Hello and welcome to class!

Last time

We discussed linear transformations, and their matrix representations.

Today

We'll review span, linear independence, and various ways to understand them, and then introduce more arithmetic operations on matrices.

A has r rows and c columns; $A : \mathbb{R}^c \to \mathbb{R}^r$

Columns below have equivalent conditions (except in parethesis)

$A\mathbf{x} = 0$ implies $\mathbf{x} = 0$	$A\mathbf{x} = \mathbf{b}$ has solutions for any $\mathbf{b}$
pivot in every column	pivot in every row
columns linearly independent	rows linearly independent
rows span all of $\mathbb{R}^c$	columns span all of $\mathbb{R}^r$
distinct points to distinct points	hits all of $\mathbb{R}^r$.
(can only happen if $c \leq r$)	(can only happen if $r \leq c$)

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.

Span and linear independence

In particular, a collection of d vectors in $\mathbb{R}^n$

can only span if $d \geq n$,

and can only be linearly independent if $d \leq n$.

Example

Are the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ linearly independent?

Definitely not, there are too many. Let's put them in a matrix and row reduce anyway.

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

The row operations do not affect linear independence, and the final matrix has a zero row, so the original rows were not linearly independent. You can also see: there's not a pivot in every row, the columns don't span, etc. But note: the columns are linearly independent.

Example

Are the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 9 \end{bmatrix}$ linearly independent?

Put them in a matrix and row reduce!

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 5 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 5 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The row operations do not affect linear independence, and the final matrix has a zero row, so the original rows were not linearly independent. You can also see: there's not a pivot in every row, the columns don't span, etc. But note: the columns are not linearly independent.

Try it yourself

Are the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 9 \\ 1 \end{bmatrix}$ linearly independent?

Put them in a matrix and row reduce!

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -1 & 1 & 1 \\ 5 & 1 & 9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 5 & 1 & 9 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The final echelon matrix has no zero row, so the original rows are linearly independent.

A has r rows and c columns; $A : \mathbb{R}^c \to \mathbb{R}^r$

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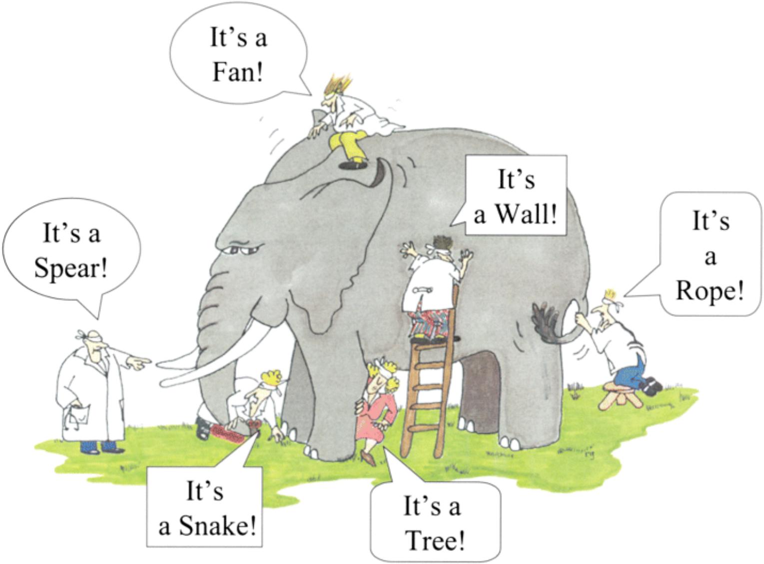
$A\mathbf{x} = 0$ implies $\mathbf{x} = 0$	$A\mathbf{x} = \mathbf{b}$ has solutions for any $\mathbf{b}$
pivot in every column	pivot in every row
columns linearly independent	rows linearly independent
rows span all of $\mathbb{R}^c$	columns span all of $\mathbb{R}^r$
distinct points to distinct points	hits all of $\mathbb{R}^r$.
(can only happen if $c \leq r$)	(can only happen if $r \leq c$)

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.


It's a Fan!
It's a Wall!
It's a Spear!
It's a Rope!
It's a Snake!
It's a Tree!

Scalar multiplication

Just like for vectors, multiplying a matrix by a scalar just means multiplying every element of the matrix by that scalar.

$$3 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

$$-2 \cdot \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 4 & -6 & -2 \end{bmatrix}$$

$$0 \cdot \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix addition

And you can add matrices of the same size by adding them termwise.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 5 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 2 & 2 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

they're not the same size

Matrix transpose

The transpose of the matrix is what you get by reflecting along a northwest-southeast diagonal. This makes the old first column into the new first row, etcetera.

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \\ -1 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 3 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

Matrix entries

We write $M_{i,j}$ or M_{ij} for the entry in the $i'th$ row and $j'th$ column of the matrix M.

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 5 & 8 \end{bmatrix}$$

$$A_{1,1} = 3, \qquad A_{2,3} = 8$$

Matrix multiplication

Given two matrices, A, B, if A has as many columns as B has rows, then there is a matrix product AB.

The matrix product is determined by the formula

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}$$

where n is the number of columns of A, or of rows of B.

AB has as many rows as A and as many columns as B.

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

Matrix multiplic

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & \\ & \\ & \end{bmatrix}$$

$$1 \times 1 + 2 \times 2 + 1 \times 1 + 2 \times 0 = 6$$

Matrix multipli

$$
\begin{bmatrix} -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}
=
\begin{bmatrix} 0 & \end{bmatrix}
$$

$$-1 \times 1 + 0 \times 2 + 1 \times 1 + 3 \times 0 = 0$$

Matrix multiplic

$$\begin{bmatrix} \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \\ 8 & \end{bmatrix}$$

$$2 \times 1 + 3 \times 2 + 0 \times 1 + 1 \times 0 = 8$$

Matrix multipli

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 0 & \\ 8 & \end{bmatrix}$$

$$1 \times 0 + 2 \times 1 + 1 \times 2 + 2 \times 3 = 10$$

Matrix multip

$$\begin{bmatrix} & & & \\ & & & \\ -1 & 0 & 1 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & \\ 0 & 11 \\ 8 & \end{bmatrix}$$

$$-1 \times 0 + 0 \times 1 + 1 \times 2 + 3 \times 3 = 11$$

Matrix multipli

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & \\ 0 & \\ 8 & 6 \end{bmatrix}$$

$$2 \times 0 + 3 \times 1 + 0 \times 2 + 1 \times 3 = 6$$

Matrix multiplication

Another way to see it:

Write the second matrix as a "row of columns"

$$B = \left[\ \mathbf{b}_1 \,\middle|\, \mathbf{b}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{b}_n\ \right]$$

Then:

$$AB = A\left[\ \mathbf{b}_1 \,\middle|\, \mathbf{b}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{b}_n\ \right] = \left[\ A\mathbf{b}_1 \,\middle|\, A\mathbf{b}_2 \,\middle|\, \cdots \,\middle|\, A\mathbf{b}_n\ \right]$$

The matrix-vector product is a special case.

Try it yourself

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = ?$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = ?$$

Matrix multiplication

The matrix product is associative, distributes over matrix addition, but is not generally commutative.

Indeed, the dimensions of A and B can be such that AB makes sense but BA does not; and we saw on the last slide that even if they both make sense, they need not be equal.

Try it yourself

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \quad ?$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \quad ?$$

The identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$$

The identity matrix

We write I_n for the matrix with 1's along the diagonal, and zeroes everywhere else.

$$I_1 = [1], \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If $I_n \cdot M$ is defined, i.e., M has n rows, then

$$I_n \cdot M = M$$

If $M \cdot I_m$ is defined, i.e., M has m columns, then

$$M \cdot I_m = M$$

The identity matrix

We write I_n for the matrix with 1's along the diagonal, and zeroes everywhere else.

$$I_1 = [1], \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the identity matrix I_n is the unique reduced echelon $n \times n$ matrix with a pivot in every row (or equivalently, every column).

Try it yourself

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = ?$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = ?$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = ?$$

Try it yourself

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 11 & 13 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix}$$

Elementary row operations

You can do an elementary row operation by multiplying on the left by the matrix which is obtained by performing that row operation on the identity matrix.

Try it yourself!

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = ?$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = ?$$

Try it yourself!

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix inversion

If A is a matrix, we say A is invertible if there is some other matrix B such that BA and AB are both identity matrices.

In this case we say that B is the inverse of A, and write it as A^{-1}.

The inverse is unique if it exists: if $BA = I$ and $AC = I$ then

$$B = BI = B(AC) = (BA)C = IC = C$$

Try it yourself!

Is the identity matrix invertible? yes

$$I_n \cdot I_n = I_n$$

Matrix inversion

If A is an invertible matrix, then the following equations are equivalent:

$$A\mathbf{x} = \mathbf{b} \qquad\qquad \mathbf{x} = A^{-1}\mathbf{b}$$

In particular,

$A\mathbf{x} = \mathbf{0}$ has only the zero solution $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$

For any $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$.

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pivot in every column	pivot in every row
columns linearly independent	rows linearly independent
rows span all of $\mathbb{R}^c$	columns span all of $\mathbb{R}^r$
one-to-one	onto
(can only happen if $c \leq r$)	(can only happen if $r \leq c$)

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.

Matrix inversion

If A is an invertible matrix, then A is square, and all the conditions on the previous slide hold.

Conversely, for a square matrix, being invertible is equivalent to any one of these conditions.

Calculating the inverse

In particular, row-reducing an invertible matrix to reduced row-echelon form gives the identity matrix.

This leads to an algorithm for calculating the inverse.

Calculating the inverse

Row reduction is implemented by elementary matrices, so if M is invertible — hence can be row reduced to the identity — there exist some elementary matrices, $E_1, \dots, E_k$ such that

$$E_k \cdots E_2 E_1 M = I$$

Multiplying by M^{-1} on both sides, (or recalling that the inverse was unique)

$$E_k \cdots E_2 E_1 = M^{-1}$$

Calculating the inverse

The equations

$$E_k \cdots E_2 E_1 \cdot M = I \qquad\qquad E_k \cdots E_2 E_1 \cdot I = M^{-1}$$

can be combined: putting the matrices M and I next to each other,

$$E_k \cdots E_2 E_1 \cdot [\, M \mid I \,] = [\, I \mid M^{-1} \,]$$

Calculating the inverse

Now remember what E_i do: they are row operations. Thus,

$$E_k \cdots E_2 E_1 \cdot [\, M \mid I \,] = [\, I \mid M^{-1} \,]$$

is simply asserting that $[\, I \mid M^{-1} \,]$ is obtained from $[\, M \mid I \,]$ by row reduction!

Calculating the inverse

To find the inverse of M,

- Form the matrix $[M|I]$
- Row reduce it
- If the result has the form $[I|X]$ then $X = M^{-1}$
- If not, M was not invertible (not enough pivots).

Calculating the inverse

Find the inverse of $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

First put it next to the identity in a matrix.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

Calculating the inverse

Row reduce this matrix.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Read off the inverse from the right of the matrix:

$$\left[\begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array}\right]^{-1} = \left[\begin{array}{cc} 3 & -2 \\ -1 & 1 \end{array}\right]$$

Try it yourself!

Find inverses for the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1} = ?$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}^{-1} = ?$$

Another way to think about calculating the inverse

The columns of the matrix A^{-1} are $A^{-1}\mathbf{e}_1, A^{-1}\mathbf{e}_2, \ldots, A^{-1}\mathbf{e}_n$.

These are the solutions to the equations $A\mathbf{x} = \mathbf{e}_1$, $A\mathbf{x} = \mathbf{e}_2$, $\ldots$

To find these solutions, we would row reduce the augmented matrixes $[A|\mathbf{e}_1]$, $[A|\mathbf{e}_2]$, $\ldots$

Do them all at once by row reducing the matrix $[A|\mathbf{e}_1|\mathbf{e}_2|\cdots|\mathbf{e}_n]$

$[\mathbf{e}_1|\mathbf{e}_2|\cdots|\mathbf{e}_n]$ is just the identity matrix, so row reduce $[A|I]$.

The inverse of a 2×2 matrix

Consider an arbitrary 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d - cb/a & -c/a & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -c/(ad - bc) & a/(ad - bc) \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & d/(ad - bc) & -b/(ad - bc) \\ 0 & 1 & -c/(ad - bc) & a/(ad - bc) \end{array}\right]$$

The inverse of a 2×2 matrix

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse if (and in fact only if) $ad - bc \neq 0$, and in this case its inverse is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called the discriminant.