Hello and welcome to class!

Last time

We discussed the matrix-vector product and corresponding formulation of linear equations. We also introduced the notions of linear dependence and linear independence.

Today

We'll see many equivalent conditions to the linear independence of the rows or columns of a matrix. Then we'll study linear transformations and the matrices which represent them.

Review from last class

Matrices multiply vectors

$$
r\left\{ \overbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix}}^{\mathrm{c}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} \right\} c = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1c}x_c \\ a_{21}x_1 + \cdots + a_{2c}x_c \\ \vdots \\ a_{r1}x_1 + \cdots + a_{rc}x_c \end{bmatrix} \right\} r
$$

and thereby define functions

$$
\begin{aligned} A : \mathbb{R}^c &\rightarrow \mathbb{R}^r \\ \mathbf{x} &\mapsto A\mathbf{x} \end{aligned}
$$

When does $A\mathbf{x} = \mathbf{b}$ have solutions for any $\mathbf{b}$?

In terms of the row reduced matrix

When every row of A has a pivot — as we saw last time, $A\mathbf{x} = \mathbf{b}$ has a solution exactly when the augmented matrix $[A|\mathbf{b}]$ has no pivots in the last column.

If there is already a pivot in every row of A, there can't be a pivot in the final column.

When does $A\mathbf{x} = \mathbf{b}$ have solutions for any $\mathbf{b}$?

In terms of the rows

When the rows are linearly independent.

Recall this means that no nonzero linear combination of the rows of A is zero.

Indeed, if there were such an expression, then by row operations, a row of the form $[0\,0\,\cdots\,0\,|1]$ can be created in the augmented matrix for some choice of $\mathbf{b}$.

When does $A\mathbf{x} = \mathbf{b}$ have solutions for any $\mathbf{b}$?

In terms of the columns

When the columns of A span the entire space.

Recall that solving $A\mathbf{x} = \mathbf{b}$ means expressing $\mathbf{b}$ as a linear combination of the columns of A.

When does $A\mathbf{x} = \mathbf{b}$ have solutions for any $\mathbf{b}$?

In terms of the associated function

When the function determined by the matrix A is onto — it hits every point in $\mathbb{R}^r$. Indeed, solving $A\mathbf{x} = \mathbf{b}$ means finding a point which maps to $\mathbf{b}$, and if every point is hit by the map, then this can always be done.

When does $A\mathbf{x} = \mathbf{b}$ have solutions for any $\mathbf{b}$?

If A has r rows and c columns, the following are equivalent

- $A\mathbf{x} = \mathbf{b}$ has solutions for any $\mathbf{b}$
- The matrix A has a pivot in every row
- The rows of A are linearly independent
- The columns of A span all of $\mathbb{R}^r$
- The function corresponding to A hits all of $\mathbb{R}^r$.

The zero solution

Homogeneous equations always have the zero solution.

$A\mathbf{x} = 0$ is always solved by $\mathbf{x} = 0$.

Inhomogenous equations do not. An inhomogenous equation need have no solutions at all.

When does $A\mathbf{x} = 0$ have only the zero solution?

In terms of the row reduced matrix

When every column of A has a pivot

As we saw last time, this means we get to introduce zero free parameters. Thus there is at most one solution. The zero solution is a solution, and there are no others.

When does $A\mathbf{x} = 0$ have only the zero solution?

In terms of the columns

When the columns are linearly independent.

A solution to $A\mathbf{x} = 0$ is a way of writing 0 as a linear combination of the columns of A. If this equations has only the zero solution that means the only way of doing this is to have all the coefficients be zero which is the definition of linear independence of the columns of A.

When does $A\mathbf{x} = 0$ have only the zero solution?

In terms of the rows

When the rows span.

I'll let you think about this one. Hint: every column has a pivot.

When does $A\mathbf{x} = 0$ have only the zero solution?

In terms of the associated function

When the function determined by the matrix A is one-to-one — no two distinct points in $\mathbb{R}^c$ are mapped to the same point in $\mathbb{R}^r$.

Indeed, if two points $\mathbf{x}, \mathbf{y}$ are sent to the same point, $A\mathbf{x} = A\mathbf{y}$, then we have $A(\mathbf{x} - \mathbf{y}) = 0$. So if zero is the only solution, then $\mathbf{x} - \mathbf{y} = 0$, or in other words, $\mathbf{x} = \mathbf{y}$. So the only way two points can be sent to the same point is if they were the same point to begin with.

When does $A\mathbf{x} = \mathbf{b}$ have solutions for any $\mathbf{b}$?

If A has r rows and c columns, the following are equivalent

- $A\mathbf{x} = 0$ has only the zero solution.
- The matrix A has a pivot in every column
- The columns of A are linearly independent
- The rows of A span all of $\mathbb{R}^c$
- The function corresponding to A carries distinct points to distinct points.

A has r rows and c columns.

$A\mathbf{x} = 0$ implies $\mathbf{x} = 0$	$A\mathbf{x} = \mathbf{b}$ has solutions for any $\mathbf{b}$
pivot in every column	pivot in every row
columns linearly independent	rows linearly independent
rows span all of $\mathbb{R}^c$	columns span all of $\mathbb{R}^r$
distinct points to distinct points	hits all of $\mathbb{R}^r$.

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.

Span and linear independence

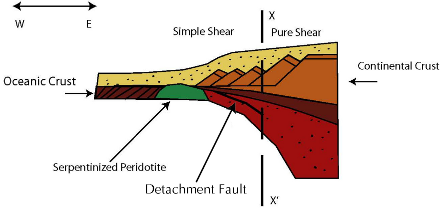
A collection of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{R}^n$ spans if every vector in $\mathbb{R}^n$ can be written as a linear combination of the $\mathbf{v}_i$.

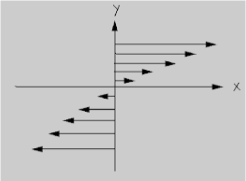
A collection of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{R}^n$ is linearly independent if, whenever $a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = 0$, then all the a_i are zero.

A collection consisting of a single vector is linearly independent so long as it's not the zero vector, and two vectors are linearly independent as long as one isn't a multiple of the other.

Pictures of linear transformations

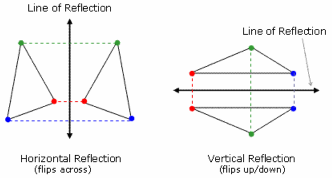
Shear





Pictures of linear transformations

Reflection



Rotation

Pictures of linear transformations

Rotation

Pictures of linear transformations

Rotation

Pictures of linear transformations

Rotation

Pictures of linear transformations

Rotation

Pictures of linear transformations

Rotation

Pictures of linear transformations

Rotation

Linear transformations

Geometrically, linear transformations take lines to lines. Our definitions will also be such that they preserve the origin.

These two properties characterize linear transformations (assuming you know what a line is), but we will prefer the following algebraic definition.

Linear transformations

Definition

A linear transformation is a function $T : \mathbb{R}^c \to \mathbb{R}^r$ such that

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

Reminder about functions

Given two sets X and Y, a function $f : X \to Y$ gives some element $f(x)$ of Y for every element x of X.

We say that the domain of the function is X, and that the codomain is Y.

The range is the subset of Y consisting of elements of the form $f(x)$ for some x in X.

The function is said to be one-to-one if no two elements of X map to the same element of Y, and is said to be onto if every element of Y is hit, i.e., the range and codomain are equal.

A has r rows and c columns; $A : \mathbb{R}^c \to \mathbb{R}^r$

Columns below have equivalent conditions (except in parethesis)

$A\mathbf{x} = 0$ implies $\mathbf{x} = 0$	$A\mathbf{x} = \mathbf{b}$ has solutions for any $\mathbf{b}$
pivot in every column	pivot in every row
columns linearly independent	rows linearly independent
rows span all of $\mathbb{R}^c$	columns span all of $\mathbb{R}^r$
one-to-one	onto
(can only happen if $c \leq r$)	(can only happen if $r \leq c$)

If A is square, i.e. $r = c$, there's a pivot in every row if and only if there's a pivot in every column so these are all equivalent.

A (nonlinear) function example

We could define a function

$$\begin{aligned} t : \text{this room} &\rightarrow \mathbb{R} \\ \text{each point} &\rightarrow \text{the temperature there} \end{aligned}$$

The domain is this room, the codomain is $\mathbb{R}$, and the range is some subset of the interval $(60^\circ F, 110^\circ F)$. The function is not onto or one-to-one.

Linear transformations

Definition

A linear transformation is a function $T : \mathbb{R}^c \to \mathbb{R}^r$ such that

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

Note that $\mathbb{R}^c$ is the domain and $\mathbb{R}^r$ is the codomain.

The range and in particular if the function is onto, and whether the function is one-to-one, depend on the details of T.

Example

Consider the following matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

It determines a linear transformation by the formula

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x + 3y \\ x + 4y \end{bmatrix}$$

Example

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x + 3y \\ x + 4y \end{bmatrix}$$

has domain $\mathbb{R}^2$ and codomain $\mathbb{R}^3$. Its range is

$$\mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$$

The columns are linearly independent, so the linear transformation is one-to-one. The columns don't span, so it's not onto.

Linear transformations

Definition

A linear transformation is a function $T : \mathbb{R}^c \to \mathbb{R}^r$ such that

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

Example

The zero function $T(\mathbf{x}) = \mathbf{0}$ for all x is linear, since

$$T(a\mathbf{v} + b\mathbf{w}) = \mathbf{0} = \mathbf{0} + \mathbf{0} = a\mathbf{0} + b\mathbf{0} = aT(\mathbf{v}) + bT(\mathbf{w})$$

Linear transformations

Definition

A linear transformation is a function $T : \mathbb{R}^c \to \mathbb{R}^r$ such that

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

Example

If A is a matrix with r rows and c columns, then we saw last time that the following function is linear.

$$\begin{aligned} A : \mathbb{R}^c &\to \mathbb{R}^r \\ \mathbf{x} &\mapsto A\mathbf{x} \end{aligned}$$

Linear transformations

Definition

A linear transformation is a function $T : \mathbb{R}^c \to \mathbb{R}^r$ such that

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

Nonexample

The function $f(x) = x^2$ is not linear. Indeed

$$f(1+1) = (1+1)^2 = 4 \neq 2 = 1^2 + 1^2 = f(1) + f(1)$$

Try it yourself

Is it a linear transformation?

$f(x) = 0$ yes

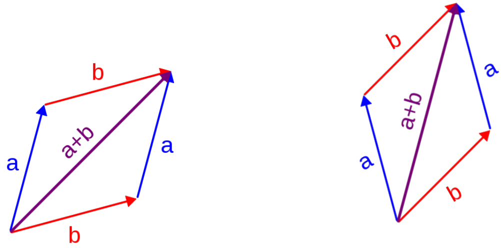
$f(x) = 2$ no

$f(x, y) = (x + 2y, y + 3x)$ yes

$f(x, y) = xy$ no

$f(x, y, z) = x + y + z$ yes

Rotation is a linear transformation

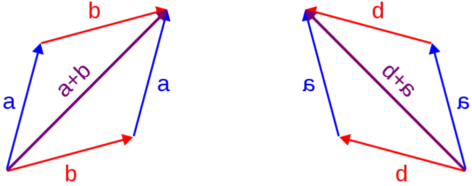


Because the sum of the rotated vectors is the rotation of the sum of the vectors, i.e.,

$$\mathrm{Rotate}(\mathbf{a}+\mathbf{b}) = \mathrm{Rotate}(\mathbf{a}) + \mathrm{Rotate}(\mathbf{b})$$

Geometrically, rotation preserves the rule for adding vectors.

Reflection is a linear transformation

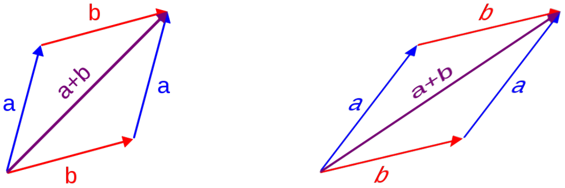


Because the sum of the rotated vectors is the rotation of the sum of the vectors, i.e.,

$$\text{Reflect}(\mathbf{a} + \mathbf{b}) = \text{Reflect}(\mathbf{a}) + \text{Reflect}(\mathbf{b})$$

Geometrically, reflection preserves the rule for adding vectors.

Shear is a linear transformation

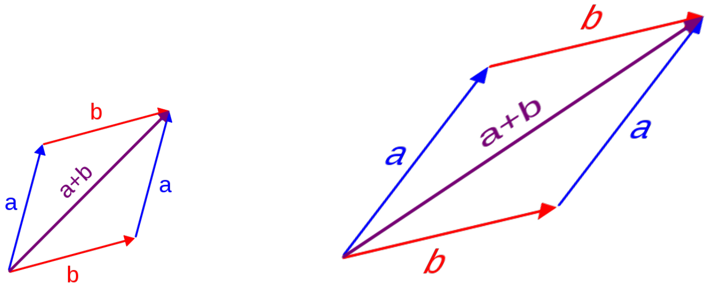


Because the sum of the sheared vectors is the shear of the sum of the vectors, i.e.,

$$\mathrm{Shear}(\mathbf{a} + \mathbf{b}) = \mathrm{Shear}(\mathbf{a}) + \mathrm{Shear}(\mathbf{b})$$

Geometrically, shearing preserves the rule for adding vectors.

Rescaling is a linear transformation



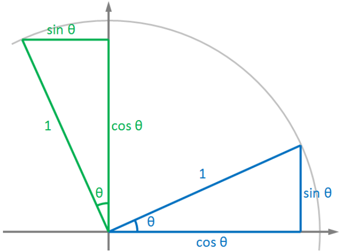
Because the sum of the scaled vectors is the scaling of the sum of the vectors, i.e.,

$$\mathrm{Scale}(\mathbf{a}+\mathbf{b}) = \mathrm{Scale}(\mathbf{a}) + \mathrm{Scale}(\mathbf{b})$$

Geometrically, scaling preserves the rule for adding vectors.

The matrix of rotation

Consider rotation of the plane by the angle θ



It takes $(1, 0)$ to $(\cos\theta, \sin\theta)$ and $(0, 1)$ to $(-\sin\theta, \cos\theta)$. A matrix which does the same is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Are these the same linear transformation?

Classifying linear transformations

Warmup

Describe all linear transformations from $\mathbb{R}$ to $\mathbb{R}$.

Suppose $T : \mathbb{R} \to \mathbb{R}$ is a linear transformation. Then $T(1)$ has some value t (in $\mathbb{R}$). If you want to know what $T(c)$ is for any other c, you can write

$$T(c) = T(c \cdot 1) = cT(1) = ct$$

Moreover the function defined by $T(c) = ct$ is linear, since

$$\begin{aligned} T(cv + dw) &= (cv + dw)t = cvt + dwt = c(vt) + d(wt) \\ &= cT(v) + dT(w) \end{aligned}$$

Thus, the linear functions from $\mathbb{R} \to \mathbb{R}$ are exactly those functions of the form $T(c) = ct$ for some $t \in \mathbb{R}$.

Classifying linear transformations

Warmup, II

Describe all linear transformations from $\mathbb{R}$ to $\mathbb{R}^n$.

Suppose $T : \mathbb{R} \to \mathbb{R}^n$ is a linear transformation. Then $T(1)$ has some value $\mathbf{t}$ (in $\mathbb{R}^n$). If you want to know what $T(c)$ is for any other c, you can write

$$T(c) = T(c \cdot 1) = cT(1) = c\mathbf{t}$$

Moreover the function defined by $T(c) = c\mathbf{t}$ is linear, since

$$T(cv+dw) = (cv+dw)\mathbf{t} = cv\mathbf{t}+dw\mathbf{t} = c(v\mathbf{t})+d(w\mathbf{t}) = cT(v)+dT(w)$$

Thus, the linear functions from $\mathbb{R} \to \mathbb{R}^n$ are exactly those functions of the form $T(c) = c\mathbf{t}$ for some $\mathbf{t} \in \mathbb{R}^n$.

The matrix of a linear transformation

For linear maps $T : \mathbb{R}^m \to \mathbb{R}^n$, we can't do the same thing, since it's no longer true that every vector in $\mathbb{R}^m$ is a scalar multiple of some given vector.

But, every vector is a linear combination of the $\mathbf{e}_i$.

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} &= v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_m \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= v_1\mathbf{e}_1 + \cdots + v_m\mathbf{e}_m \end{aligned}$$

The matrix of a linear transformation

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}\right) = v_1 T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + \cdots + v_m T\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right)$$

$$= v_1 T(\mathbf{e}_1) + \cdots + v_m T(\mathbf{e}_m)$$

$$= \left[\begin{array}{c|c|c|c} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_m) \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

The matrix of a linear transformation

Thus the linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is the linear transformation associated to the matrix

$$\left[\; T(\mathbf{e}_1) \;\middle|\; T(\mathbf{e}_2) \;\middle|\; \cdots \;\middle|\; T(\mathbf{e}_m) \;\right]$$

whose columns are the $T(\mathbf{e}_i)$.

Example

The matrix of the linear transformation

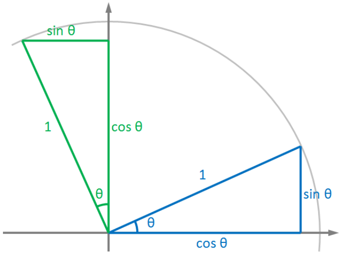
$$f(x, y) = (3x + 5y, 2x + 4y, x + 2y)$$

We are supposed to evaluate $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$ and stick them in as the columns of the matrix. We have $f(1, 0) = (3, 2, 1)$ and $f(0, 1) = (5, 4, 2)$, so the matrix is

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 5y \\ 2x + 4y \\ x + 2y \end{bmatrix}$$

The matrix of rotation

Consider rotation of the plane by the angle θ



It takes $(1,0)$ to $(\cos\theta, \sin\theta)$ and $(0,1)$ to $(-\sin\theta, \cos\theta)$. The matrix which does the same is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Try it yourself!

Find the matrix which performs the reflection in the x axis in 2 dimensions.

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Composing linear transformations.

If $S : \mathbb{R}^a \to \mathbb{R}^b$ and $T : \mathbb{R}^b \to \mathbb{R}^c$ are linear transformations, then so is their composition $T \circ S : \mathbb{R}^a \to \mathbb{R}^c$.

Indeed,

$$\begin{aligned}
(T \circ S)(c\mathbf{v} + d\mathbf{w}) &= T(S(c\mathbf{v} + d\mathbf{w})) \\
&= T(cS(\mathbf{v}) + dS(\mathbf{w})) \\
&= cT(S(\mathbf{v})) + dT(S(\mathbf{w})) \\
&= c(T \circ S)(\mathbf{v}) + d(T \circ S)(\mathbf{w})
\end{aligned}$$

The transformations S, T, and $T \circ S$ are each determined by a matrix. What's the relation between these matrices?