Hello and welcome to class!

For some time now

We have been studying linear ordinary differential equations.

This time

We turn to linear partial differential equations. These are equations in which the function for which we are solving may depend on many variables, and we may take derivatives with respect to all of them. Rather than develop a systematic theory, we will mostly focus on a few examples of physical importance.

The heat equation

Let us try and understand how heat flows in a substance.

We will first discuss the one dimensional case.

This means either you should imagine the substance is one dimensional, or close to it, e.g. a very thin wire, or just that the temperature is constant in two of the dimensions, and only varies along the third. We will moreover imagine that the material in question is uniform.

Temperature facts

Imagine that we add energy (heat) to the system. The temperature will change proportionally. That is, per unit time:

$$\text{change in temperature} \sim \frac{\text{change in heat}}{\text{amount of stuff}}$$

Also, as it turns out, having a temperature difference causes heat to flow, proportionally to the difference

$$\frac{\text{heat flow}}{\text{per unit time}} \sim \frac{\text{change in temperature}}{\text{distance to flow}}$$

The heat equation

Let's rewrite things in terms of symbols.

We'll write $u(x,t)$ for the function giving the temperature at location x and time t, and $H(x,t)$ for the heat flowing. Then the first equation

$$\frac{\text{change in temperature}}{\text{per unit time}} \sim \frac{\text{change in heat}}{\text{(amount of stuff)(per unit time)}}$$

says

$$\frac{\delta u(x,t)}{\delta t} \sim \frac{\delta H}{\delta x \delta t}$$

The heat equation

The second equation

$$\frac{\text{heat flow}}{\text{per unit time}} \sim \frac{\text{change in temperature}}{\text{distance to flow}}$$

says

$$\frac{\delta H}{\delta t} \sim \frac{\delta u}{\delta x}$$

Combining this with the previous equation $\frac{\delta u}{\delta t} \sim \frac{\delta H}{\delta x \delta t}$, and replacing difference with derivative, we get the heat equation

$$\frac{\partial u}{\partial t} \sim \frac{\partial^2 u}{\partial x^2}$$

The heat equation

We restore now the proportionality constant. It will depend on various properties of the substance; the heat capacity, the density, the thermal conductivity; but is a positive constant.

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

The analogue of the "initial value problem" in this setting is the following. One should specify the temperature distribution in the wire at some initial time t_0. This is the data of a function in one variable, $u(x, t_0)$. One should also say how the ends of the wire will behave during the whole evolution.

Separation of variables

Recall that we understand linear vector ODE, i.e., equations like

$$\frac{d}{dt}\mathbf{u} = A\mathbf{u}$$

Given a function of two variables like $u(x, t)$, one can specify one of the variables to get a function in the other. I.e., $u(x, 0)$ and $u(x, 1)$ are just functions in x.

So, one way to think of a function of two variables like $u(t, x)$ is as a function of t valued in the vector space of functions in x.

Separation of variables

Now our equation

$$\frac{d}{dt}u(x,t) = \beta \frac{d^2}{dx^2}u(x,t)$$

looks a lot like

$$\frac{d}{dt}\mathbf{u}(t) = A\mathbf{u}(t)$$

where now the vector space V in which $\mathbf{u}(t)$ takes values is replaced by the vector space of functions in x,

and the linear transformation A on V is replaced by the linear transformation $\frac{d^2}{dx^2}$ on the space of functions of x.

Separation of variables

We know how to solve $\frac{d}{dt}\mathbf{u}(t) = A\mathbf{u}(t)$: given an eigenvector $\mathbf{v}$ of A with eigenvalue λ, a solution is given by $\mathbf{v}e^{\lambda t}$.

To solve $\frac{d}{dt}u(x,t) = \beta\frac{d^2}{dx^2}u(x,t)$, we want an eigenvector $v(x)$ of the operator $\beta\frac{d^2}{dx^2}$ (in this context, often called an eigenfunction), with eigenvalue λ, and then a solution will be given by $v(x)e^{\lambda t}$.

Finding an eigenfunction of $\beta\frac{d^2}{dx^2}$ of eigenvalue λ means solving

$$\beta\frac{d^2}{dx^2}v(x) = \lambda v(x)$$

Separation of variables

The general solution of $\beta \frac{d^2}{dx^2} v(x) = \lambda v(x)$ is

$$v(x) = Ae^{x\sqrt{\lambda/\beta}} + Be^{-x\sqrt{\lambda/\beta}}$$

where we will have to replace these by sin and cos in case $\lambda/\beta < 0$, (i.e., since $\beta > 0$, when $\lambda < 0$).

So some solutions to $\frac{d}{dt} u(x,t) = \beta \frac{d^2}{dx^2} u(x,t)$ are given by

$$u(x,t) = e^{\lambda t}(A_\lambda e^{x\sqrt{\lambda/\beta}} + B_\lambda e^{-x\sqrt{\lambda/\beta}}) \qquad \lambda > 0$$

$$u(x,t) = e^{\lambda t}(A_\lambda + B_\lambda x) \qquad \lambda = 0$$

$$u(x,t) = e^{\lambda t}(A_\lambda \cos(x\sqrt{-\lambda/\beta}) + B_\lambda \sin(x\sqrt{-\lambda/\beta})) \qquad \lambda < 0$$

Separation of variables

As for the equation $\frac{d}{dt}\mathbf{u}(t) = A\mathbf{u}(t)$, we can get more solutions by taking linear combinations of these solutions:

$$
\begin{aligned}
u(x,t) &= \sum_{\lambda>0} e^{\lambda t}(A_\lambda e^{x\sqrt{\lambda/\beta}} + B_\lambda e^{-x\sqrt{\lambda/\beta}}) \\
&+ A_0 + B_0 x \\
&+ \sum_{\lambda<0} e^{\lambda t}(A_\lambda \cos(x\sqrt{-\lambda/\beta}) + B_\lambda \sin(x\sqrt{-\lambda/\beta}))
\end{aligned}
$$

Boundary values

Suppose our system system is a wire of length L in which the temperature at the ends is fixed at zero. Let us coordinatize our wire to run from 0 to L.

This constrains which functions in the previous expression are allowed.

One can think of it as restricting saying that we are considering solving the equations inside the vector space of functions with the above properties.

Boundary values

So, let us see which of the expressions $A_\lambda e^{x\sqrt{\lambda/\beta}} + B_\lambda e^{-x\sqrt{\lambda/\beta}}$ and $A_\lambda \cos(x\sqrt{-\lambda/\beta}) + B_\lambda \sin(x\sqrt{-\lambda/\beta})$ can possibly vanish at $x = 0$ and $x = L$.

$A_\lambda e^{x\sqrt{\lambda/\beta}} + B_\lambda e^{-x\sqrt{\lambda/\beta}}$ evaluates to

$$A_\lambda + B_\lambda = 0$$

$$A_\lambda e^{L\sqrt{\lambda/\beta}} + B_\lambda e^{-L\sqrt{\lambda/\beta}} = 0$$

Hence $A_\lambda = B_\lambda = 0$, since otherwise $e^{L\sqrt{\lambda/\beta}} = e^{-L\sqrt{\lambda/\beta}}$, which can never happen (take a log).

Boundary values

Trying $A_0 + B_0 x$, we find this says $A_0 = 0$ and $A_0 + B_0 L = 0$, hence $A_0 = B_0 = 0$.

Finally, we consider $A_\lambda \cos(x\sqrt{-\lambda/\beta}) + B_\lambda \sin(x\sqrt{-\lambda/\beta})$. The vanishing at $x = 0$ implies that $A_\lambda = 0$. The vanishing at $x = L$ implies that $L\sqrt{-\lambda/\beta}$ must be some integer multiple of π, let us say $N\pi$. In other words,

$$\sqrt{-\lambda/\beta} = N\pi/L$$

$$\lambda = -\beta\left(\frac{N\pi}{L}\right)^2$$

Boundary values

In sum, the solutions which obey the boundary value conditions are:

$$u(x,t) = \sum_{N=1}^{\infty} c_N e^{-\beta\left(\frac{N\pi}{L}\right)^2 t} \sin\left(\frac{N\pi}{L}x\right)$$

An initial-boundary value problem

Consider a wire of length π (from $x = 0$ to $x = \pi$), diffusivity $\beta = 1$, and with initial temperature ($t = 0$) distributed as

$$u(x, 0) = \sin(x) + 3\sin(2x) + 5\sin(3x)$$

What is the temperature distribution at time $t = 1$?

Setting $\beta = 1$ and $L = \pi$ in our general solutions gives

$$u(x, t) = \sum_{N=1}^{\infty} c_N e^{-N^2 t} \sin(Nx)$$

Asking for this to agree with the initial condition means that $c_1 = 1, c_2 = 3, c_3 = 5$ and all other coefficients vanish. So,

$$u(x, 1) = e^{-1}\sin(x) + 3e^{-4}\sin(2x) + 5e^{-9}\sin(3x)$$

The wave equation

Consider a string stretched horizontally between two points, but free to vibrate up and down. We will write $y(x, t)$ for the height at horizontal position x and time t.

It turns out that the vibration of the string is governed by the equation

$$\frac{\partial^2}{\partial t^2} y(x, t) = \alpha^2 \frac{\partial^2}{\partial x^2} y(x, t)$$

In fact, one can derive this equation by imagining that the string is made up of infinitely many tiny springs, but I won't do this here.

The wave equation

To solve this equation, let's again compare it to the ODE

$$\frac{d^2}{dt^2}\mathbf{y}(t) = A\mathbf{y}(t)$$

Given an eigenvector $\mathbf{v}$ for A with eigenvalue λ, we would get solutions of the form $Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$ for $\lambda > 0$, or solutions $A\cos(t\sqrt{-\lambda}) + B\sin(t\sqrt{-\lambda})$ for $\lambda < 0$.

For the vibrating string with endpoints fixed at say $0, L$, the operator A above is replaced with the operator $\alpha^2 \frac{d^2}{dx^2}$ on the vector space of functions in x which vanish at $0, L$.

The wave equation

We saw that the eigenfunctions for $\alpha^2 \frac{d^2}{dx^2}$ with these boundary conditions are given by $\sin\left(\frac{n\pi x}{L}\right)$, which has eigenvalue $-\alpha^2 \left(\frac{n\pi}{L}\right)^2$

The corresponding solution to the wave equation will be

$$y(x,t) = \left(A\cos\left(\frac{n\pi\alpha t}{L}\right) + B\sin\left(\frac{n\pi\alpha t}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

More general solutions will be given as linear combinations of these.