Hello, and welcome to class!

Last time

We introduced the notion of Fourier series, and discussed how to expand a function into one.

This time

Having developed this tool, we return to studying the heat equation.

Heat equation review

The heat equation in one variable is:

$$\frac{\partial}{\partial t} u(x,t) = \beta \frac{\partial^2}{\partial x^2} u(x,t)$$

We saw that some solutions are given by

$$u(x,t) = e^{\lambda t}(A_\lambda e^{x\sqrt{\lambda/\beta}} + B_\lambda e^{-x\sqrt{\lambda/\beta}}) \qquad \lambda > 0$$

$$u(x,t) = A_\lambda + B_\lambda x \qquad \lambda = 0$$

$$u(x,t) = e^{\lambda t}(A_\lambda \cos(x\sqrt{-\lambda/\beta}) + B_\lambda \sin(x\sqrt{-\lambda/\beta})) \qquad \lambda < 0$$

Heat equation review

Last week, we considered a wire of length L,

whose endpoints were kept at temperature zero.

In other words, we imposed boundary conditions

$$u(0, t) = 0 = u(L, t)$$

Heat equation review

Of our above solutions, the only ones which take this form are the

$$e^{\lambda t} \sin(x\sqrt{-\lambda/\beta}) \qquad \text{when } \sqrt{-\lambda/\beta} = N\pi/L$$

This led to a general solution of the form

$$u(x,t) = \sum_{N=1}^{\infty} c_N e^{-\beta\left(\frac{N\pi}{L}\right)^2 t} \sin\left(\frac{N\pi}{L}x\right)$$

Initial conditions and Fourier expansion

Finally suppose we are given the initial temperature in the form of some function $u(x,0)$. Our job now is to express

$$u(x,0) = \sum_{N=1}^{\infty} c_N \sin\left(\frac{N\pi}{L}x\right)$$

In other words, to find values c_N making the above formula true. Because then the solution will be given by

$$u(x,t) = \sum_{N=1}^{\infty} c_N e^{-\beta\left(\frac{N\pi}{L}\right)^2 t} \sin\left(\frac{N\pi}{L}x\right)$$

Initial conditions and Fourier expansion

The expression

$$u(x,0) = \sum_{N=1}^{\infty} c_N \sin\left(\frac{N\pi}{L}x\right)$$

looks much like a Fourier expansion.

Two differences from last time: first, the function is only defined on the interval $[0, L]$, and second, we want to expand it only in sin rather than in sin and cos.

Fourier series review

The Fourier series of a function $f(x)$ defined on $[-L, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Initial conditions and Fourier expansion

To absorb these differences, we extend the function f to $[-L, L]$ simply by defining $f(-x) = -f(x)$.

This has the virtue of ensuring that the extension is an odd function, which therefore has a Fourier expansion consisting only of $\sin \frac{n\pi x}{L}$ waves, exactly as we wanted.

In interpreting the answer, we just ignore the value of the function on $[-L, 0]$.

Example

Consider a wire of length L and diffusivity β in which the initial temperature is described by the function

$$u(x) = \begin{cases} x & 0 \leq x \leq L/2 \\ L - x & L/2 \leq x \leq L \end{cases}$$

and in which the temperature at the endpoints is kept at zero.

Let us determine the temperature in the wire as a function of time.

First, we should expand out $u(x)$ into its sin Fourier series.

Example

That is, we want to compute

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{L}\left(\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right) \\
&= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left(\int_0^{n\pi/2} u\sin u\,du + \int_{n\pi/2}^{\pi}(n\pi-u)\sin u\,du\right)
\end{aligned}
$$

Noting $\int u\sin u du = \sin u - u\cos u$, this is

$$
\frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left(\left[\sin u - u\cos u\right|_0^{n\pi/2} - \left[\sin u - u\cos u\right|_{n\pi/2}^{n\pi} - \left[n\pi\cos u\right|_{n\pi/2}^{n\pi}\right)
$$

Example

$$\frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left(\left[\sin u - u\cos u\right|_0^{n\pi/2} - \left[\sin u - u\cos u\right|_{n\pi/2}^{n\pi} - \left[n\pi\cos u\right|_{n\pi/2}^{n\pi}\right)$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left(2\sin\frac{n\pi}{2}\right) = \frac{4L}{(n\pi)^2}\sin\frac{n\pi}{2}$$

Thus the Fourier expansion of the original function $u(x)$ is

$$u(x) = \frac{4L}{\pi^2}\left(\sin(x) - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \frac{1}{49}\sin 7x + \cdots\right)$$

Example

And finally, having written the initial condition as

$$u(x) = \frac{4L}{\pi^2}\left(\sin(x) - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \frac{1}{49}\sin 7x + \cdots\right)$$

we see that the time evolution is given by

$$u(x,t) = \frac{4L}{\pi^2}\left(e^{-\beta\left(\frac{\pi}{L}\right)^2 t}\sin(x) - \frac{1}{9}e^{-\beta\left(\frac{3\pi}{L}\right)^2 t}\sin 3x + \frac{1}{25}e^{-\beta\left(\frac{\pi}{5L}\right)^2 t}\sin 5x - \cdots\right)$$

Example

Another sort of boundary condition we might impose is, instead of fixing the initial and final temperatures to be zero, that the wire is insulated, i.e., the derivatives $\frac{\partial}{\partial x}u(x,t)$ vanish identically at $0, L$.

Let's revisit our possible solutions.

$$u(x,t) = e^{\lambda t}(A_\lambda e^{x\sqrt{\lambda/\beta}} + B_\lambda e^{-x\sqrt{\lambda/\beta}}) \qquad \lambda > 0$$

$$u(x,t) = A_\lambda + B_\lambda x \qquad \lambda = 0$$

$$u(x,t) = e^{\lambda t}(A_\lambda \cos(x\sqrt{-\lambda/\beta}) + B_\lambda \sin(x\sqrt{-\lambda/\beta})) \qquad \lambda < 0$$

Try it yourself: which satisfy the boundary conditions?

Example

Yet a third scenario: we could consider ask that at 0 and at L, the temperature is fixed to be some given constants U_1 and U_2, possibly nonzero.

Again, we should look at our possible solutions,

$$u(x,t) = e^{\lambda t}(A_\lambda e^{x\sqrt{\lambda/\beta}} + B_\lambda e^{-x\sqrt{\lambda/\beta}}) \qquad \lambda > 0$$

$$u(x,t) = A_\lambda + B_\lambda x \qquad \lambda = 0$$

$$u(x,t) = e^{\lambda t}(A_\lambda \cos(x\sqrt{-\lambda/\beta}) + B_\lambda \sin(x\sqrt{-\lambda/\beta})) \qquad \lambda < 0$$

Try it yourself: which satisfy the boundary conditions?