i^{\check{\alpha}} s_i)\, n(u)$ $A(u) = \prod_i g_i(u)^{\check{\alpha}_i} e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}$

q-Opers and q-Langlands

[Frenkel, PK, Zeitlin, Sage, **JEMS 2023**]

Miura (G,q)-oper with singularities $A(u) = \prod_i g_i(u)^{\check{\alpha}_i} e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}$

Theorem: There is a 1-to-1 correspondence between the set of nondegenerate Z-twisted (G,q)-opers on $\mathbb{P}^1$ and the set of nondegenerate polynomial solutions of the QQ-system based on $\widehat{{}^L\mathfrak{g}}$

$$\widetilde{\xi}_i\, Q^i_-(u)Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u)Q^i_+(u) = \Lambda_i(u)\prod_{j>i}\Big[Q^j_+(\hbar u)\Big]^{-a_{ji}}\prod_{j<i}\Big[Q^j_+(u)\Big]^{-a_{ji}}, \qquad i=1,\ldots,r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i}\zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1}\prod_{j<i}\zeta_j^{-a_{ji}}$$

q-Langlands Correspondence

[Aganagic Frenkel Okounkov]

Two types of solutions of the qKZ equation:

Analytic in chamber of equivariant parameters $\{a_i\}$– conformal blocks of $U_\hbar(\widehat{g})$

Analytic in chamber of quantum parameters (twists) $\{\zeta_i\}$– conformal blocks for deformed W-algebra $W_{q,\hbar}({}^L\widehat{g})$

$$\begin{tikzcd}
U_\hbar(\widehat{g}) \arrow[rr, leftrightarrow, "\text{The q-Langlands correspondence}"] \arrow[d] & & W_{q,\hbar}({}^L\widehat{g}) \arrow[d, "q \to 1"] \\
\text{XXZ Bethe} & & (G,\hbar)\text{-oper}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Dmod}_\kappa(Bun_G) \arrow[r, leftrightarrow, "\text{Equivalence of categories}"] & \mathrm{Dmod}_{-\frac{1}{m\kappa}}(Bun_{{}^LG})
\end{tikzcd}$$

n-particle tCM from ϵ-opers

The QQ-system $\xi_{i+1}Q_i^+(z+\epsilon)Q_i^-(z)-\xi_iQ_i^+(z)Q_i^-(z+\epsilon)=(\xi_{i+1}-\xi_i)\Lambda_i(z)Q_{i-1}(z)Q_{i+1}(z)$

Theorem: Qs can be represented using *twisted* Wronskians

$$Q_j^+(z)=\frac{det\Big(M_{1,\dots,j}\Big)}{det\Big(V_{1,\dots,j}\Big)}\,,\qquad Q_j^-(z)=\frac{det\Big(M_{1,\dots,j-1,j+1}\Big)}{det\Big(V_{1,\dots,j-1,j+1}\Big)}$$

$$M_{i_1,\dots,i_j}(z)=\begin{bmatrix} s_{i_1}(z) & \xi_{i_1}s_{i_1}(z+\epsilon) & \cdots & \xi_{i_1}^{j-1}s_{i_1}(z+\epsilon(j-1)) \\ \vdots & \vdots & \ddots & \vdots \\ s_{i_j}(z) & \xi_{i_j}s_{i_j}(z+\epsilon) & \cdots & \xi_{i_j}^{j-1}s_{i_j}(z+\epsilon(j-1)) \end{bmatrix}$$

$$V_{i_1,\dots,i_j}=\begin{bmatrix} 1 & \xi_{i_1} & \cdots & \xi_{i_1}^{j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_{i_j} & \cdots & \xi_{i_j}^{j-1} \end{bmatrix}$$

The QQ-system is equivalent to the Desnanot-Jacobi-Lewis Carrol identity

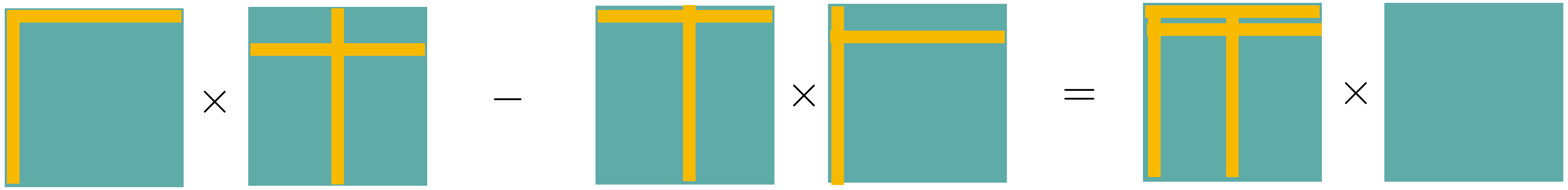
$$\det M_1^1 \det M_{k+1}^2 - \det M_{k+1}^1 \det M_1^2 = \det M_{1,k+1}^{1,2} \det M$$

Cluster Algebras

[PK, Zeitlin, **Crelle (2023)**]

The QQ-system $\xi_{i+1}\, Q^i_-(u)Q^i_+(u+\epsilon) - \xi_i Q^i_-(u+\epsilon)Q^i_+(u) = \Lambda_i(u)Q^{i+1}_+(u+\epsilon)Q^{i+1}_+(u)$

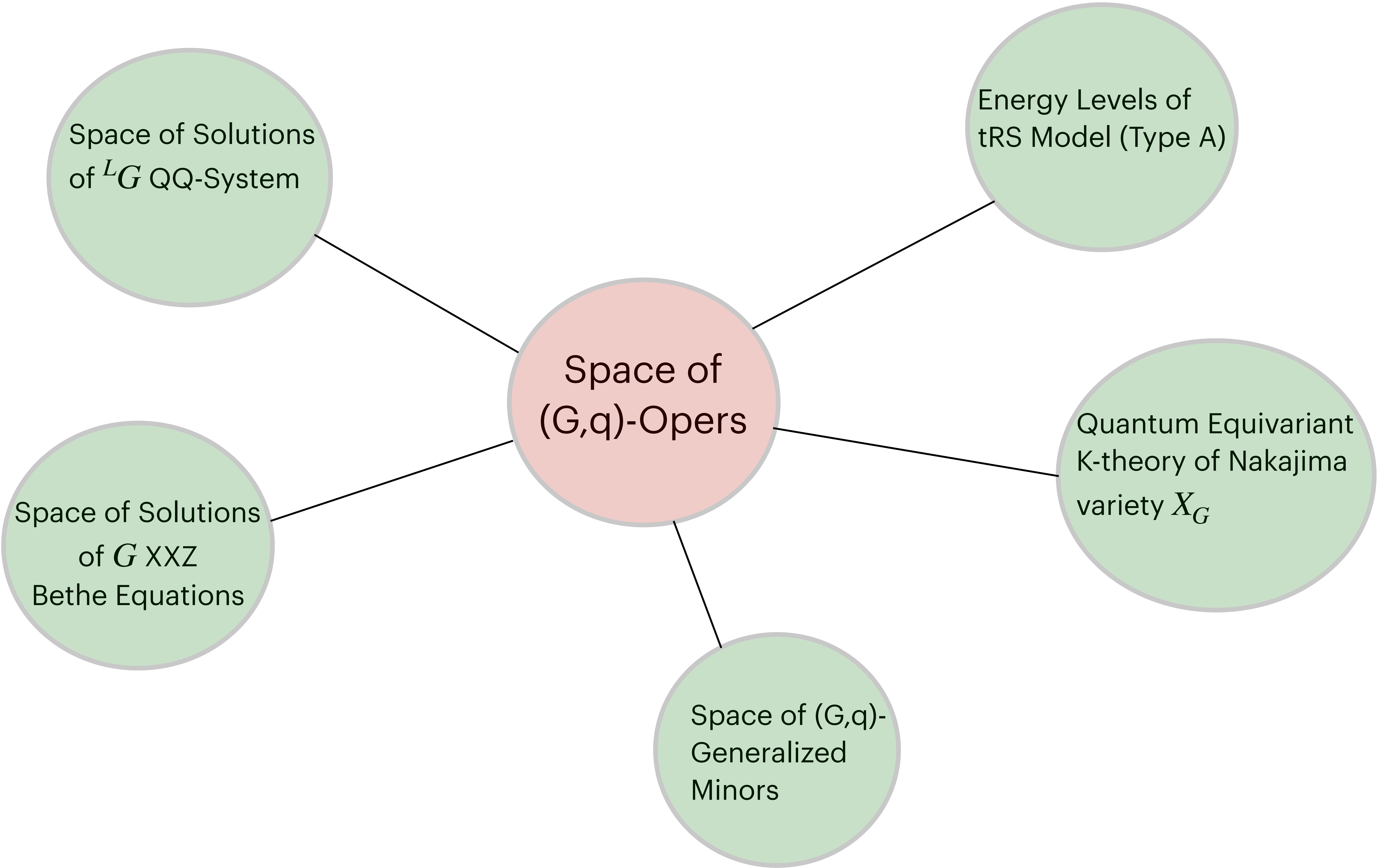
For $G = SL(n)$ obtain Lewis Carrol (Desnanot-Jacobi-Trudi) identity $M^1_1 M^2_i - M^1_i M^2_1 = M^{12}_{1i} M$



For general G obtain relation on generalized minors $\Delta^{\omega_i}(v(u)) = Q^i_+(u)$ [Fomin Zelevinsky]

$$\Delta_{u\cdot\omega_i,v\cdot\omega_i}\Delta_{uw_i\cdot\omega_i,vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i,v\cdot\omega_i}\Delta_{u\cdot\omega_i,vw_i\cdot\omega_i} = \prod_{j\neq i}\Delta^{-a_{ji}}_{u\cdot\omega_j,v\cdot\omega_j},$$

$u, v \in W_G$



Space of (G,q)-Opers

Space of Solutions of ${}^{L}G$ QQ-System

Energy Levels of tRS Model (Type A)

Quantum Equivariant K-theory of Nakajima variety X_G

Space of (G,q)-Generalized Minors

Space of Solutions of G XXZ Bethe Equations