Hello and welcome to class!

Last time

We studied the formal properties of determinants, and how to compute them by row reduction.

Today

We'll see some more formulas involving the determinant — minor expansion and Cramer's rule — and discuss the interpretation of the determinant as a signed volume.

Review: computing determinants by row reduction

To compute the determinant of a matrix, row reduce it, and keep track of any row switches or rescalings of rows.

At the end, multiply together:

- the inverses of the row rescaling factors
- the diagonal entries of the final echelon matrix
- $(-1)^{\#\text{rowswaps}}$

That's the determinant of the original matrix.

This method is much much faster than summing all the terms.

Example

Let's compute the determinant of this matrix

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{bmatrix}$$

First, we row reduce, keeping track of rescalings and row switches

Example

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -3 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -4 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -7 & 7 \end{bmatrix} \xrightarrow{-1}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -7 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1/7} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the determinant is $(-1)^2 \cdot (-7) \cdot (1 \cdot 1 \cdot 1 \cdot 1) = -7$.

Try it yourself!

Compute the determinant of this matrix:

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Row reduce, keeping track of rescalings and row switches:

Try it yourself!

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & 1 & 2 & 1 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 4 & 2 & -5 \\ 0 & 5 & 1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{-1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 5 & 1 & -2 \\ 0 & 4 & 2 & -5 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -9 & -17 \\ 0 & 0 & -6 & -17 \end{bmatrix}$$

The determinant is 51.

Review: terms in the determinant

In the 2x2 case:

$$\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix} \qquad \begin{bmatrix} a & \boxed{b} \\ \boxed{c} & d \end{bmatrix}$$

$$+ad \qquad\qquad -bc$$

Review: terms in the determinant

In the 3x3 case:

$$\begin{bmatrix} \boxed{a} & b & c \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix} \quad \begin{bmatrix} a & \boxed{b} & c \\ d & e & \boxed{f} \\ \boxed{g} & h & i \end{bmatrix} \quad \begin{bmatrix} a & b & \boxed{c} \\ \boxed{d} & e & f \\ g & \boxed{h} & i \end{bmatrix}$$

$$+aei \qquad +bfg \qquad +cdh$$

$$\begin{bmatrix} \boxed{a} & b & c \\ d & e & \boxed{f} \\ g & \boxed{h} & i \end{bmatrix} \quad \begin{bmatrix} a & \boxed{b} & c \\ \boxed{d} & e & f \\ g & h & \boxed{i} \end{bmatrix} \quad \begin{bmatrix} a & b & \boxed{c} \\ d & \boxed{e} & f \\ \boxed{g} & h & i \end{bmatrix}$$

$$-afh \qquad -bdi \qquad -ceg$$

Another perspective

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$+aei - afh \qquad -bdi + bfg \qquad +cdh - ceg$$

$$+a\begin{vmatrix} e & f \\ h & i \end{vmatrix} \qquad -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} \qquad +c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

no orange-green inversions

one orange-green inversions

two orange-green inversions

Minor expansion

For a matrix A, I'll write $A_{\not{i}\not{j}}$ for the matrix formed by omitting row i and column j. For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We have:

$$\begin{aligned} |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}|A_{\not{1}\not{1}}| - a_{12}|A_{\not{1}\not{2}}| + a_{13}|A_{\not{1}\not{3}}| \end{aligned}$$

Minor expansion

More generally, by the same argument, for a square $n \times n$ matrix A with entry $a_{i,j}$ in row i and column j,

for any k in $1, \ldots, n$, there is a minor expansion along the k'th row

$$|A| = \sum_{j=1}^{n} (-1)^{j+k} a_{kj} |A_{\not{k}\not{j}}|$$

and a minor expansion along the k'th column

$$|A| = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} |A_{\not{j}\not{k}}|$$

The sign $(-1)^{\text{row}+\text{column}}$

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix}$$

Example

Compute by minor expansion along the second row:

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{vmatrix}$$

Example

$$
\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 3 & 7 & 8 & -2 \end{vmatrix}
= -2 \begin{vmatrix} \text{[redacted]} & 2 & 3 & -1 \\ \text{[redacted]} & \text{[redacted]} & \text{[redacted]} & \text{[redacted]} \\ \text{[redacted]} & 1 & -1 & 2 \\ \text{[redacted]} & 7 & 8 & -2 \end{vmatrix}
+ 0 \begin{vmatrix} 1 & \text{[redacted]} & 3 & -1 \\ \text{[redacted]} & \text{[redacted]} & \text{[redacted]} & \text{[redacted]} \\ 0 & \text{[redacted]} & -1 & 2 \\ 3 & \text{[redacted]} & 8 & -2 \end{vmatrix}
$$

$$
-3 \begin{vmatrix} 1 & 2 & \text{[redacted]} & -1 \\ \text{[redacted]} & \text{[redacted]} & \text{[redacted]} & \text{[redacted]} \\ 0 & 1 & \text{[redacted]} & 2 \\ 3 & 7 & \text{[redacted]} & -2 \end{vmatrix}
+ 1 \begin{vmatrix} 1 & 2 & 3 & \text{[redacted]} \\ \text{[redacted]} & \text{[redacted]} & \text{[redacted]} & \text{[redacted]} \\ 0 & 1 & -1 & \text{[redacted]} \\ 3 & 7 & 8 & \text{[redacted]} \end{vmatrix}
$$

Example

$$-2\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 7 & 8 & -2 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & -1 \\ 0 & -1 & 2 \\ 3 & 8 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 7 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 7 & 8 \end{vmatrix}$$

Now we minor-expand each of these 3×3 determinants.

We'll use the second row for each (to catch the zero).

Example

$$\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 7 & 8 & -2 \end{vmatrix} = -1\begin{vmatrix} 3 & -1 \\ 8 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix}$$

$$= -2 - 3 + 10 = 5$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 7 & -2 \end{vmatrix} = -0\begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}$$

$$= 0 + 1 - 2 = -1$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 7 & 8 \end{vmatrix} = -0\begin{vmatrix} 2 & 3 \\ 7 & 8 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}$$

$$= 0 - 1 + 1 = 0$$

Example

$$-2\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 2 \\ 7 & 8 & -2 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & -1 \\ 0 & -1 & 2 \\ 3 & 8 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 7 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 3 & 7 & 8 \end{vmatrix}$$

$$(-2 \times 5) + (0 \times ?) + (-3 \times -1) + (1 \times 0) = -7$$

That's the same as we got doing this the other way.

Which was easier?

Try it yourself!

Compute by minor expansion the determinant of the matrix.

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

A formula for the inverse

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} A_{\not{1}\not{1}} & -A_{\not{2}\not{1}} & A_{\not{3}\not{1}} \\ -A_{\not{1}\not{2}} & A_{\not{2}\not{2}} & -A_{\not{3}\not{2}} \\ A_{\not{1}\not{3}} & -A_{\not{2}\not{3}} & A_{\not{3}\not{3}} \end{bmatrix}$$

$$A \cdot Adj(A) = \begin{bmatrix} a_{11}A_{\not{1}\not{1}} - a_{12}A_{\not{1}\not{2}} + a_{13}A_{\not{1}\not{3}} & -a_{11}A_{\not{2}\not{1}} + a_{12}A_{\not{2}\not{2}} - a_{13}A_{\not{2}\not{3}} & a_{11}A_{\not{3}\not{1}} - a_{12}A_{\not{3}\not{2}} + a_{13}A_{\not{3}\not{3}} \\ a_{21}A_{\not{1}\not{1}} - a_{22}A_{\not{1}\not{2}} + a_{23}A_{\not{1}\not{3}} & -a_{21}A_{\not{2}\not{1}} + a_{22}A_{\not{2}\not{2}} - a_{23}A_{\not{2}\not{3}} & a_{21}A_{\not{3}\not{1}} - a_{22}A_{\not{3}\not{2}} + a_{23}A_{\not{3}\not{3}} \\ a_{31}A_{\not{1}\not{1}} - a_{32}A_{\not{1}\not{2}} + a_{33}A_{\not{1}\not{3}} & -a_{31}A_{\not{2}\not{1}} + a_{32}A_{\not{2}\not{2}} - a_{33}A_{\not{2}\not{3}} & a_{31}A_{\not{3}\not{1}} - a_{32}A_{\not{3}\not{2}} + a_{33}A_{\not{3}\not{3}} \end{bmatrix}$$

The diagonal terms, e.g., $a_{11}A_{\not{1}\not{1}} - a_{12}A_{\not{1}\not{2}} + a_{13}A_{\not{1}\not{3}}$, are minor expansions of $\det(A)$.

A formula for the inverse

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} A_{\not{1}\not{1}} & -A_{\not{2}\not{1}} & A_{\not{3}\not{1}} \\ -A_{\not{1}\not{2}} & A_{\not{2}\not{2}} & -A_{\not{3}\not{2}} \\ A_{\not{1}\not{3}} & -A_{\not{2}\not{3}} & A_{\not{3}\not{3}} \end{bmatrix}$$

Let's look at an off-diagonal term of $A \cdot Adj(A)$, say

$$a_{21}A_{\not{1}\not{1}} - a_{22}A_{\not{1}\not{2}} + a_{23}A_{\not{1}\not{3}}$$

Expanding this out from the definition,

$$a_{21}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

A formula for the inverse

The quantity

$$a_{21}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

is the minor expansion of the determinant

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The matrix has a repeated row, so the determinant is zero! The same is true for all the off diagonal terms.

A formula for the inverse

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} A_{\not{1}\not{1}} & -A_{\not{2}\not{1}} & A_{\not{3}\not{1}} \\ -A_{\not{1}\not{2}} & A_{\not{2}\not{2}} & -A_{\not{3}\not{2}} \\ A_{\not{1}\not{3}} & -A_{\not{2}\not{3}} & A_{\not{3}\not{3}} \end{bmatrix}$$

$$A \cdot Adj(A) = \det(A) \cdot I = Adj(A) \cdot A$$

This holds for any square matrix A, where

$$Adj(A)_{ij} = (-1)^{i+j} |A_{\not{j}\not{i}}|$$

The entry in row i, column j of $Adj(A)$ is the determinant of the matrix formed by removing column i and row j of A, times $(-1)^{i+j}$.

Try it yourself!

For the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, determine $Adj(A)$, and verify

$$A \cdot Adj(A) = \det(A) \cdot I = Adj(A) \cdot A$$

$$Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Cramer's rule

Consider a matrix equation $A\mathbf{x} = \mathbf{b}$ where A is square. Then

$$\det(A) \cdot \mathbf{x} = (Adj(A) \cdot A)\mathbf{x} = Adj(A) \cdot \mathbf{b}$$

Take the i'th row of the column vector on both sides:

$$\det(A) \cdot x_i = \sum_j Adj(A)_{ij} b_j = \sum_j (-1)^{i+j} |A_{\not{j}\not{i}}| b_j$$

I.e., the minor expansion along the i'th column of the determinant of the matrix formed by replacing the i'th column of A by $\mathbf{b}$.

Cramer's rule

Consider a matrix equation $A\mathbf{x} = \mathbf{b}$ where A is square.

Then if $\det(A) \neq 0$,

$$x_i = \frac{\det(\text{replace column } i \text{ of } A \text{ by } \mathbf{b})}{\det(A)}$$

Never use these formulas to compute

As we saw, taking the determinant of a 4×4 matrix by minor expansion was more difficult than by row reduction.

It only gets worse as the size of the matrix grows.

Likewise, row reduction beats computing Adj for inverting matrices, and beats Cramer's rule for solving systems.

Why learn these formulas at all?

It's conceptually satisfying to know that, not only is there a procedure for solving systems or inverting matrices,

there's in fact a closed form formula.

The properties of the formula reveal facts about the solutions.

Integer inverses and solutions

Say you have an invertible matrix M with integer entries.

Does its inverse also have integer entries?

It does, if and only $\det(M) = \pm 1$.

Observe $\det(M)\det(M^{-1}) = \det(MM^{-1}) = 1$. The determinant of an integer matrix is always an integer — it's made by additions and multiplications. If M^{-1} has integer entries, then $\det(M)$ and $\det(M^{-1})$ are two integers which multiply to 1, hence both ± 1.

Similarly, the Adj of an integer matrix is an integer: it's made by additions and multiplications. So, if $\det M = \pm 1$, then $M^{-1} = Adj(M)/\det M$ is an integer matrix as well.

Integer inverses and solutions

Similarly, consider the equation $A\mathbf{x} = \mathbf{b}$.

Assume

- A is square and has integer entries
- $\mathbf{b}$ has integer entries
- $\det(A) = \pm 1$

We saw that A^{-1} has integer entries, so the (unique) solution $\mathbf{x} = A^{-1}\mathbf{b}$ also has integer entries.

Volumes

You probably have an intuitive notion of what volume means: the amount of stuff that can fit inside something. For our purposes, the stuff is going to be cubes of a fixed side length:

$$\mathrm{Volume}(S) \sim \text{number of cubes that fit inside } S$$

To be more precise,

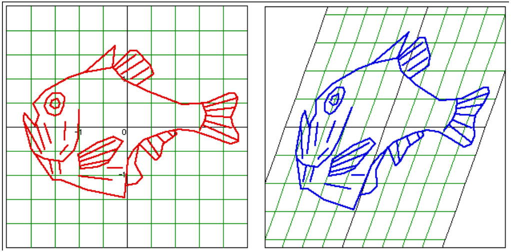
$$\mathrm{Volume}(S) = \lim_{\epsilon \to 0} \epsilon^{-\mathrm{dim}} \cdot \begin{pmatrix} \text{number of cubes} \\ \text{of side length } \epsilon \\ \text{that fit inside } S \end{pmatrix}$$

By this we mean: for $S \subset \mathbb{R}^n$, we overlay the ϵ-mesh grid on S, and count the number of cubes which fall completely inside.

Linear transformations and volumes

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Given a set X, we want to think about how the volumes of X and $T(X)$ compare.

The key observation is that the number of cubes in X is the same as the number of T-transformed such cubes in $T(X)$.



Linear transformations and volumes

Filling the transformed cubes with yet smaller regular cubes, and observing that the failure of these to pack correctly at the boundary is washed out as $\epsilon \to 0$, we conclude:

$$\frac{\mathrm{Volume}(X)}{\mathrm{Volume}(\mathrm{cube})} = \frac{\mathrm{Volume}(T(X))}{\mathrm{Volume}(T(\mathrm{cube}))}$$

Rearranging,

$$\mathrm{Volume}(T(X)) = \mathrm{Volume}(T(\mathrm{cube})) \cdot \mathrm{Volume}(X)$$

But what's $\mathrm{Volume}(T(\mathrm{cube}))$?

Linear transformations and volumes

Suppose now given two linear transformations, $T, S : \mathbb{R}^n \to \mathbb{R}^n$.

We apply the formula

$$\mathrm{Volume}(T(X)) = \mathrm{Volume}(T(\mathrm{cube})) \cdot \mathrm{Volume}(X)$$

to the set $X = S(\mathrm{cube})$:

$$\mathrm{Volume}(T(S(\mathrm{cube}))) = \mathrm{Volume}(T(\mathrm{cube})) \cdot \mathrm{Volume}(S(\mathrm{cube}))$$

Linear transformations and volumes

In other words, the function

$$\begin{aligned} V : \text{linear transformations} &\to \mathbb{R} \\ T &\mapsto \text{Volume}(T(\text{unit cube})) \end{aligned}$$

respects multiplication in the sense that

$$V(T \circ S) = V(T)V(S)$$

Note also that $V(\text{Identity}) = 1$, and $V(\text{non invertible matrix}) = 0$.

Linear transformations and volumes

Now consider any linear transformation T.

If it's not invertible, $V(T) = 0$.

If it is invertible, then by row reduction we can expand it as a product of elementary matrices,

$$T = E_n \cdots E_1$$

Since volume scaling is multiplicative,

$$V(T) = V(E_n) \cdots V(E_1)$$

Linear transformations and volumes

It remains to understand volume scaling of elementary matrices.

Rescaling a row — stretching a coordinate — rescales volume by the same factor: the volume of a box is the product of its side lengths, and we rescaled one of them.

Switching two rows doesn't change volume at all — we're just renaming the sides of the box.

Adding a multiple of one row to another takes a box to a parallelopiped with the same base and the same height, so again doesn't change volume.

Determinants and volumes

So for an elementary linear transformation,

$$\mathrm{Volume}(T(\mathrm{unit\,cube})) = |\det(T)|$$

Volume scaling is multiplicative, so this holds for any linear transformation.

Collecting these observations, for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and any set $X \subset \mathbb{R}^n$,

$$\mathrm{Volume}(T(X)) = |\det(T)| \cdot \mathrm{Volume}(X)$$