Hello and welcome to class!

Last time

We looked at the heat and wave equations and found that, at least for initial conditions which can be decomposed into sums of sines, we could describe a solution.

This time

We learn that all functions can be decomposed into sines and cosines.

Periodicity

A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic with period T if

$$f(x + T) = f(x) \qquad \text{for all } x$$

Periodicity

Some functions which are periodic with period 2π:

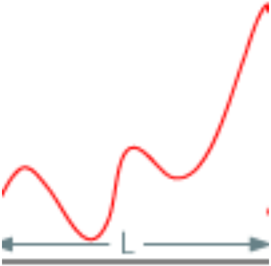
$$\sin(x), \qquad \cos(x), \qquad \sin(x) + \cos(x), \qquad \sin(x + 3).$$

They are also periodic with period 4π. They are also periodic with period 6π. They are also periodic with period 8π.

We say that 2π is the fundamental period of these functions.

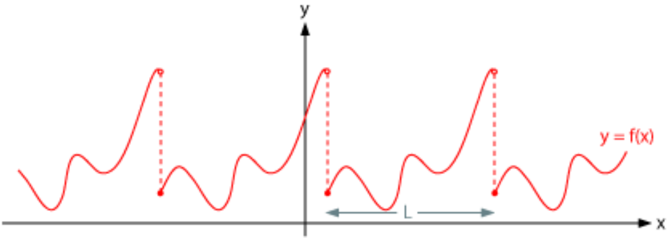
Periodic functions?

Often, we are interested in the behavior of some finite region of space, e.g. $[0, L]$. Perhaps the most natural thing to do would be to consider functions that are only defined on this region.



Periodic functions?

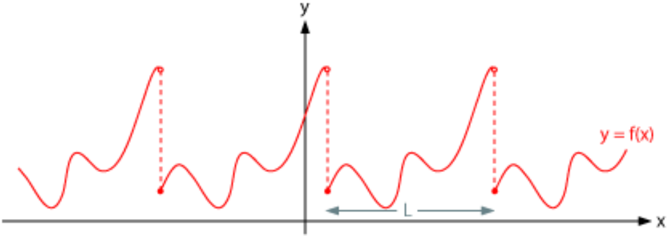
We will see it is more convenient to instead consider functions which are defined on all of $\mathbb{R}$ but are periodic with period L.



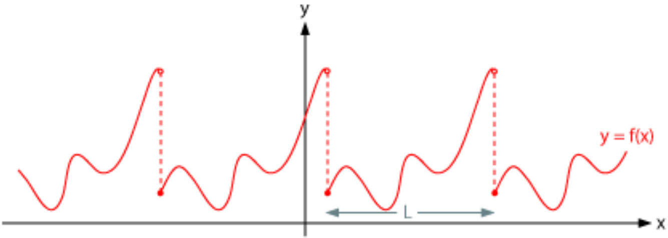
Periodic functions?

That is, if we start out with a function $\overline{f}$ with domain $[0, L]$, we can get a function with domain $\mathbb{R}$ by setting

$$f(x) := f(x \pm \text{whatever multiple of } L \text{ is required to put it in } [0, L])$$



Periodic functions?



Note the result can be discontinuous. That's ok, we'll allow ourselves functions with finitely many discontinuities.

Fourier series

Given a periodic function, let us say with period $2L$,

We will try and express it as a sum of the periodic functions we know with period $2L$,

Namely, $\sin(\frac{n\pi}{L}x)$ and $\cos(\frac{n\pi}{L}x)$ for integers $n > 0$, and the constant function.

Such an expression is called a Fourier series.

Fourier series

Intuitively, the idea is that, from very far away, the graph of a periodic function just looks like a straight line, some constant function. Subtracting off this constant and zooming in, we see oscillations at some characteristic frequency. The simplest such oscillations look like sin and cos waves; we estimate the function by these, subtract this off, zoom in further, and repeat the process.

Orthogonality

The above description of iteratively subtracting off successive approximations may remind you of taking orthogonal projections.

Thus our first step is finding an inner product on the space of functions with respect to which $\sin(\frac{n\pi}{L}x)$ and $\cos(\frac{n\pi}{L}x)$ and the constant function are orthogonal.

Fortunately, we do not have to look very hard.

Orthogonality

Theorem

The functions $\sin(\frac{n\pi}{L}x)$*, and* $\cos(\frac{n\pi}{L}x)$ *(for integers* $n > 0$*), and the constant function are orthogonal with respect to the inner product*

$$\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)dx$$

Their lengths-squared are

$$\langle \sin(\frac{n\pi}{L}x), \sin(\frac{n\pi}{L}x)\rangle = \int_{-L}^{L} \sin(\frac{n\pi}{L}x)\sin(\frac{n\pi}{L}x)dx = L$$

$$\langle \cos(\frac{n\pi}{L}x), \cos(\frac{n\pi}{L}x)\rangle = \int_{-L}^{L} \cos(\frac{n\pi}{L}x)\cos(\frac{n\pi}{L}x)dx = L$$

$$\langle 1, 1\rangle = \int_{-L}^{L} dx = 2L$$

Orthogonality

Let's check some of the assertions of this theorem.

For instance, it is saying

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = 0$$

This is true because the integrand is the product of an even function with an odd function, hence odd and we are integrating it over a region symmetric under $x \to -x$.

Orthogonality

It also is saying, when $n \neq m$,

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = 0$$

The identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, converts the above into the (more visibly true) formula

$$\frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(n+m)\pi}{L}x\right) = 0$$

Note that when $n = m$, the first of the above cos is $\cos(0) = 1$, so the integral yields L, as asserted by the theorem.

Orthogonal projections

Recall that, in an inner product space, the orthogonal projection of a vector $\mathbf{v}$ to the space spanned by the vector $\mathbf{w}$ was given by

$$\left(\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\right) \mathbf{w}$$

And, given an orthogonal set $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$, the orthogonal projection of $\mathbf{v}$ to the space they span is given by

$$\left(\frac{\langle \mathbf{v}, \mathbf{w}_1 \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\right) \mathbf{w}_1 + \left(\frac{\langle \mathbf{v}, \mathbf{w}_2 \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\right) \mathbf{w}_2 + \cdots + \left(\frac{\langle \mathbf{v}, \mathbf{w}_n \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\right) \mathbf{w}_n$$

If the $\mathbf{w}_i$ were a basis, then this is the expression of $\mathbf{v}$ in that basis.

Fourier series

Now we apply that to the orthogonal set we have just found. If f is a function with domain $[-L, L]$, then its Fourier series is:

$$\left(\frac{\int_{-L}^{L} f(x)dx}{2L}\right) \cdot 1 + \sum_{n=1}^{\infty} \left(\frac{\int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L}x\right) dx}{L}\right) \cos\left(\frac{n\pi}{L}x\right)$$

$$+ \sum_{n=1}^{\infty} \left(\frac{\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx}{L}\right) \sin\left(\frac{n\pi}{L}x\right)$$

Note the complicated expressions in parenthesis are just numbers, and are the analogues of the $\left(\frac{\langle \mathbf{v}, \mathbf{w}_i \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\right)$.

Fourier series

Said with less symbols on each line, the Fourier series of a function is an expression of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Fourier series

If we had been working in a finite dimensional vector space, and if it had been true that the functions $\sin \frac{n\pi x}{L}$, $\cos \frac{n\pi x}{L}$, plus the constant function gave a basis, then, by orthogonality, the Fourier series would be the expansion in this basis.

In particular, a function would be equal to its Fourier series.

In fact, this is true here as well, in a certain sense and under appropriate conditions.

Fourier series

Theorem

If f is piecewise continuous on $[-L, L]$, then $\int_{-L}^{x} f(x)dx$ can be computed termwise from the Fourier series.

If in addition f' is piecewise continuous, then the Fourier series converges pointwise to f away from the discontinuities.

If in fact f is continuous (and takes the same value at $\pm L$), and f' is piecewise continuous, then the convergence is uniform.

Finally, if f is continuous (and takes the same value at $\pm L$), and f' and f'' are both piecewise continuous, then the Fourier series for f can be differentiated term-by-term to get the Fourier series of f'.

Computing the Fourier coefficients

Consider the "square wave" function $f(x)$ with period 2π which takes value -1 on $[0, -\pi]$ and value 1 on $[0, \pi]$. Let us determine its Fourier series.

We should compute the coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

First note the coefficients a_n all vanish: $f(x)$ is odd, hence $\cos(x) f(x)$ is odd, hence its integral from $-\pi$ to π is zero.

Computing the Fourier coefficients

As for the b_n, we have

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx &= \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)dx \\ &= -\frac{2}{n\pi}\cos(nx)\Big|_{0}^{\pi} \\ &= -\frac{2}{n\pi}((-1)^n - 1) \end{aligned}$$

Thus the Fourier series is given by

$$f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Computing the Fourier coefficients

Consider the periodic function with period 2π given by $|x|$ in $[-\pi, \pi]$. Let us determine its Fourier series.

We should compute the coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

First note the coefficients b_n all vanish: $f(x)$ is even, hence $\sin(x) f(x)$ is odd, hence its integral from $-\pi$ to π is zero.

Computing the Fourier coefficients

We have $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx = \pi^2$; the other a_n are:

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)dx &= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx \\
&= \frac{2}{\pi}\left(\left[x\frac{\sin(nx)}{n}\right|_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}dx\right) \\
&= \frac{2}{\pi n^2}\left[\cos(nx)\right|_0^{\pi} \\
&= \frac{2}{\pi n^2}\left((-1)^n - 1\right)
\end{aligned}$$

Thus the Fourier series is given by

$$f(x) = \frac{\pi^2}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \cdots\right)$$