Hello and welcome to class!

Last time

We learned more about row reduction, and interpreted linear equations in terms of the linear span of vectors.

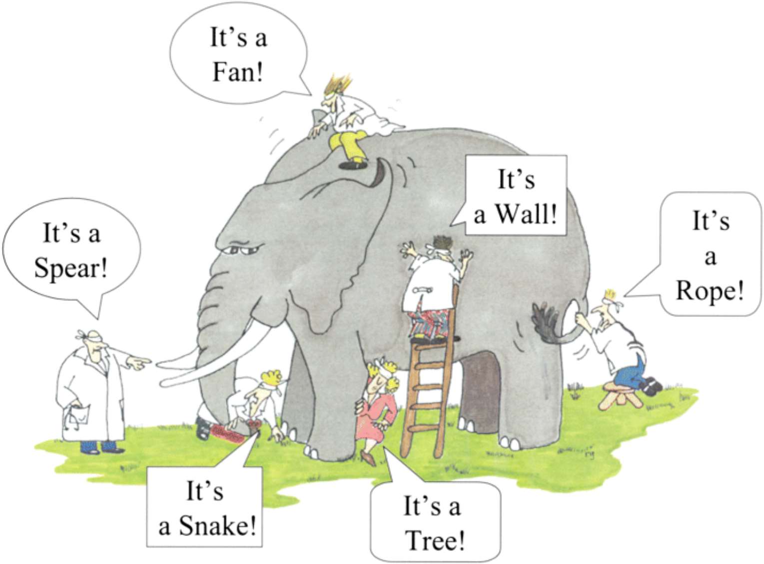
Today

I'll talk about the matrix-vector product, and how linear equations can be formulated in these terms.

I'll then explain more about the algebraic and geometric description of solution sets to linear equations.

Finally, I'll start explaining the fundamental notions of linear dependence and independence.

A parable



Systems of linear equations

$$\begin{aligned} 3w + 5x + 4y + 3z &= 2 \\ 2w + x + y + z &= 6 \\ w + x - y + z &= -3 \end{aligned}$$

can be abbreviated via an augmented matrix

$$\left[\begin{array}{cccc|c} 3 & 5 & 4 & 3 & 2 \\ 2 & 1 & 1 & 1 & 6 \\ 1 & 1 & -1 & 1 & -3 \end{array}\right]$$

or written in vector form as

$$w \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

The matrix-vector product

There is one more way to write systems of linear equations:

$$\begin{aligned} 3w + 5x + 4y + 3z &= 2 \\ 2w + x + y + z &= 6 \\ w + x - y + z &= -3 \end{aligned}$$

can be written as

$$\begin{bmatrix} 3 & 5 & 4 & 3 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

The matrix-vector product

In general, a column vector with c entries (in $\mathbb{R}^c$) can be multiplied on the left by a matrix with r rows and c columns — the matrix has as many columns as the original vector has rows — to obtain a vector with r rows (in $\mathbb{R}^r$) — the new vector will have as many rows as the matrix.

[matrix with r rows and c columns] [vector with c rows]

=

[vector with r rows]

The matrix-vector product

The formula

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1c}x_c \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2c}x_c \\ \vdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rc}x_c \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 3-4 \\ 5-6 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Try it yourself!

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$ These ones are the wrong size.

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 * 3 + (-1) * 4 \\ (-1) * 3 + 2 * 4 \\ 3 * 3 + 2 * 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 17 \end{bmatrix}$$

The matrix-vector product

So the equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}$$

is equivalent to the equation

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1c}x_c \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2c}x_c \\ \vdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rc}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}$$

An example with numbers

$$\begin{bmatrix} 3 & 5 & 4 & 3 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

Expanding the product on the left, we find

$$\begin{bmatrix} 3w + 5x + 4y + 3z \\ 2w + x + y + z \\ w + x - y + z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

or in other words

$$\begin{aligned} 3w + 5x + 4y + 3z &= 2 \\ 2w + x + y + z &= 6 \\ w + x - y + z &= -3 \end{aligned}$$

$A\mathbf{x} = \mathbf{b}$

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}$$

Then the equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}$$

becomes simply $A\mathbf{x} = \mathbf{b}$. Here A is called the matrix of coefficients.

The matrix-vector product

The product $A\mathbf{x}$ can also be written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{r1} \end{bmatrix} + \cdots + x_c \begin{bmatrix} a_{1c} \\ a_{2c} \\ \vdots \\ a_{rc} \end{bmatrix}$$

That is, $A\mathbf{x}$ is a linear combination of the columns of A, with coefficients given by the entries of the vector $\mathbf{x}$.

The equation $A\mathbf{x} = \mathbf{b}$ asserts that the vector $\mathbf{b}$ is equal to this linear combination of the columns of A.

An example with numbers

$$\begin{bmatrix} 3 & 5 & 4 & 3 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

The product on the left means a linear combination of the columns of the matrix, weighted by the entries of the vector.

$$w \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

or in other words

$$\begin{aligned} 3w + 5x + 4y + 3z &= 2 \\ 2w + x + y + z &= 6 \\ w + x - y + z &= -3 \end{aligned}$$

A function from $\mathbb{R}^4$ to $\mathbb{R}^3$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 3 & 5 & 4 & 3 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3w + 5x + 4y + 3z \\ 2w + x + y + z \\ w + x - y + z \end{bmatrix}$$

This map brought to you by the matrix

$$\begin{bmatrix} 3 & 5 & 4 & 3 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

Functions from matrices

If A is a matrix with r rows and c columns, and $\mathbf{x} \in \mathbb{R}^c$ is any vector, then $A\mathbf{x} \in \mathbb{R}^r$.

So the matrix A defines a function (also called A):

$$\begin{aligned} A : \mathbb{R}^c &\to \mathbb{R}^r \\ \mathbf{x} &\mapsto A\mathbf{x} \end{aligned}$$

The equation $A\mathbf{x} = \mathbf{b}$ can be read as: which vectors $\mathbf{x}$ have the property that their image under the function A is the vector $\mathbf{b}$?

Functions from matrices

This is similar to the way that a number a determines a function "multiplication by a"

$$\begin{aligned} a : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto ax \end{aligned}$$

Indeed, this is the case $n = m = 1$.

Linearity

The matrix-vector product has the following properties:

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= A\mathbf{x}+A\mathbf{y} \\ A(c\mathbf{x}) &= c(A\mathbf{x}) \end{aligned}$$

Functions with these two properties are said to be linear. In fact, every linear function from $\mathbb{R}^c$ to $\mathbb{R}^r$ is multiplication by some matrix. We will see why this is next time.

Thinking about linear equations

We have seen that linear equations can be interpreted as

- Asking for the intersection locus of lines or planes
- Asking how to write one vector as a linear combination of given others
- Asking for vectors mapping to a given one under the linear function associated to the coefficient matrix

We know how to compute the solutions (row reduction).

Now we'll discuss qualitative properties of the solution set.

Homogenous and inhomogenous equations

If $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$, then

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0$$

That is, the difference between two solutions to the inhomogenous equation $A\mathbf{x} = b$ is a solution to the homogenous equation $A\mathbf{x} = 0$.

Differently said, given one solution $\mathbf{x} = \mathbf{x}_0$ to the inhomogenous equation $A\mathbf{x} = \mathbf{b}$, all other solutions are given by the sum of $\mathbf{x}_0$ and a solution to the homogenous equation $A\mathbf{x} = 0$.

Try it yourself

Find all solutions to the homogenous equation $x + y = 0$ and to the inhomogenous equation $x + y = 1$. Graph your answers. How do they compare?

Homogenous equations and linearity

If $A\mathbf{x}_1 = 0$ and $A\mathbf{x}_2 = 0$, then for any scalars c_1, c_2,

$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2 = 0$$

In other words, any linear combination of solutions to a homogenous equation is again a solution.

This is not true for inhomogenous equations! (Try $x = 1$.)

Homogenous equations and linearity

The solution set to a homogenous equation can be described as a linear span.

To do this, as always, the first step is row reducing your system.

You are all expert row reducers now, so I'll skip that step and just start with an already reduced one.

Homogenous equations and linearity

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This has the following augmented matrix:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Homogenous equations and linearity

We read off the solution from the augmented matrix

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Introduce free parameters for the variables whose column has no pivot; s for x_1 and t for x_5. Now we read off the solutions: these are $(s, -t, -t, 0, t)$ for any s, t, or in other words,

$$\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \text{ any } s, t \right\} = \text{Linear Span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Inhomogenous equations

The solution set of the inhomogenous equation $A\mathbf{x} = \mathbf{b}$, if nonempty, is a translate of the solution set of the homogenous equation $A\mathbf{x} = 0$. Indeed, if $A\mathbf{x}_0 = \mathbf{b}$ then for any $\mathbf{x}$ such that $A\mathbf{x} = 0$, we have

$$A(\mathbf{x}_0 + \mathbf{x}) = A\mathbf{x}_0 + A\mathbf{x} = \mathbf{b} + 0 = \mathbf{b}$$

Symbolically,

$$\mathbf{x}_0 + \{\text{solutions of } A\mathbf{x} = 0\} = \{\text{solutions of } A\mathbf{x} = \mathbf{b}\}$$

Try it yourself!

Solve the following three systems of equations.

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Try it yourself!

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has solution set the linear span of $(-2, 1)$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

has the empty solution set

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

has solution set $(2, 0) + \text{Linear span}(-2, 1)$.

Linear dependence and independence

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is said to be linearly dependent if one of the $\mathbf{v}_i$ can be written as a linear combination of the others. Otherwise, it's said to be linearly independent.

An equivalent characterization: the vectors are linearly dependent if there is some collection of scalars a_i, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_k\mathbf{v}_k = \mathbf{0}$$

Example

The vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent.

The zero vector

The set containing only the zero vector, $\{\mathbf{0}\}$ is linearly dependent

$$1 \times \mathbf{0} = \mathbf{0}$$

More generally, any collection of vectors which includes the zero vector is linearly dependent.

$$0 \times \mathbf{v}_1 + 0 \times \mathbf{v}_2 + \cdots 0 \times \mathbf{v}_n + \boxed{1} \times \mathbf{0} = \mathbf{0}$$

Elementary operations do not change linear independence

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependendent. Then so too are any re-arrangement of these vectors and also any rescaling by nonzero vectors.

Also, so are $\mathbf{v}_1, \mathbf{v}_2 + c\mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_n$. Indeed, if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0$$

then so too

$$(a_1 - ca_2)\mathbf{v}_1 + a_2(\mathbf{v}_2 + c\mathbf{v}_1) + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n = 0$$

Moreover, $a_1 - ca_2$ and a_2 are both zero if and only if a_1 and a_2 are both zero.

Row reduction and linear independence

The rows of a reduced echelon matrix are linearly independent if and only if there is no zero row. This is because each pivot sits in a column with only zeros, so any non-zero linear combination of the rows will see one of the pivot entries. So to check if a collection of vectors is linearly dependent or linearly independent make them the rows of a matrix, row reduce, and then look for a zero row!.

Is this true for an echelon matrix?
Think about it!

Try it yourself!

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ linearly dependent}$$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \text{ linearly dependent}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \text{ linearly dependent}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ linearly independent}$$