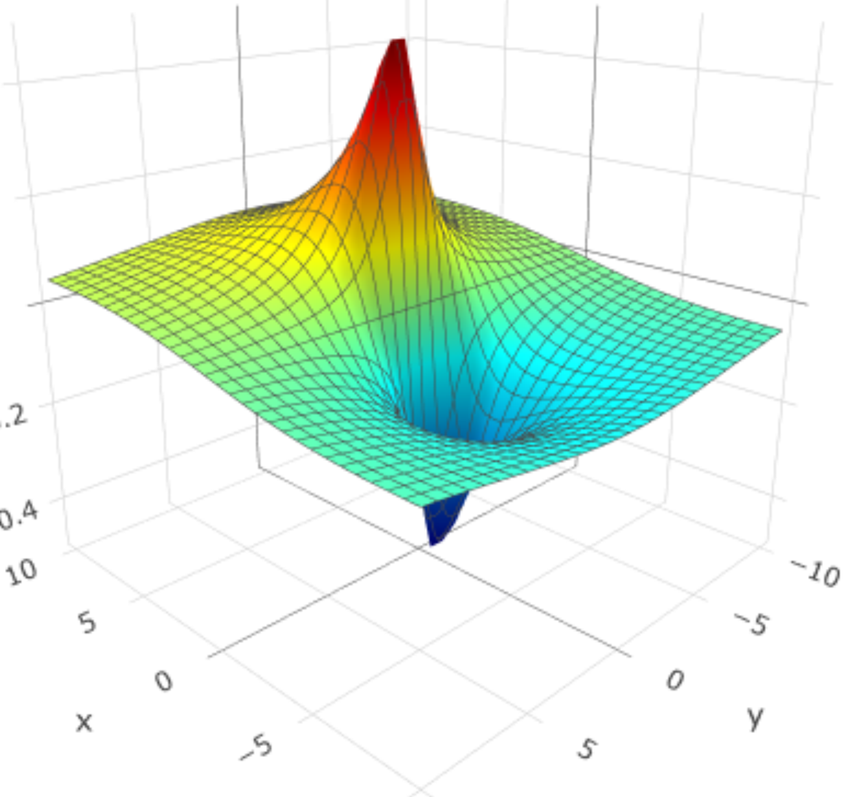


From Instantons to Vortices via Double Scaling

with A. Gorsky, O. Koroteeva, A. Vainshtein

[arXiv:1910.02606] [arXiv:2110.02157]

Peter Koroteev

Literature

[arXiv:2110.02157] Phys.Lett.B **826** (2022) 136919
Double Inozemtsev Limits of the Quantum DELL System
A. Gorsky, P. Koroteev, O. Koroteeva, S. Shakirov

[arXiv:1910.02606] J.Math.Phys. **61** (2020) 082302
On Dimensional Transmutation in 1+1D Quantum Hydrodynamics
A. Gorsky, P. Koroteev, O. Koroteeva, A. Vainshtein

[arXiv:1805.00986] Commun.Math.Phys. **381** (2021) 175
A-type Quiver Varieties and ADHM Moduli Spaces
P. Koroteev

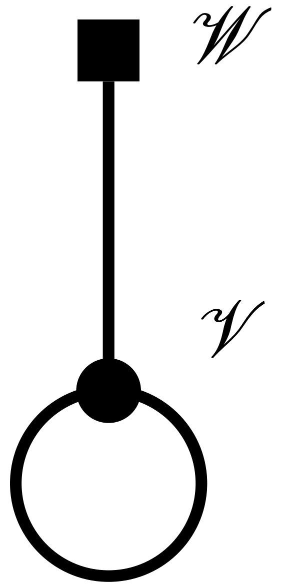
Instantons vs Vortices

$F = \star F$

Moduli space $\mathscr{I}_{k,n}$ of k instantons on $\mathbb{R}^4$

ADHM space

Hilb$^k[\mathbb{C} \times \mathbb{C}]$



dim$\mathscr{I}_{k,n} = 4kn$

HyperKähler

Moduli space of rank-N torsion-free sheaves on $\mathbb{P}^2$
With framing at infinity. $c_2(\mathscr{F}) = -k \cdot [\mathrm{pt}]$

Lagrangian embedding

[Hanany Tong]

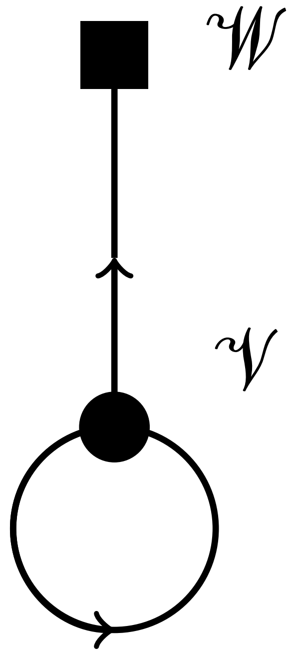
$\mathscr{V}_{k,n} \subset \mathscr{I}_{k,n}$

$F = \nabla \phi$

Moduli space $\mathscr{V}_{k,n}$ of k vortices on $\mathbb{R}^2$

1/2 ADHM space

Hilb$^k[\mathbb{C}]$

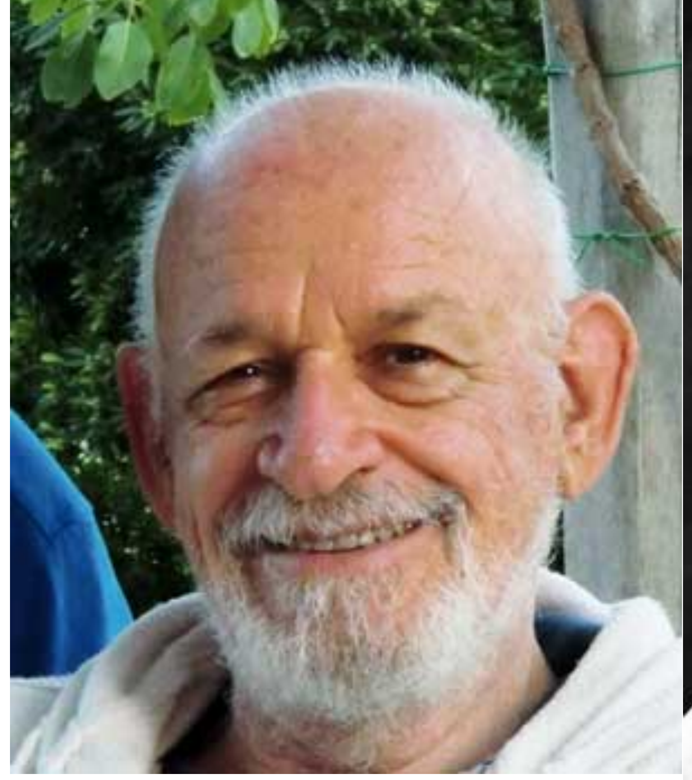


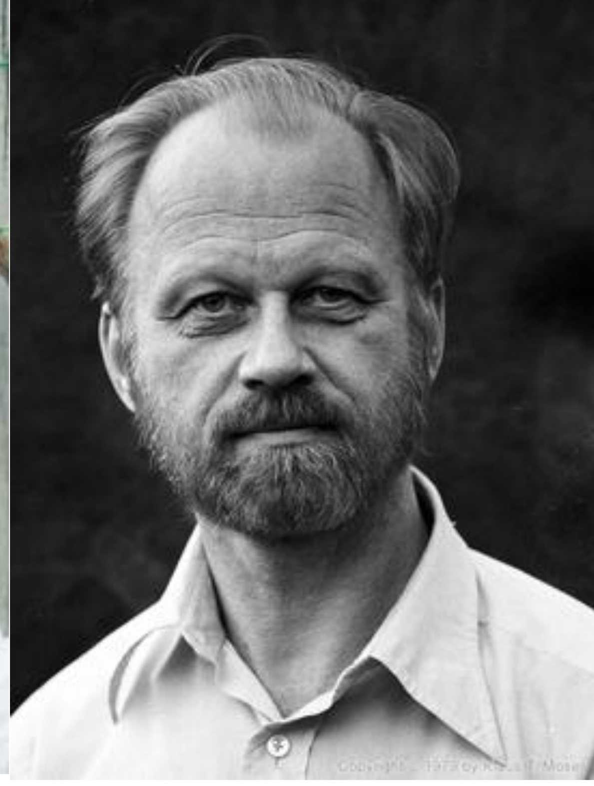
dim$\mathscr{V}_{k,n} = 2kn$

Kähler

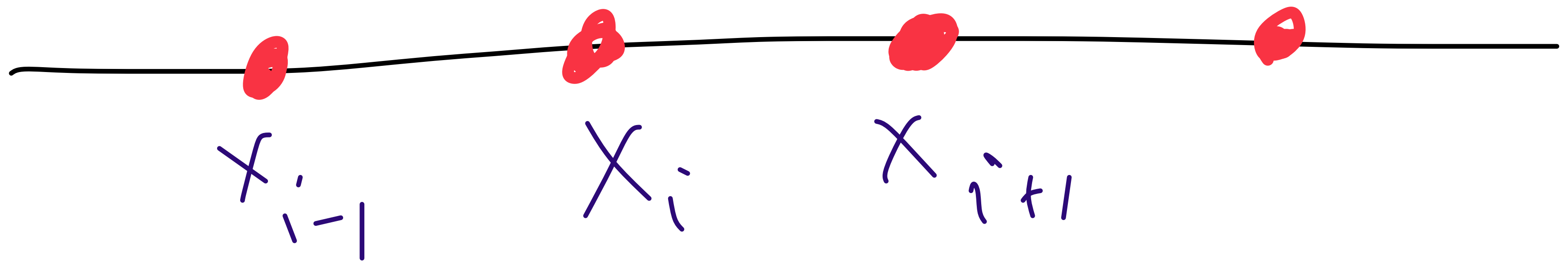
Ideals scheme-theoretically
supported on $\mathbb{C} \subset \mathbb{C}^2$

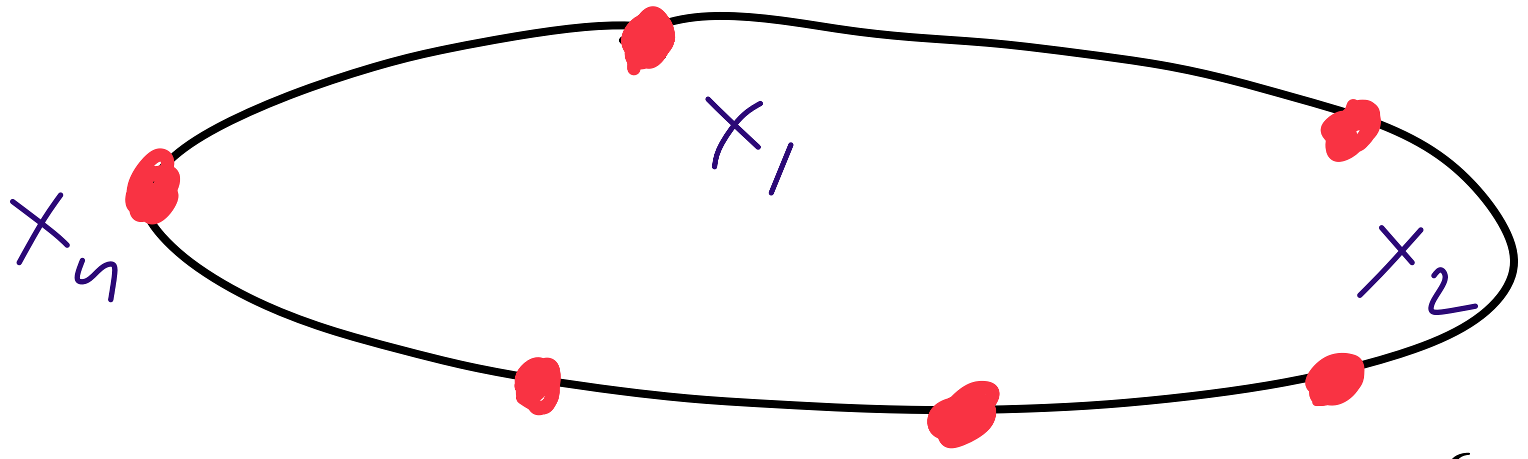
Integrable Many-Body Systems





Calogero in 1971 introduced a new integrable system. Moser in 1975 proved its integrability using Lax pair





$$H_{CM} = \sum_{i=1}^{n} \frac{p_i^2}{2m} + g^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}$$

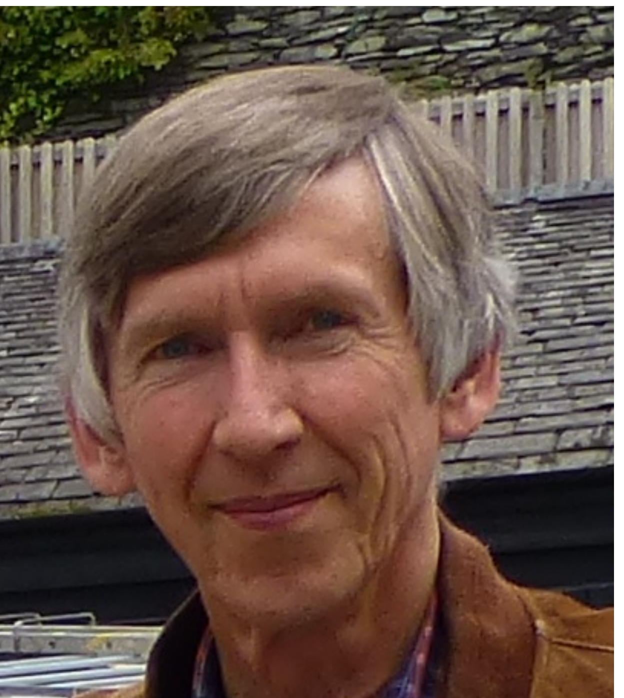
$$\{H_i, H_j\} = 0$$
$$i = 1, \ldots, n$$

$$V(z) \simeq \frac{1}{z^2} \qquad \wp(x_j - x_i)$$

The **Calogero-Moser (CM)** system has several generalizations

rCM —> tCM —> eCM

$$V(z) \simeq \frac{1}{\sinh z^2}$$



Another relativistic generalization called **Ruijsenaars-Schneider (RS)** family

rRS —> tRS —> eRS

Geometrically described by Hamiltonian reduction of T*GL(n)

$$H_{CM} = \lim_{c \to \infty} H_{RS} - nmc^2$$

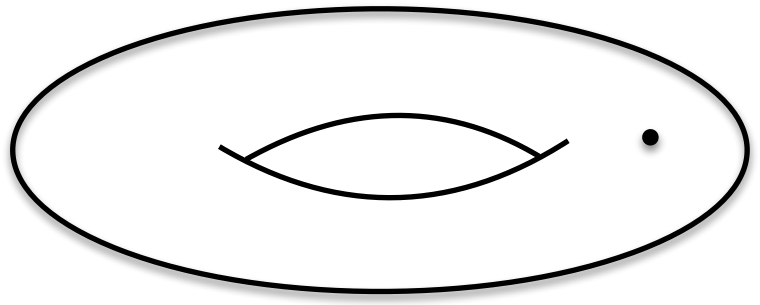
Algebraic Integrable Systems

- These are examples of complex algebraic integrable systems with n degrees of freedom whose phase space is a Lagrangian fibration of complex dimension $2n$ equipped with holomorphic symplectic 2-form $\Omega = \sum_{i=1}^{n} dp_i \wedge dx_i$ over a smooth base whose fibers are Abelian varieties (admit group law)
- There are n Poisson commuting Hamiltonians $H_1, \ldots, H_n$
- In *action-angle variables,* Hamiltonian evolution is linearized on the fibers which serve as level sets of the Hamiltonians

Hitchin Integrable System

[Donagi Witten]
[Gorsky Nekrasov]
[Nekrasov Pestun Shatashvili]

Seiberg-Witten solution of $\mathcal{N} = 2^*$ gauge theory leads to Hitchin integrable system $(\mathscr{E}, \varphi)$



Holomorphic G vector bundle over C_p with holomorphic section φ (Higgs field) of $K_{C_p} \otimes \mathrm{ad}(E) \otimes \mathcal{O}(p)$

$$A = \alpha_p \, d\vartheta + \cdots$$

$$\varphi = \frac{1}{2}(\beta_p + i\gamma_p)\frac{dz}{z} + \cdots$$

$$\mathcal{E} \to \mathcal{M}_{\text{vac}}(\mathbb{R}^3 \times S^1)$$ Hyperkähler (3d Coulomb branch)

$$\downarrow$$

$$\mathcal{B} = \mathcal{M}_{\text{vac}}(\mathbb{R}^4)$$ Special Kähler (4d Coulomb branch) $\{a_1, \ldots, a_n\}$

The n-dimensional Abelian variety is parameterized by the period matrix $\tau_{ij} = \dfrac{\partial \mathcal{F}}{\partial a_i \partial a_j}$

Liouville tori can be found inside the Jacobians of the algebraic curve $\det(z - \varphi) = 0$

Coordinates $x_i = \displaystyle\sum_{j=1}^{N-1} \int_{P_0}^{P_j} \omega_i$

The Abelian nature of Lagrangian fibers suggests that coordinates and momenta take values in $\mathbb{C}$, $\mathbb{C}^\times$, $\mathcal{E}$

Seiberg-Witten Solution

[Seiberg Witten 1994]

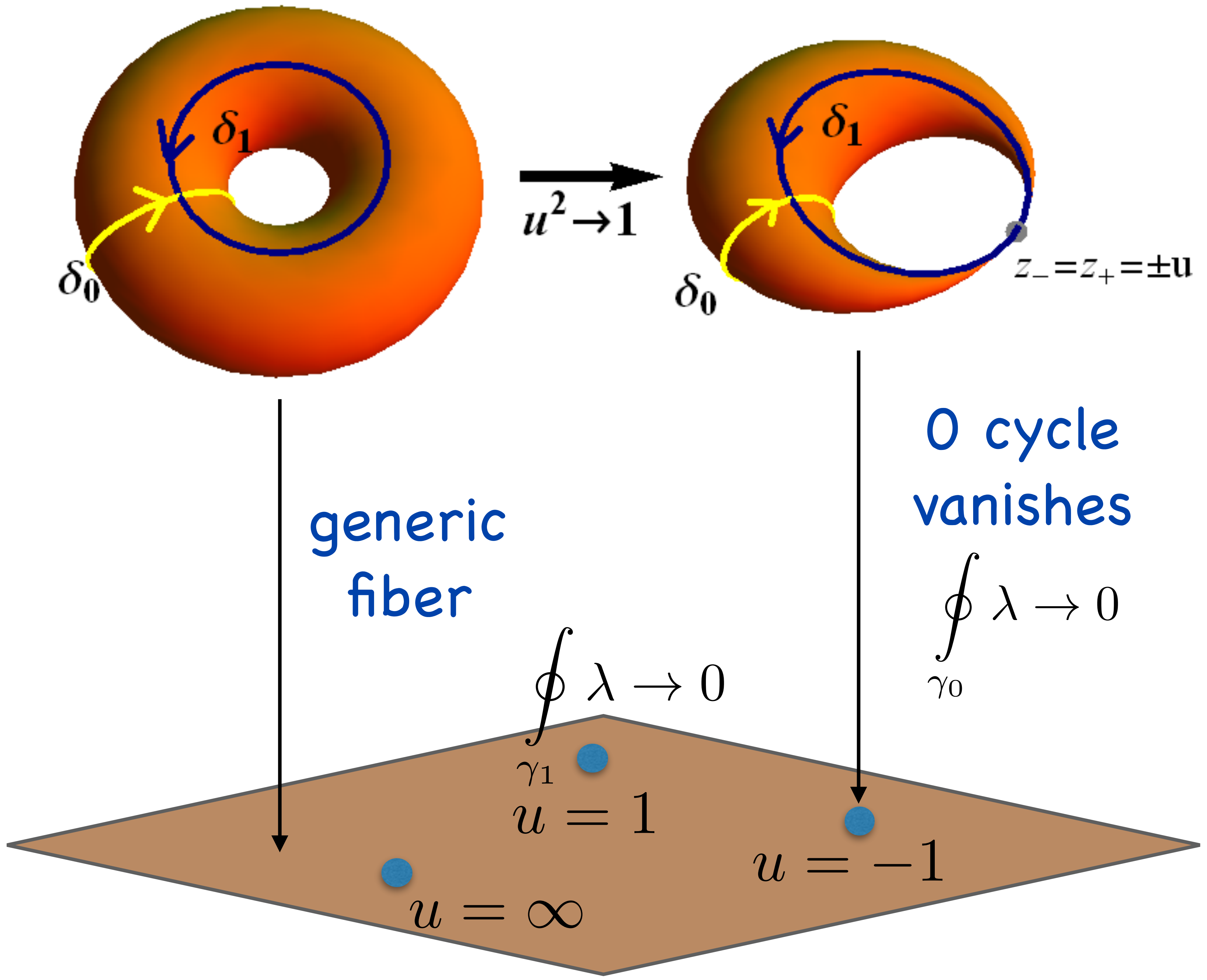
Provides mass spectrum of BPS particles of $\mathcal{N}$=2 gauge theory in 4d in the infrared

Potential	$V \sim \mathrm{Tr}\lvert[\phi,\phi]\rvert^2$
UV vacuum	$\langle\phi\rangle = a\sigma_3/2$
Coordinate on the moduli space	$u = \langle \mathrm{tr}\phi^2\rangle$
Using S-duality define dual magnetic variables	$(a, a_D) \quad a_D = \dfrac{\partial\mathcal{F}(a)}{\partial a}$
One-loop correction in the semi-classical region	$a_D \sim \dfrac{ia}{\pi}\left(1 + \ln\dfrac{a^2}{\Lambda^2}\right)$ $a \sim \sqrt{u}$
Monodromy around infinity	$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

In IR spectrum given by period integrals of the curve

$$2u = p^2 - \left(z + \frac{1}{z}\right) \qquad \lambda = p\frac{dz}{z}$$

Masses of BPS particles $\quad S_j(u) = \oint_{\gamma_j}\lambda$



Many-Body Systems of CM/RS type

p \ q	rational	trigonometric	elliptic
r	rational elliptic Calogero-Moser-Sutherland *2d N=(2,2) quiver theory*	trigonometric Calogero-Moser-Sutherland *2d N=(2,2)* quiver theory*	elliptic Calogero-Moser-Sutherland *4d N=2**
t	rational Ruijsenaars-Schneider (dual trig. CMS)	trigonometric Ruijsenaars-Schneider *3d N=2* quiver theory*	elliptic Ruijsenaars-Schneider *5d N=1**
e	dual elliptic CMS	dual elliptic RS *'dual' 5d N=1** *'dual' 3d defect*	Double elliptic model *6d (1,0)**

[Oblomkov]
[Etingof]

The Calogero-Moser Space

Let V be an N-dimensional vector space over $\mathbb{C}$. Let $\mathscr{M}'$ be the subset of $GL(V) \times GL(V) \times V \times V^*$ consisting of elements (M, T, u, v) such that

$$\hbar MT - TM = u \otimes v^T$$

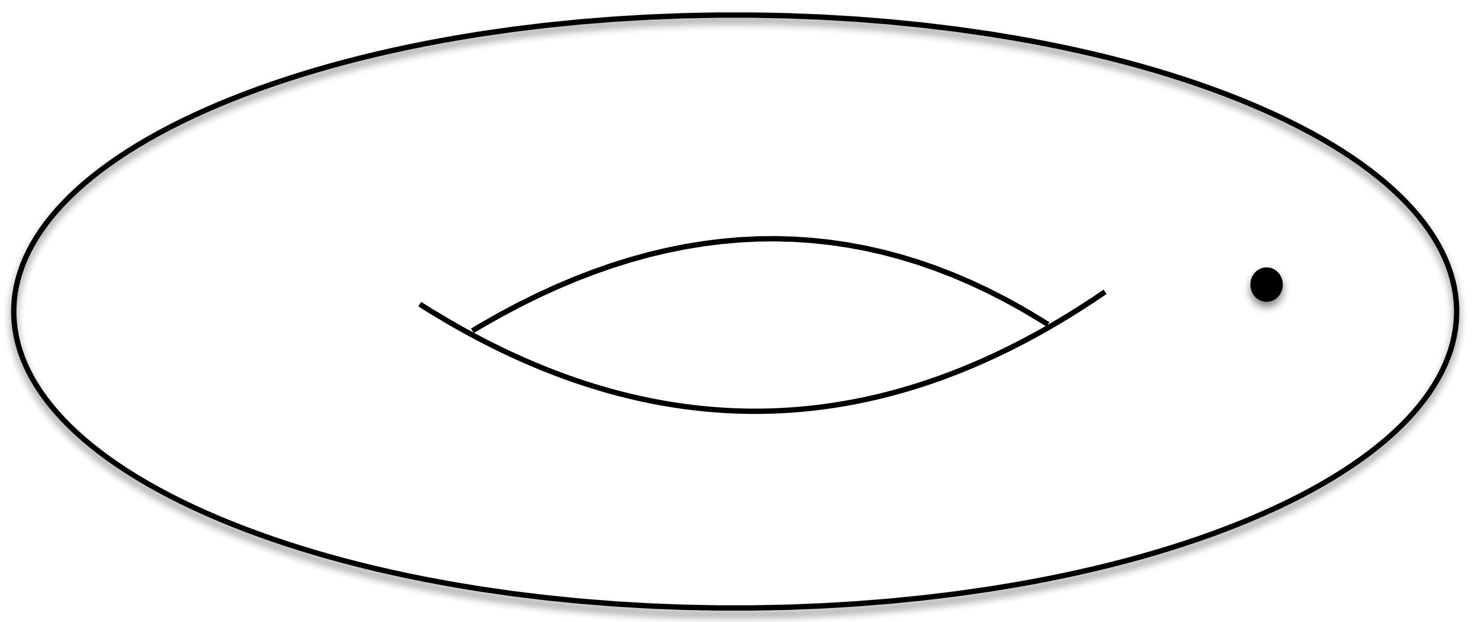
The group $GL(N; \mathbb{C}) = GL(V)$ acts on $\mathscr{M}'$ by conjugation

$$(M, T, u, v) \mapsto (gMg^{-1}, gTg^{-1}, gu, vg^{-1})$$

The quotient of $\mathscr{M}'$ by the action of $GL(V)$ is called **Calogero-Moser space $\mathscr{M}$**

Also can be understood as moduli space of flat connections on punctured torus

Integrable Hamiltonians are ~TrT^k



$$\mathcal{M}_n = \{A, B, C\}/GL(n; \mathbb{C})$$

$$ABA^{-1}B^{-1} = C$$

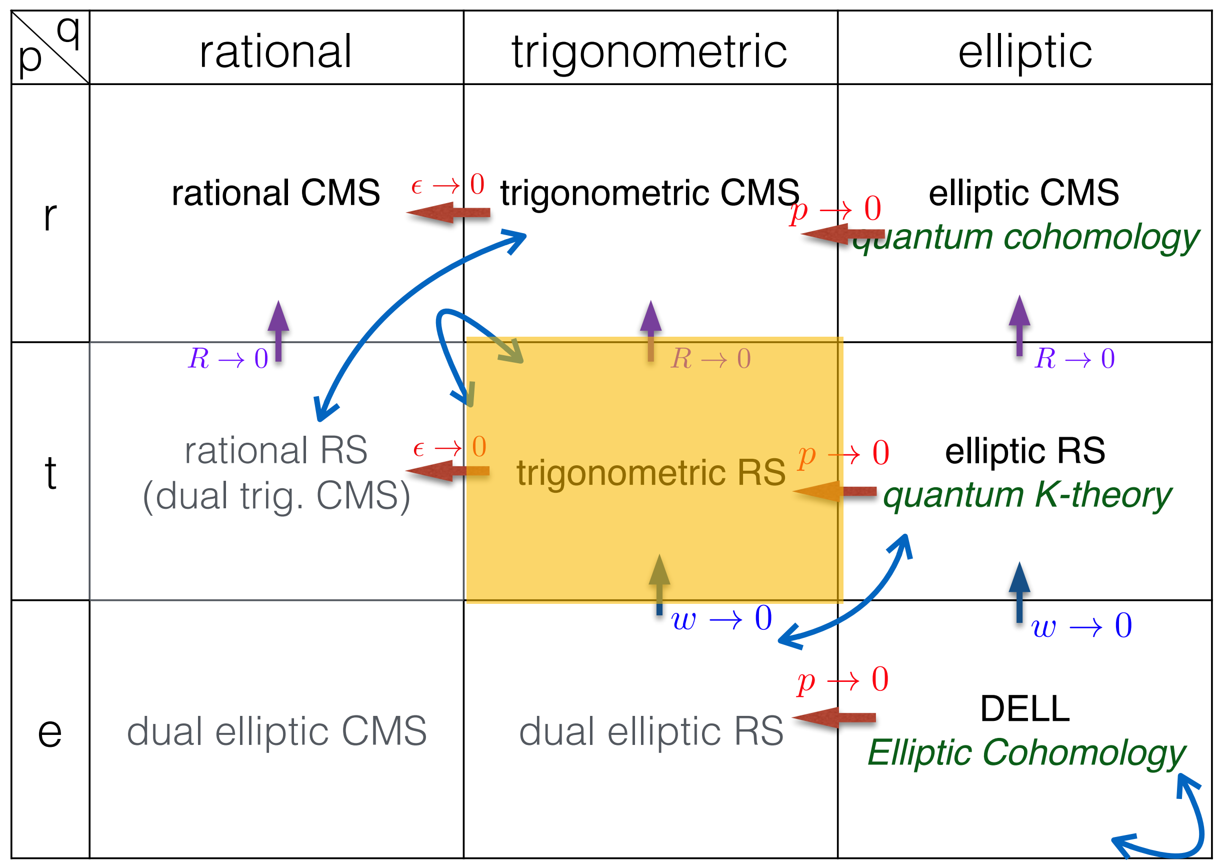
$$C = \text{diag}(\hbar, \ldots, \hbar, \hbar^{n-1})$$

Trigonometric RS Model

Flatness condition $\hbar MT - TM = u \otimes v^T$

In the basis where M is diagonal with eigenvalues $\xi_1, \dots, \xi_n$ matrix T

$$T_{ij} = \frac{u_i v_j}{\hbar \xi_i - \xi_j}$$



Define tRS momenta

$$p_i = -u_i v_i \frac{\prod_{k \neq i}(\xi_i - \xi_k)}{\prod_k (\xi_i - \hbar \xi_k)}$$

The tRS Lax matrix reads

$$T_{ij} = \frac{\prod_{k \neq j}(\xi_i - \hbar \xi_k)}{\prod_{k \neq i}(\xi_j - \xi_k)} p_i$$

Two particles

$$H_1 = \frac{\hbar \xi_1 - \xi_2}{\xi_1 - \xi_2} p_1 + \frac{\hbar \xi_2 - \xi_1}{\xi_2 - \xi_1} p_1$$

$$H_2 = p_1 p_2$$

Characteristic polynomial of T generates tRS Hamiltonians

Eigenproblem

$$\sum_{\substack{\mathcal{I} \subset 1,\dots,n \\ |\mathcal{I}|=k}} \prod_{\substack{i \in \mathcal{I} \\ j \notin \mathcal{I}}} \frac{\hbar \xi_i - \xi_j}{\xi_i - \xi_j} \prod_{m \in \mathcal{I}} p_m = e_k(a_i)$$

Quantum tRS Spectrum

Difference operators $p_i f(\xi_i) = f(q\xi_i)$ $p_i \xi_j = q^{\delta_{ij}} \xi_i p_j$

tRS eigenvalue problem

$$H_i(\xi, p) V(\xi, a) = e_i(a) V(\xi, a)$$

What is the geometric meaning of V?

Answering this question will help us to understand elliptic models

Before we answer this question notice the symmetry of the flatness condition

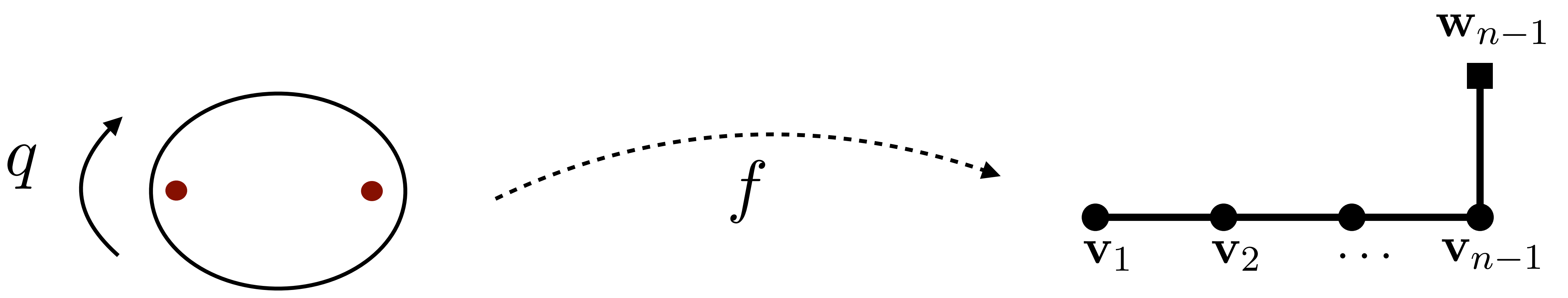
$$\hbar MT - TM = u \otimes v^T$$

$$\hbar \mapsto \hbar^{-1} \quad M \leftrightarrow T$$

3d mirror symmetry

Enumerative AG/Integrable Systems

Quantum equivariant K-theory of Nakajima quiver varieties



$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathbf{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}\left(\widehat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \tau|_{p_1}, \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{nonsing}\, p_2}\right) \boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}^\times_q}(X)_{loc}[[\boldsymbol{z}]]$$

Saddle point limit yields Bethe equations for XXZ

Quantum classes satisfy interesting difference equations in equivariant parameters and Kahler parameters

[Okounkov, Smirnov]

qKZ, Dynamical equation

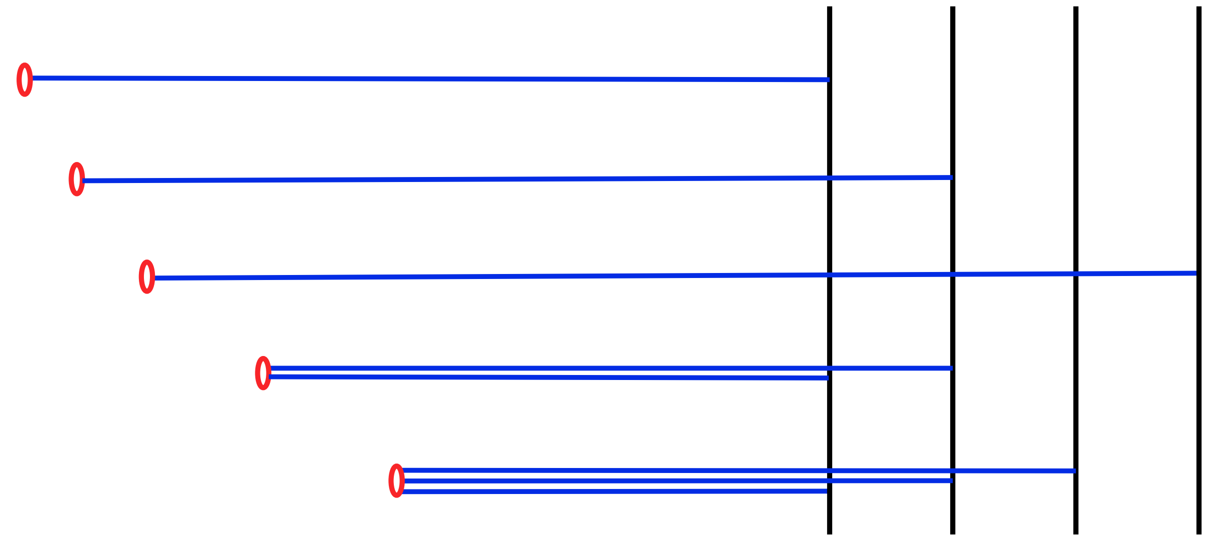
After symmetrization, they can be rewritten as eigenvalue equations for ***the tRS*** system

[PK, Zeitlin] [PK]

mirror frame

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\} \\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I} \\ j\notin\mathcal{I}}} \frac{t\, a_i - a_j}{a_i - a_j} \prod_{i\in\mathcal{I}} p_i$$

$$T_r(\boldsymbol{a})\mathrm{V}(\boldsymbol{a},\vec{\zeta}) = S_r(\vec{\zeta},t)\mathrm{V}(\boldsymbol{a},\vec{\zeta})$$



In terms of string/gauge theory tRS eigenproblem is Ward identity

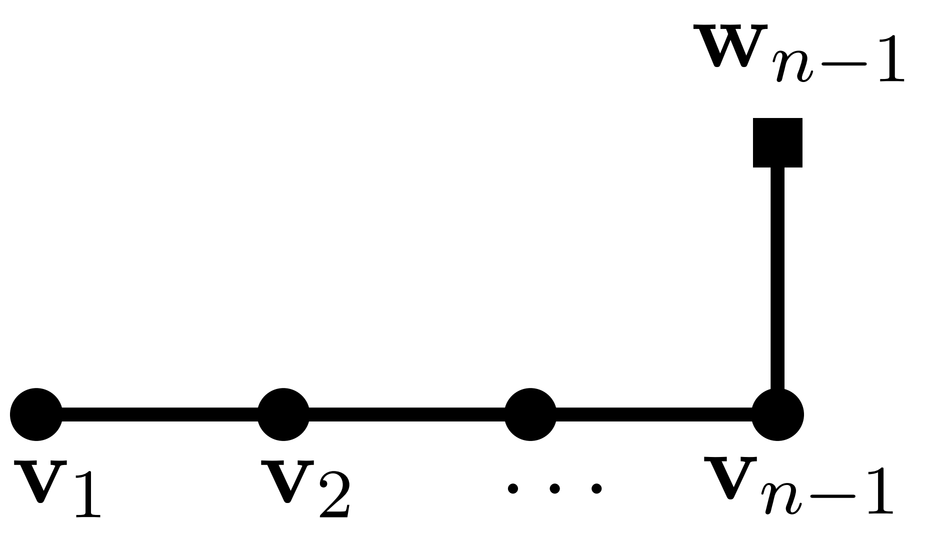
[Gaiotto, PK] [Bullimore, Kim, PK]

Vertex/Vortex Functions

After classifying fixed points of space of nonsingular quasimaps we can compute the vertex using the localization theorem

$$V^{(\tau)}_{\boldsymbol{p}}(z) = \sum_{d_{i,j}\in C} z^{\mathbf{d}} q^{N(\mathbf{d})/2}\, EHG \quad \tau(x_{i,j}q^{-d_{i,j}})$$

$$E = \prod_{i=1}^{n-1}\prod_{j,k=1}^{\mathbf{v}_i} \{x_{i,j}/x_{i,k}\}^{-1}_{d_{i,j}-d_{i,k}} \qquad x_{i,j}\in\{a_1,\dots a_{\mathbf{w}_n}\}$$



Vertex (trivial class)

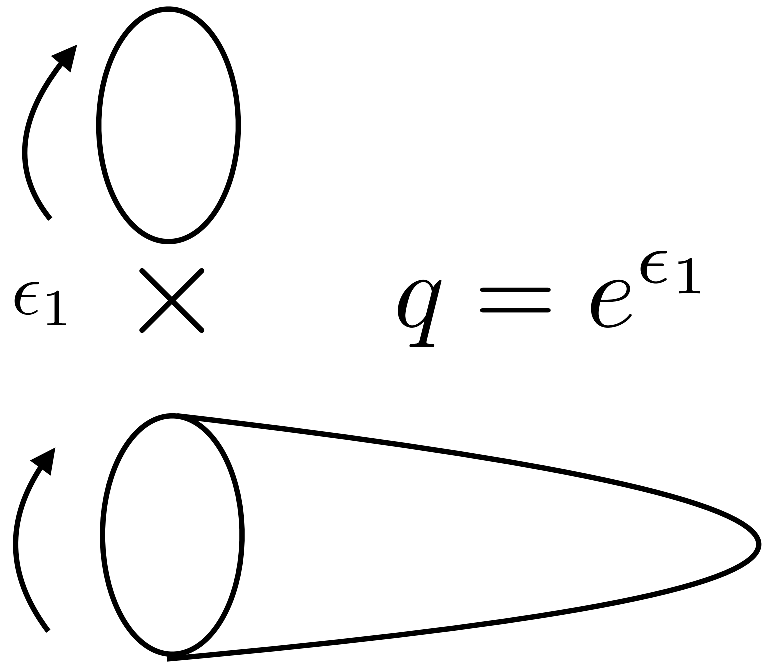
$$V = {}_2\phi_1\left(\hbar, \hbar\frac{a_1}{a_2}, q\frac{a_1}{a_2}; q; z\right) \qquad \mathbf{v}_1 = 1,\ \mathbf{w}_1 = 2$$

Vortex (defet partition function)

$\mathcal{N} = 2^*$ quiver gauge theory on $X_3 = \mathbb{C}_{\epsilon_1}\times S^1_\gamma$

Lagrangian depends on twisted masses a_1, a_2

FI parameter z and U(1) R-symmetry $\log\hbar$



Quantum tRS Spectrum

[PK]

Theorem 2.10. *Let $V^{(1)}_{\boldsymbol{p}}$ be the coefficient for the vertex function for X Define*

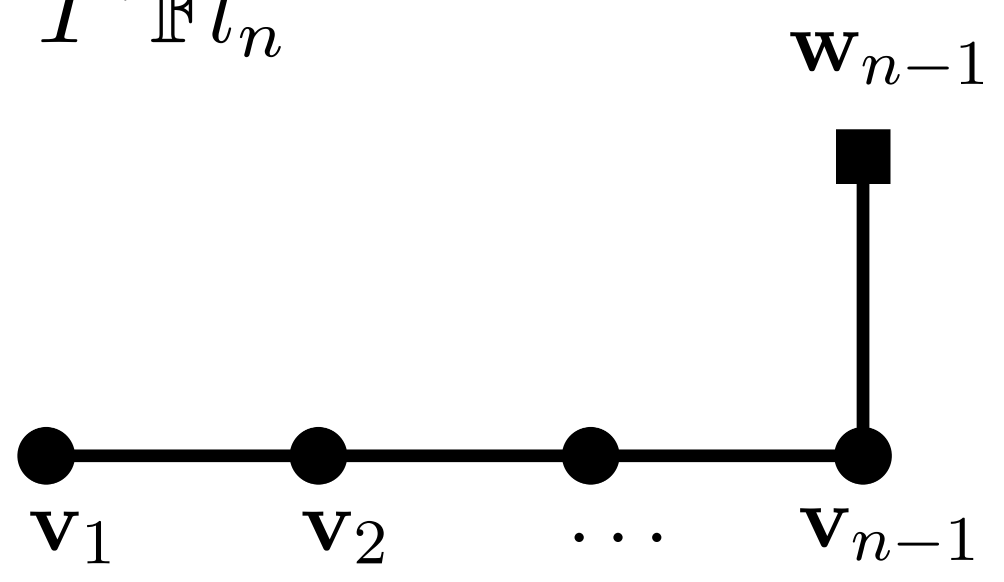
$$\mathsf{V}^{(1)}_{\boldsymbol{p}} = \prod_{i=1}^{n} \frac{\theta(\hbar^{i-n}\zeta_i, q)}{\theta(a_i\zeta_i, q)} \cdot V^{(1)}_{\boldsymbol{p}}\,, \tag{2.9}$$

where $\theta(x,q) = (x,q)_\infty(qx^{-1},q)_\infty$ is basic theta-function. Then $\mathsf{V}_{\boldsymbol{p}}$ are eigenfunctions for tRS difference operators (2.8) for all fixed points $\boldsymbol{p}$

$$T_r(\boldsymbol{\zeta})\mathsf{V}^{(1)}_{\boldsymbol{p}} = e_r(\mathbf{a})\mathsf{V}^{(1)}_{\boldsymbol{p}}\,, \qquad r = 1,\dots,n\,, \tag{2.10}$$

where e_r is elementary symmetric polynomial of degree r of $a_1,\dots,a_n$.

$T^*\mathbb{F}l_n$



tRS momenta $$p_i = \frac{s_{i+1,1}\cdot\dots\cdot s_{i+1,i+1}}{s_{i,1}\cdot\dots\cdot s_{i,i}}$$

Quantum multiplication by class

$$\widehat{\Lambda^i\mathscr{V}_i} \otimes \widehat{\Lambda^{i+1}\mathscr{V}^*_{i+1}}$$

Chern roots $s_{i,a}$ satisfy XXZ Bethe Ansatz equations

Macdonald Polynomials

Proposition 2.11. *Consider coefficient functions for K-theory of QM to X_n (2.9) for all fixed points of the maximal torus. Let λ be a partition of k elements of length n and $\lambda_1 \geq \cdots \geq \lambda_n$. Let*

$$\frac{a_{i+1}}{a_i} = q^{\ell_i}\hbar\,, \quad \ell_i = \lambda_{i+1} - \lambda_i\,, \quad i = 1, \ldots, n-1\,. \tag{2.18}$$

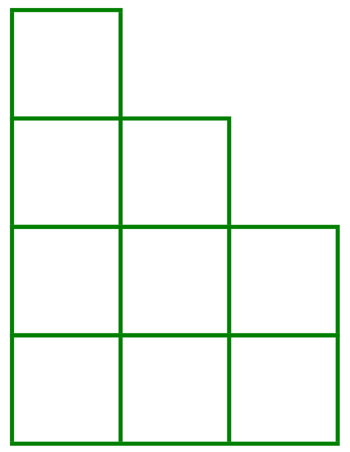
Then there exists a fixed point $\boldsymbol{q}$ for which

$$\mathsf{V}_{\boldsymbol{q}} = P_\lambda(\boldsymbol{\zeta}; q, \hbar)\,. \tag{2.19}$$

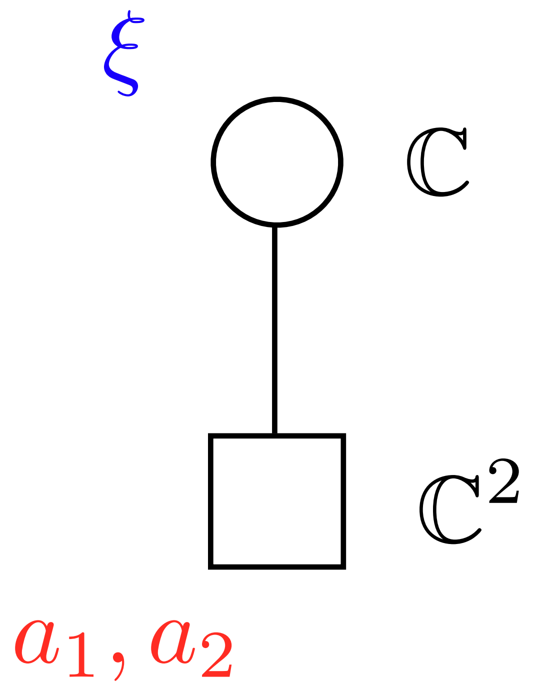
$$a_i = a q^{\lambda_i}\hbar^{i-n}\,, \quad i = 1, \ldots, n$$

$$T_1(\boldsymbol{\zeta})P_\lambda(\boldsymbol{\zeta}; q, \hbar) = \left(\sum_{i=1}^{n} q^{\lambda_i}\hbar^{i-n}\right) P_\lambda(\boldsymbol{\zeta}; q, \hbar)$$

$\lambda =$



Example: $T^*\mathbb{P}^1$



$$H_1 = \frac{q\xi_1 - \xi_2}{\xi_1 - \xi_2} p_1 + \frac{q\xi_2 - \xi_1}{\xi_2 - \xi_1} p_2$$

$$H_2 = p_1 p_2$$

$$V_{\mathbf{p}}^{(1)} = \sum_{d>0} \left(\frac{\zeta_1}{\zeta_2}\right)^d \prod_{i=1}^{2} \frac{\left(\frac{q}{\hbar}\frac{a_{\mathbf{p}}}{a_i}; q\right)_d}{\left(\frac{a_{\mathbf{p}}}{a_i}; q\right)_d} = {}_2\phi_1\left(\hbar, \hbar\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}, q\frac{a_{\mathbf{p}}}{a_{\bar{\mathbf{p}}}}; q; \frac{q}{\hbar}\frac{\zeta_1}{\zeta_2}\right)$$

Macdonald polynomials

$$a_1 = a q^{\lambda_1} \hbar^{-1}, a_2 = a q^{\lambda_2}$$

$$\mathrm{V}_{\square} = \zeta_1 + \zeta_2,$$

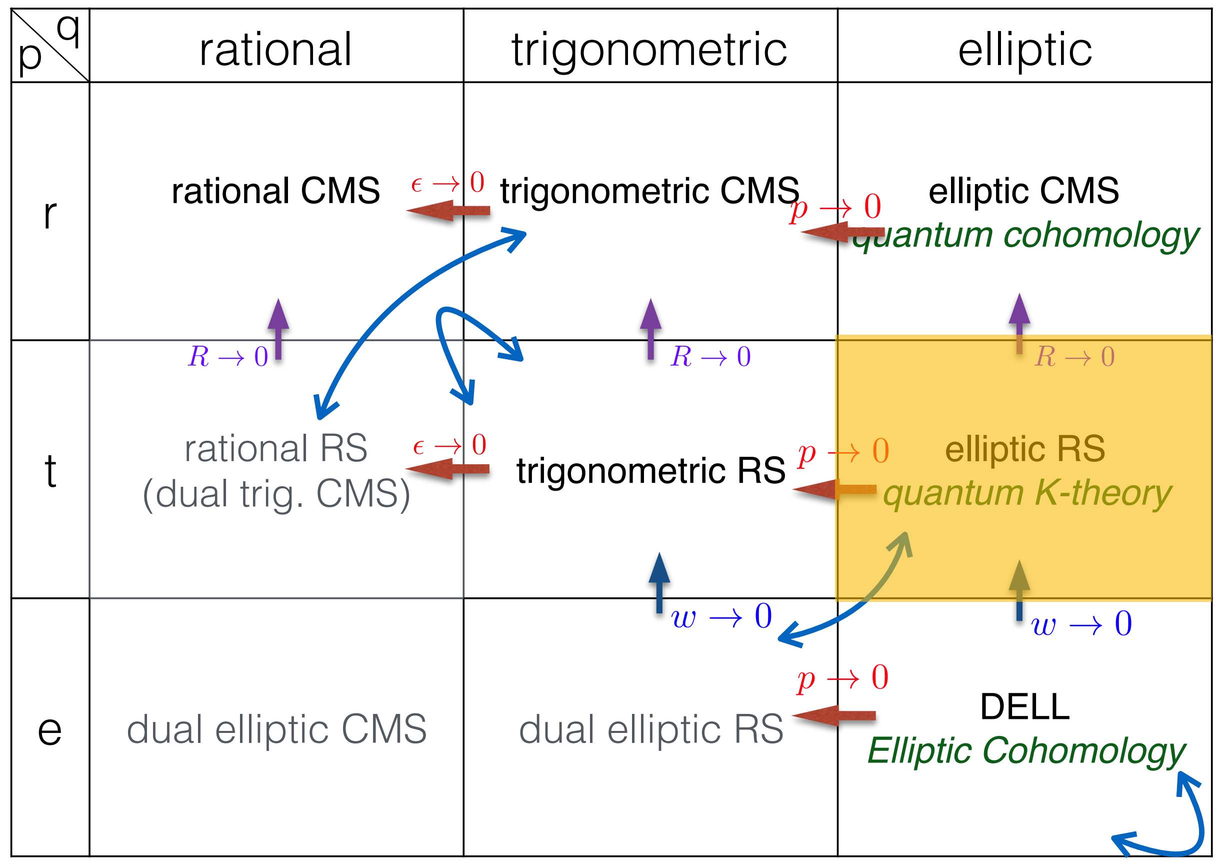
$$\mathrm{V}_{\square\square} = \zeta_1^2 + \zeta_2^2 + \frac{(q+1)(\hbar-1)}{q\hbar - 1}\zeta_1\zeta_2,$$

$$\mathrm{V}_{\square\square\square} = \zeta_1^3 + \zeta_2^3 + \frac{\left(q^2+q+1\right)(\hbar-1)}{q^2\hbar - 1}\zeta_2\zeta_1^2 + \frac{\left(q^2+q+1\right)(\hbar-1)}{q^2\hbar - 1}\zeta_2^2\zeta_1$$

Elliptic RS Model

$$E_r(\zeta) = \sum_{\substack{\mathcal{I}\subset\{1,\ldots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})}{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})} \prod_{i\in\mathcal{I}} p_k$$

$$E_r(\zeta)\mathcal{Z} = \mathscr{E}_r \mathcal{Z} \qquad (1)$$



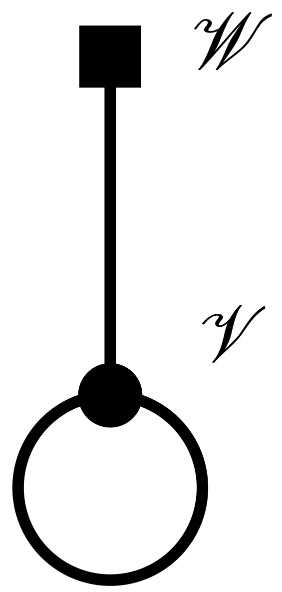
Conjecture 5.1. *The solution of* (1) *is given by the K-theoretic holomorphic equivariant Euler characteristic of the affine Laumon space*

$$\mathcal{Z} = \sum_{\boldsymbol{d}} \vec{\mathfrak{q}}^{\boldsymbol{d}} \int_{\mathcal{L}_{\boldsymbol{d}}} 1\,,$$

where $\vec{\mathfrak{q}} = (\mathfrak{q}_1,\ldots,\mathfrak{q}_n)$ *is a string of* $\mathbb{C}^\times$*-valued coordinates on the maximal torus of* $\mathcal{L}^{aff}_{\boldsymbol{d}}$. *The eigenvalues* $\mathscr{E}_r$ *are equivariant Chern characters of bundles* $\Lambda^r\mathscr{W}$, *where* $\mathscr{W}$ *is the constant bundle of the corresponding ADHM space. In other words they have the following form*

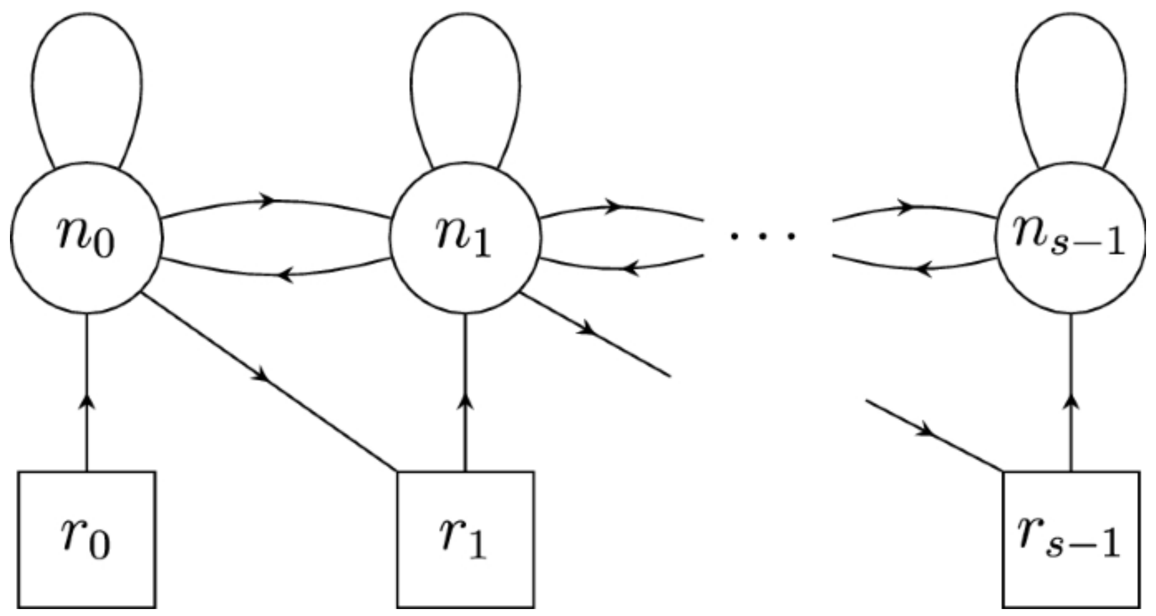
$$\mathscr{E}_r = e_r + \sum_{l=1}^{\infty} \mathfrak{p}^l \mathcal{E}_r^{(l)}\,,$$

where e_r *are symmetric functions of the equivariant parameters* $a_1,\ldots,a_N$.



eRS Spectrum

Euler characteristic of ***affine Laumon space*** (representation space of a chain-saw quiver)



$$\mathcal{Z} = \sum_{\vec{\mu}} \prod_{l=1}^{n} \mathfrak{q}_l^{k_l(\vec{\mu})} \, z_{\vec{\mu}}(\vec{a}, \hbar, q) \qquad\qquad \mathfrak{p} = \mathfrak{q}_1 \cdot \dots \cdot \mathfrak{q}_n$$

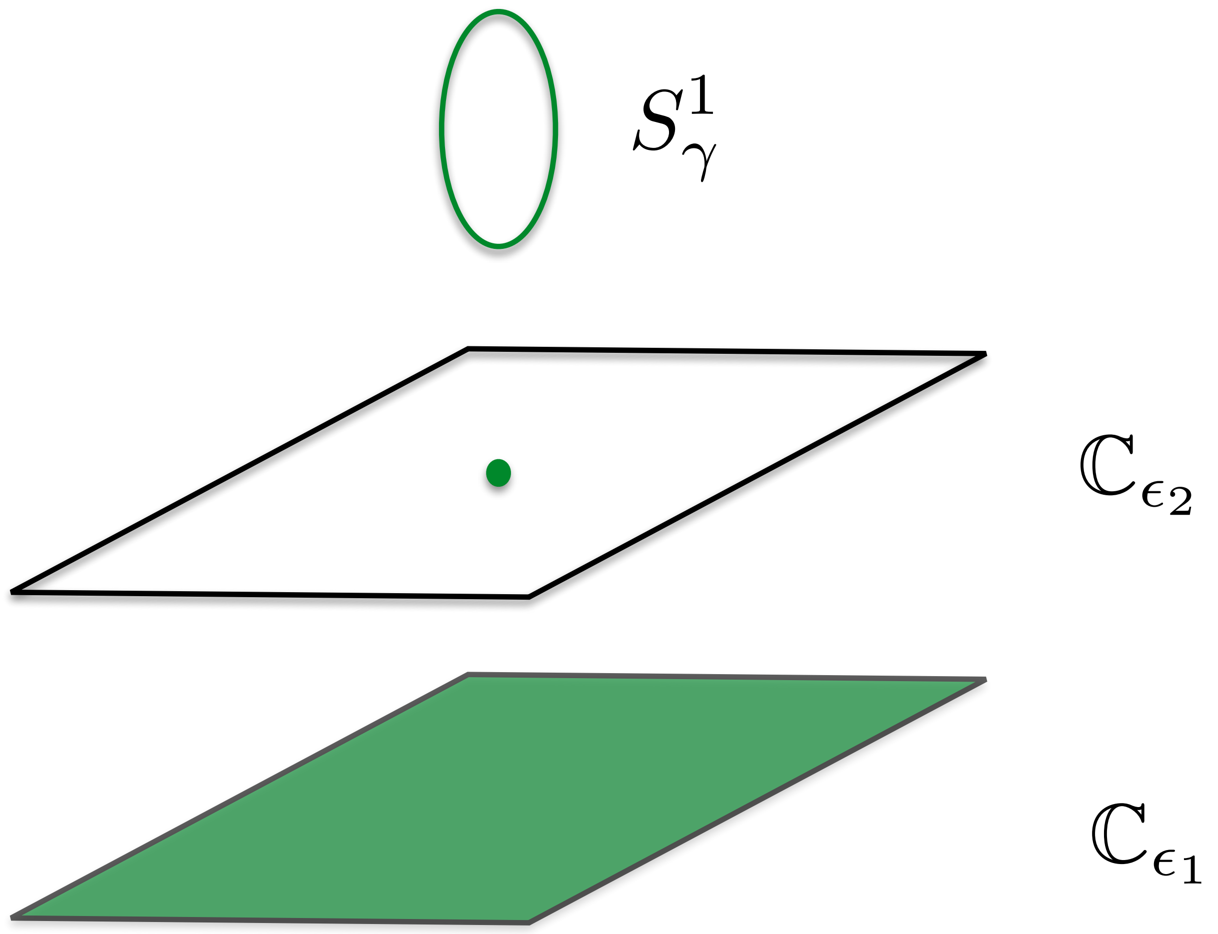
Theorem 1.1. *Let* $\boldsymbol{x} = (x_1, \dots x_N)$ *be the position vector of the eRS model and* $\mathcal{Z}^{RS}(\boldsymbol{a}, \boldsymbol{x}) = \lim_{w \to 0} \mathcal{Z}^{6d/4d}_{inst}(w, p, \boldsymbol{x})$ *is its wavefunction. Then the following equality holds*

$$\mathcal{H}_k \mathcal{Z}^{RS}(\boldsymbol{a}, \boldsymbol{x}) = \lambda_k(\boldsymbol{a}) \mathcal{Z}^{RS}(\boldsymbol{a}, \boldsymbol{x}) \,, \qquad k = 1, \dots, N-1 \,. \tag{1.6}$$

where the eigenvalues read

$$\lambda_k(\boldsymbol{a}) = \prod_{n=0}^{k-1} \frac{\theta(t^{N-n})}{\theta(t^{n+1})} \cdot \frac{\mathcal{Z}^{RS}(\boldsymbol{a}, t^{\vec{\rho}} q^{\vec{\omega_k}})}{\mathcal{Z}^{RS}(\boldsymbol{a}, t^{\vec{\rho}})} \,, \qquad k = 1, \dots, N-1 \tag{1.7}$$

where $\vec{\omega_k}$ *is the* k*-th fundamental weight of representation of* $SU(N)$ *and* $\vec{\rho} = \big((N-1)/2, (N-3)/2, \dots, (3-N)/2, (1-N)/2\big)$ *is the* $SU(N)$ *Weyl vector.*



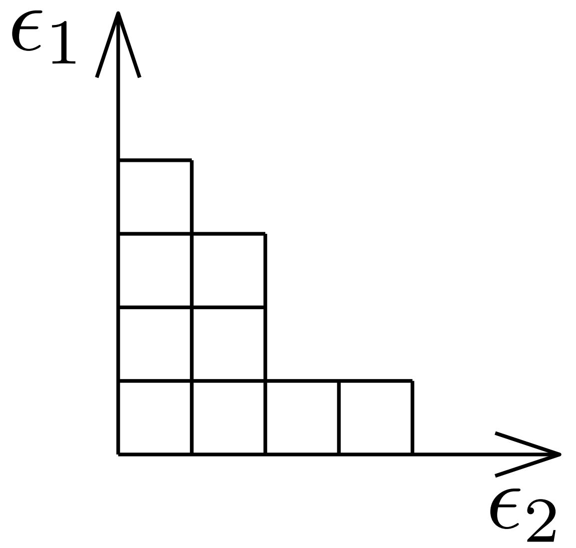
$$\oint_{|z_2| = \epsilon} A^a = 2\pi m^a \,, \quad a = 1, \dots, N$$

Fock Space

Back to Macdonald polynomials

$$q = e^{\epsilon_1}$$

$$\hbar = e^{\epsilon_2}$$



Power-symmetric variables $\quad p_m = \sum_{l=1}^{n} z_l^m$

Macdonald polynomials depend only on k and the partition

$$P_{\square\square} = \frac{1}{2}(p_1^2 - p_2)\,, \qquad P_{\substack{\square\\ \square}} = \frac{1}{2}(p_1^2 - p_2) + \frac{1-qt}{(1+q)(1-t)}p_2$$

Starting with Fock vacuum $\quad |0\rangle$

Construct Hilbert space $\quad a_{-\lambda}|0\rangle \longleftrightarrow p_\lambda$

for each partition $\quad a_{-\lambda}|0\rangle = a_{-\lambda_1} \cdots a_{-\lambda_l}|0\rangle$

Commutators $\quad [a_m, a_n] = m\frac{1-q^{|m|}}{1-\hbar^{|m|}}\delta_{m,-n}$

Ding-Iohara-Miki algebra

Free boson representation of tRS operators

$$\eta(z) =: \exp\left(-\sum_{k\neq 0} \frac{1-t^k}{k} a_k z^{-k}\right):$$

From localization we can compute

$$\phi(z) = \exp\left(\sum_{n>0} \frac{1-t^n}{1-q^n} a_{-n} \frac{z^n}{n}\right)$$

$$\frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(1)}_{\Box} \right\rangle \quad E^{(\lambda;n)}_{eRS}(p) = \left\langle W^{U(n)}_{\Box} \right\rangle \Big|_\lambda$$

Define $\phi_n(\tau) = \prod_{i=1}^n \phi(\tau_i)$ then ($t=\hbar^{-1}$)

$$[\eta(z)]_1 \phi_n(\tau)|0\rangle = \left[t^{-n} + t^{-n+1}(1-t^{-1}) D^{(1)}_{n,\vec{\tau}}(q,t)\right] \phi_n(\tau)|0\rangle$$

Assuming |t|<1

$$\mathcal{E}^{(\lambda)}_1 = \lim_{n\to\infty} \left[t^{-n+1}(1-t^{-1}) E^{(\lambda;n)}_{tRS}\right]$$

$$\mathcal{E}^{(\lambda)}_1(p) = \lim_{n\to\infty} \left[t^{-n+1}(1-t^{-1}) \frac{(pt^{-1};p)_\infty (ptq^{-1};p)_\infty}{(p;p)_\infty (pq^{-1};p)_\infty} E^{(\lambda;n)}_{eRS}(p)\right]$$

For elliptic model replace

$$\eta(z;pq^{-1}t) = \exp\left(\sum_{n>0} \frac{1-t^{-n}}{n} \frac{1-(pq^{-1}t)^n}{1-p^n} a_{-n} z^n\right) \exp\left(-\sum_{n>0} \frac{1-t^n}{n} a_n z^{-n}\right)$$

[Feigin Hashizume Hoshino Shiraishi Yanagida]

DAHA Action

[PK 1805.00986]

Vertex functions or quantum classes for X are elements of quantum K-theory of X. Equivalently we can view them as elements of equivariant K-theory of the space of quasimaps from $\mathbb{P}^1$ to X

$V \in K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)$ with maximal torus $T = \mathbb{T}(U(n) \times U(1)_\hbar \times U(1)_q)$.

Specification $a_k = q^{\lambda_k} \hbar^{n-k}$ restricts us to the Fock space representation of (q,h)-Heisenberg algebra which is a DAHA module

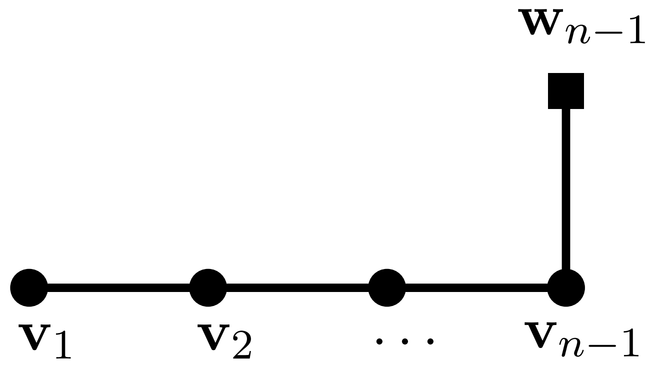
In other words, we can define the following action

$\mathcal{A}_n \longleftarrow \mathfrak{E}$ [Schiffmann Vaserot]

$\mathcal{A}_n$ acts on $K_T(\mathbb{P}^1 \to T^*\mathbb{F}_n)\Big|_{a_k = q^{\lambda_k}\hbar^{n-k}} \longrightarrow K_{q,\hbar}(\oplus_k \mathrm{Hilb}^k[\mathbb{C}^2])$, on which $\mathfrak{E}$ acts

λ not more than n columns

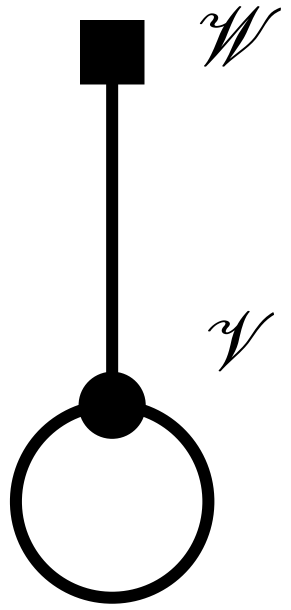
$\mathbb{C}_q^\times \times \mathbb{C}_\hbar^\times$ fixed points are Macdonald polynomials



Flags vs ADHM

[PK Sciarappa]
[PK]

$K_T(\mathbf{QM}(\mathbb{P}^1, X))$	$K_{q,\hbar}(\mathrm{Hilb}(\mathbb{C}^2))$
Kähler/quantum parameters of X $z_1, z_2 \dots$	Ring generators $x_1, x_2, \dots$
Vertex function $\mathsf{V}_{\mathbf{q}}$	Classes of $(\mathbb{C}^\times)^2$ fixed points $[\mathcal{J}]$
$\mathbb{C}_q^\times$ acting on base curve	$\mathbb{C}_q^\times$ acting on $\mathbb{C} \subset \mathbb{C}^2$
$\mathbb{C}_\hbar^\times$ acting on cotangent fibers of X	$\mathbb{C}_\hbar^\times$ acting on another $\mathbb{C} \subset \mathbb{C}^2$
Eigenvalues e_r of tRS operators T_r	Chern polynomials $\mathcal{E}_r$ of $\Lambda^r \mathcal{U}$



quantum deformation:

Eigenvalues of **elliptic** RS model at large n

Eigenvalues of **quantum** multiplication by

$$\mathscr{U} = \mathscr{W} + (1-q)(1-\hbar)\mathscr{V}|_{\mathcal{J}_{\vec{\lambda}}}$$

$$E_r(\vec{\zeta}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})}{\theta_1(\hbar\zeta_i/\zeta_j|\mathfrak{p})} \prod_{i\in\mathcal{I}} p_k$$

Chern roots obey

$$\prod_{l=1}^{N} \frac{s_a - \mathrm{a}_l}{s_a - q^{-1}\hbar^{-1}\mathrm{a}_l} \cdot \prod_{\substack{b=1\\ b\neq a}}^{k} \frac{s_a - q s_b}{s_a - q^{-1}s_b} \frac{s_a - \hbar s_b}{s_a - \hbar^{-1}s_b} \frac{s_a - q^{-1}\hbar^{-1}s_b}{s_a - q\hbar s_b} = \mathfrak{z}$$

M-theory Description

Recall that $\mathrm{Hilb}^k[\mathbb{C}^2] = \mathcal{M}^{\mathrm{inst}}_{1,k}$

How did U(1) 5d SYM appear?

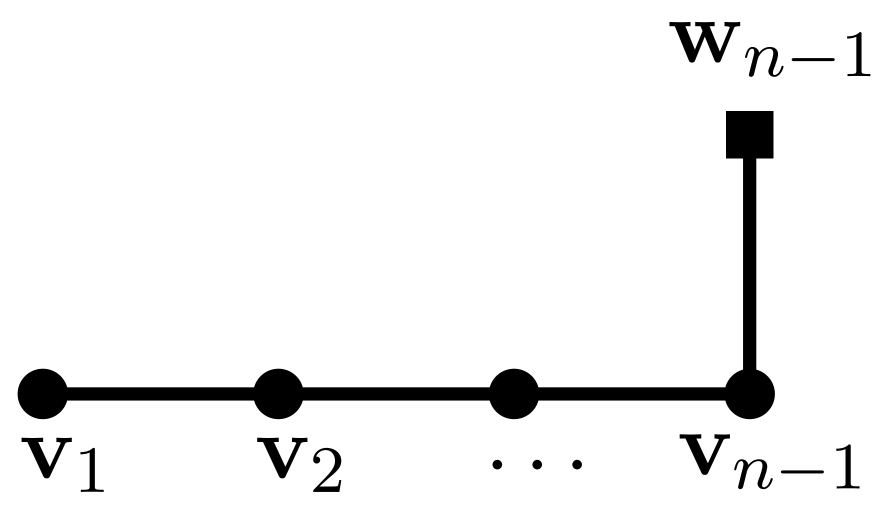
Starting with M-theory on

n M5 branes wrapping

$$S^1 \times \mathbb{C}_q \times \mathbb{C}_\hbar \times T^*S^3$$

$$S^1 \times \mathbb{C}_q \times S^3 \subset$$

Upon compactification on three sphere will get *3d quiver gauge theory* on T*Fln



When n becomes large the background undergoes through the **conifold transition** and the *resolved* conifold becomes a *deformed* conifold Y:

$$S^1 \times \mathbb{C}_q \times \mathbb{C}_t \times Y$$

Reduction on Y leads us to a 5d U(1) theory with 8 supercharges

Spectrum

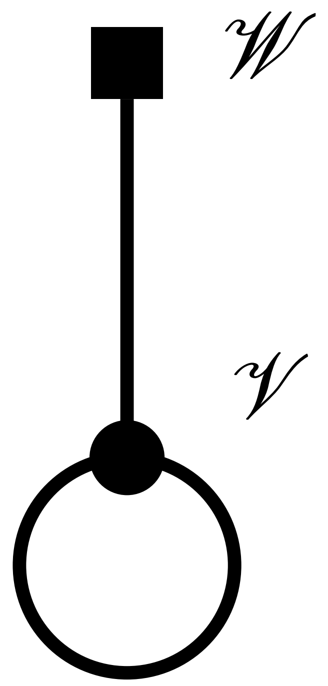
***eRS Hamiltonian** eigenvalues coincide with eigenvalues of the **quantum multiplication** operator in **quantum K-theory** ring of the instanton moduli space (Hilbert Scheme of points).*

$$\left\langle W_{\square}^{U(n)} \right\rangle \Big|_{\lambda} \sim \mathcal{E}_1^{(\lambda)} = 1 - (1-q)(1-t^{-1}) \sum_s \sigma_s \Big|_{\lambda}$$

sigmas are determined by Bethe Ansatz equations for ADHM quiver

Elliptic deformation — Quantization

Fields	χ	B_1	B_2	I	J
gauge group $U(k)$	Adj	Adj	Adj	$\mathbf{k}$	$\bar{\mathbf{k}}$
flavor $U(N) \times U(1)^2$	$\mathbf{1}_{(-1,-1)}$	$\mathbf{1}_{(1,0)}$	$\mathbf{1}_{(0,1)}$	$\bar{\mathbf{N}}_{(0,0)}$	$\mathbf{N}_{(1,1)}$
flavor parameters	$(q\hbar)^{-1}$	q	$\hbar$	a_j	$\mathrm{a}_j^{-1} q\hbar$
R-charge	2	0	0	0	0



Inozemtsev Limit — Trigonometric Case

Consider tRS difference operators

$$T_r(\vec{\zeta}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\} \\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I} \\ j\notin\mathcal{I}}} \frac{\hbar^{-1/2}\,\zeta_i - \hbar^{1/2}\zeta_j}{\zeta_i-\zeta_j} \prod_{i\in\mathcal{I}} p_k$$

$$T_r(\vec{\zeta})\mathsf{V}_{\mathbf{p}} = e_r(\mathbf{a})\mathsf{V}_{\mathbf{p}}\,, \qquad r=1,\dots,n$$

Double Scaling

$$\mathfrak{z}_i = \hbar^{-i}\zeta_i\,, \qquad \mathfrak{p}_i = \hbar^{-i+1/2}p_i\,, \qquad \mathfrak{a}_i = \hbar^{-\frac{n}{2}}\alpha_i = a_i$$

$$\hbar \to \infty$$

Obtain q-Toda Hamiltonians

$$H_r^{\text{q-Toda}} = \sum_{\substack{\mathcal{I}=\{i_1<\dots<i_r\} \\ \mathcal{I}\subset\{1,\dots,n\}}} \prod_{\ell=1}^{r} \left(1-\frac{\mathfrak{z}_{i_\ell-1}}{\mathfrak{z}_{i_\ell}}\right)^{1-\delta_{i_\ell-i_{\ell-1},1}} \prod_{k\in\mathcal{I}} \mathfrak{p}_k$$

$$H_r^{\text{q-Toda}}(\mathfrak{z}_1,\dots\mathfrak{z}_n;\mathfrak{p}_1,\dots,\mathfrak{p}_n) = e_r(\mathfrak{a}_1,\dots,\mathfrak{a}_n)$$

The first Hamiltonian

$$H_1^{\text{open}} = \mathfrak{p}_1 + \sum_{i=2}^{n} \mathfrak{p}_i\left(1-\frac{\mathfrak{z}_{i-1}}{\mathfrak{z}_i}\right)$$

Inozemtsev Limit — Elliptic Case

Theta function $$\theta_1(e^{iz}|\mathfrak{p}) = 2\mathfrak{p}^{\frac{1}{4}} \sum_{k=0}^{+\infty} (-1)^k \mathfrak{p}^{k(k+1)} \sin((k+1/2)z)$$

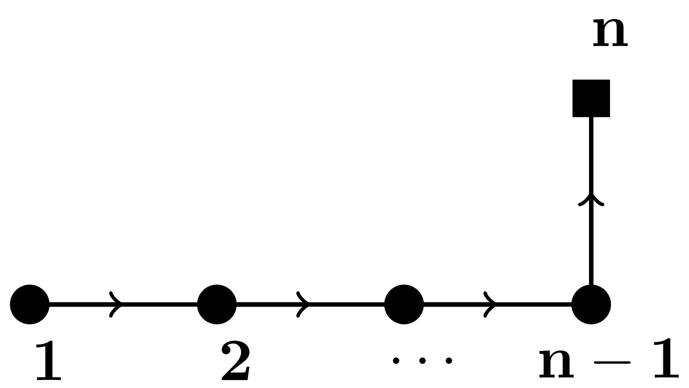
Ratio of thetas $$\frac{\theta_1\left(\frac{\zeta_1}{\hbar\zeta_2}|\mathfrak{p}\right)}{\theta_1\left(\frac{\zeta_1}{\zeta_2}|\mathfrak{p}\right)} = \frac{\frac{\sqrt{\frac{\zeta_1}{\zeta_2}}}{\sqrt{\hbar}} - \frac{\sqrt{\hbar}}{\sqrt{\frac{\zeta_1}{\zeta_2}}} + \mathfrak{p}^2\left(\frac{\left(\frac{\zeta_1}{\zeta_2}\right)^{3/2}}{\hbar^{3/2}} - \frac{\hbar^{3/2}}{\left(\frac{\zeta_1}{\zeta_2}\right)^{3/2}}\right)}{\sqrt{\frac{\zeta_1}{\zeta_2}} - \frac{1}{\sqrt{\frac{\zeta_1}{\zeta_2}}} + \mathfrak{p}^2\left(\frac{1}{\left(\frac{\zeta_1}{\zeta_2}\right)^{3/2}} - \left(\frac{\zeta_1}{\zeta_2}\right)^{3/2}\right)} + O(\mathfrak{p}^5)$$

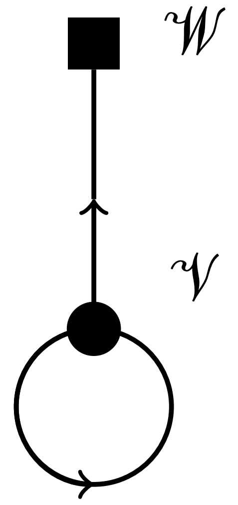
Two-body Hamiltonian becomes in the limit $\hbar \to \infty$ $$\frac{\theta_1\left(\frac{\zeta_1}{\hbar\zeta_2}|\mathfrak{p}\right)}{\theta_1\left(\frac{\zeta_1}{\zeta_2}|\mathfrak{p}\right)}p_1 + \frac{\theta_1\left(\frac{\zeta_2}{\hbar\zeta_1}|\mathfrak{p}\right)}{\theta_1\left(\frac{\zeta_1}{\zeta_2}|\mathfrak{p}\right)}p_2 \to \mathfrak{p}_1\left(1 - \mathfrak{q}\frac{\mathfrak{z}_2}{\mathfrak{z}_1}\right) + \mathfrak{p}_2\left(1 - \frac{\mathfrak{z}_1}{\mathfrak{z}_2}\right)$$

We send $\mathfrak{p} \to 0$ such that $\mathfrak{q} = \mathfrak{p}\hbar^n$ Is finite

Two-body Hamiltonian $$H_1^{\text{aff q-Toda}} = \mathfrak{p}_1\left(1 - \mathfrak{q}\frac{\mathfrak{z}_n}{\mathfrak{z}_1}\right) + \sum_{i=2}^{n} \mathfrak{p}_i\left(1 - \frac{\mathfrak{z}_{i-1}}{\mathfrak{z}_i}\right)$$

Vortex Moduli Space





After taking the $n \to \infty$ limit one gets

$$\lim_{n\to\infty} \hbar^n \mathscr{E}_1^{\mathrm{Toda}}\Big|_\lambda = \mathcal{E}_1^\Lambda(\lambda)|_\lambda$$

$$\mathcal{E}_1^\Lambda(\lambda) = a - (1-q)e_1(s_1,\ldots,s_k)$$

$$\mathfrak{a}_i = a q^{\lambda_i}\,, \quad i = 1,\ldots,n$$

Bethe equations

$$\prod_{l=1}^{N}(s_a - \mathrm{a}_l)\cdot\prod_{\substack{b=1\\ b\neq a}}^{k}\frac{q s_a - s_b}{s_a - q s_b} = \widetilde{\mathfrak{p}}^\Lambda\,, \quad a = 1,\ldots,k$$

Dimensional coupling

$$\widetilde{\mathfrak{p}}^\Lambda = \widetilde{\mathfrak{p}}\, q^{1/2}\hbar^{1/2}\prod_{l=1}^{N}(-q\hbar a_l)$$

affine q-Toda model	5d/3d $\mathcal{N}=2$ SYM theory	3d $\frac{1}{2}$-ADHM theory
Coordinates z_i	Kähler parameters	K-ring generators x_i
Eigenfunctions	Defect partition functions	$\frac{1}{2}$-ADHM Coulomb branch vacua
Planck constant $\log q$	equivariant parameter q	$\mathbb{C}_q^\times$ acting on $\mathbb{C}$
Affine parameter $\mathfrak{q}$	5d dynamical scale $\mathfrak{p}^\Lambda$	FI coupling $\widetilde{\mathfrak{p}}^\Lambda$
Eigenvalues $\mathscr{E}_r^{\mathrm{Toda}}$	VEVs of Wilson loop $\langle W_{\Lambda^r\square}\rangle$	Chern polynomials $\mathcal{E}_r^\Lambda$ of $\Lambda^r\mathcal{U}$

From ADHM to 1/2 ADHM

[PK Koroteeva Gorsky Vainshtein]

$K_\hbar(T^*\mathbb{F}l_n)$ ⟷ ADHM (**instanton** moduli space)

$$\lim_{n\to\infty}\left[\hbar^{n-1}(1-\hbar)\left\langle W_{\square}^{U(n)}\right\rangle\right]\Big|_\lambda = a-(1-q)(1-\hbar)e_1(s_1,\dots,s_k)|_\lambda$$

Claim: $\hbar\to\infty$ retracting the fibers, dimensional transmutation

[Hanany Tong]

$K(\mathbb{F}l_n)$ ⟷ 1/2 ADHM (**vortex** moduli space)

Eigenvalues of **affine** qToda lattice at large n ⟷ Eigenvalues of **quantum** multiplication by

$$H_1^{\mathrm{aff}}=\mathfrak{p}_1\left(1-\mathfrak{p}^\Lambda\frac{\mathfrak{z}_n}{\mathfrak{z}_1}\right)+\sum_{i=2}^n\mathfrak{p}_i\left(1-\frac{\mathfrak{z}_{i-1}}{\mathfrak{z}_i}\right)$$

$$\mathcal{E}_1^\Lambda(\lambda)=a-(1-q)e_1(s_1,\dots,s_k)$$

Chern roots obey

$$\prod_{l=1}^N(s_a-\mathrm{a}_l)\cdot\prod_{\substack{b=1\\b\neq a}}^k\frac{qs_a-s_b}{s_a-qs_b}=\widetilde{\mathfrak{p}}^\Lambda$$

Subscheme $\mathcal{Z}_k\subset \mathrm{Hilb}^k[\mathbb{C}^2]$

q-Heisenberg algebra preserving $\oplus_k K_q(\mathscr{Z}_k)$

Quantum K-theory

Theorem 6.1 ([KPSZ]). *The quantum equivariant K-theory of the complete n-dimensional flag variety is given by*

$$QK_{T'}(\mathbb{F}l_n) = \frac{\mathbb{C}[\mathfrak{z}_1^{\pm 1}, \dots, \mathfrak{z}_n^{\pm 1}; \mathfrak{a}_1^{\pm 1}, \dots, \mathfrak{a}_n^{\pm 1}; \mathfrak{p}_1^{\pm 1}, \dots, \mathfrak{p}_n^{\pm 1}]}{\left(H_r^{q\text{-}Toda}(\mathfrak{z}_i, \mathfrak{p}_i) = e_r(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\right)}, \tag{6.10}$$

where $H_r^{q\text{-}Toda}$ are given by (4.3) and T' is the maximal torus of $GL(n)$ with equivariant parameters $\mathfrak{a}_1, \dots, \mathfrak{a}_n$.

Theorem 6.2. *For $n > k$ there is the following embedding of Hilbert spaces*

$$\bigoplus_{l=0}^{k} K_q(Hilb^l(\mathbb{C})) \hookrightarrow \mathcal{P}_n \tag{6.13}$$

$$[\lambda] \mapsto \mathsf{I}_{\boldsymbol{q}},$$

$$\mathcal{P}_n := K_{\mathsf{T}}(\mathbf{QM}(\mathbb{P}^1, \mathbb{F}l_n))$$

where $\mathsf{I}_{\boldsymbol{q}}$ is the K-theory vertex function for some fixed point $\boldsymbol{q}$ of maximal torus T'. The statement also holds in the limit $n \to \infty$

$$\bigoplus_{l=0}^{\infty} K_q(Hilb^l(\mathbb{C})) \hookrightarrow \mathcal{P}_\infty, \tag{6.14}$$

$$\mathsf{I}_{\mathbf{q}} = \lim_{\hbar \to \infty} \mathsf{V}_{\mathbf{q}}$$

where $\mathcal{P}_\infty$ is defined as a stable limit of $\mathcal{P}_n$ as $n \to \infty$.

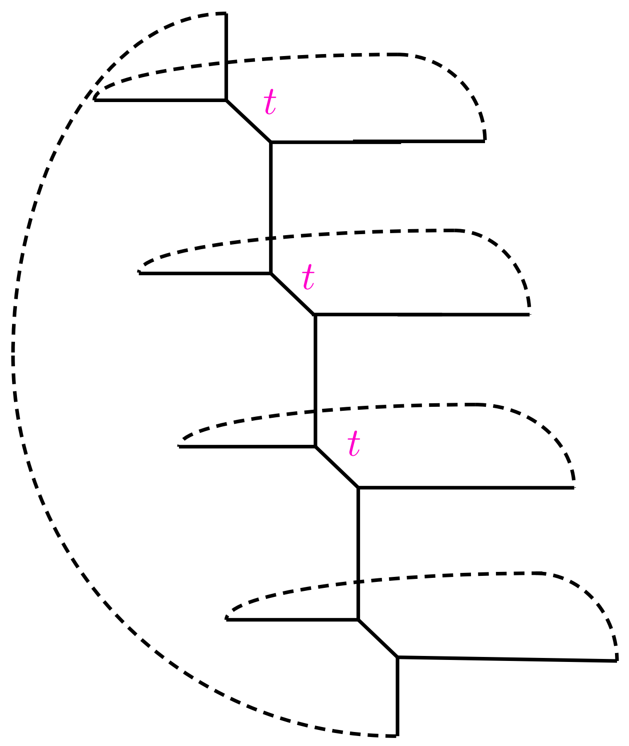
Inosemtsev Limit from DELL

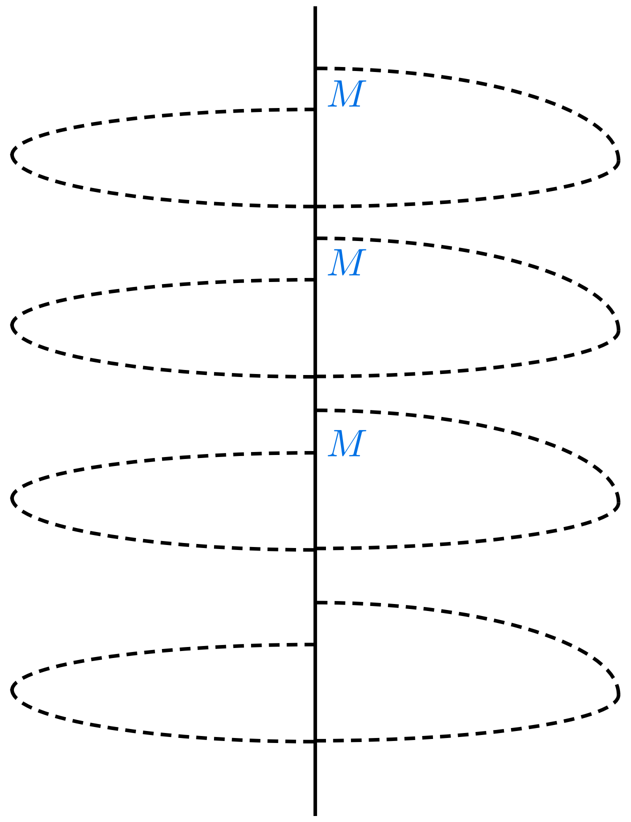
p \ q	rational	trigonometric	elliptic
r	rational CMS	trigonometric CMS	elliptic CMS *quantum cohomology*
t	rational RS (dual trig. CMS)	trigonometric RS	elliptic RS *quantum K-theory*
e	dual elliptic CMS	dual elliptic RS	DELL *Elliptic Cohomology*

$\epsilon \to 0$ $\epsilon \to 0$
$p \to 0$ $p \to 0$ $p \to 0$
$R \to 0$ $R \to 0$ $R \to 0$
$w \to 0$ $w \to 0$

The DELL

$$\mathcal{O}(z) = \sum_{n\in\mathbb{Z}} \mathcal{O}_n\, z^n = \sum_{n_1,\ldots,n_N=-\infty}^{\infty} (-z)^{\sum n_i}\, w^{\sum \frac{n_i(n_i-1)}{2}} \prod_{i<j} \theta\left(t^{n_i-n_j}\frac{x_i}{x_j}\Big|p\right)\, p_1^{n_1}\ldots p_N^{n_N}$$





N-particle DELL Hamiltonians

$$\mathcal{H}_a = \mathcal{O}_0^{-1}\mathcal{O}_a\,, \qquad a = 1,\ldots,N-1$$

Double Inosemtsev limit from DELL

$$x_a \mapsto t^{-a}x_a, \qquad p_a \mapsto t^{-a-1/2}p_a, \qquad p = t^{\alpha}\Lambda, \qquad w = t^{\beta}M$$

$\alpha = N,\ \beta = 1$

$$\widetilde{\mathcal{O}}_k = c_N : \sum_{i_1<\cdots<i_k} \prod_{a<b} \theta\left(M^{b-a+m_b-m_a}\frac{x_a}{x_b}\frac{p_b}{p_a}\Big|\widetilde{p}\right) : p_{i_1}\cdots p_{i_k}$$

$\widetilde{p} = \Lambda M^N$ and $m_a = \delta_{a\in I}$

Act with T-transform $\quad x_a \mapsto x_a p_a, \qquad p_a \mapsto p_a \quad$ and rescale $\quad x_a \mapsto M^{-a}x_a$

Yield eRS model after a conjugation by a Gaussian

$$\widetilde{\mathcal{O}}_k = c_N \sum_{i_1<\cdots<i_k} \prod_{a<b} \theta\left(M^{m_b-m_a}\frac{x_a}{x_b}\Big|\widetilde{p}\right) p_{i_1}\cdots p_{i_k}$$